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Polynomial and rational matrix interpolation: theory and control applications

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A generalization of polynomial interpolation to the matrix case is introduced and applied to problems in systems and control. It is shown that this generalization is most general and it includes all other such interpolation schemes that have appeared in the literature. The polynomial matrix interpolation theory is developed and then applied to solving matrix equations; solutions to the diophantine equation are also derived. The relation between a polynomial matrix and its characteristic values and vectors is established and it is used in pole assignment and other control problems. Rational matrix interpolation is discussed; it can be seen as a special case of polynomial matrix interpolation. It is then used to solve rational matrix equations including the model matching problem.

1. Introduction

A theory of polynomial and rational matrix interpolation is introduced in this paper and its application to certain systems and control problems is discussed at length. Note that many system and control problems can be formulated in terms of matrix equations where polynomial or rational solutions with specific properties are of interest. It is known that equations involving just polynomials can be solved by either equating coefficients of equal power of the indeterminate s or, equivalently, by using the values obtained when appropriate values for s are substituted in the given polynomials; in the latter case one uses results from the theory of polynomial interpolation. Similarly, one may solve polynomial matrix equations using the theory of polynomial matrix interpolation presented here; this approach has significant advantages and these are discussed below. In addition to equation solving, there are many instances where interpolation-type constraints are being used in systems and control without adequate justification; the theory presented here provides such justification and thus it clarifies and builds confidence into these methods.

Polynomial interpolation has fascinated researchers and practitioners alike. This is probably due to the mathematical simplicity and elegance of the theory complemented by the wide applicability of its results to areas such as numerical analysis among others. Note that, although for the scalar polynomial case, interpolation is an old and very well studied problem, only recently has polynomial matrix interpolation appeared to have been addressed in any systematic way (Antsaklis 1980, 1983, Antsaklis and Lopez 1984, Antsaklis and Gao 1990, Lopez 1984). Rational, mostly scalar interpolation has been of interest to systems and control researchers recently. Note that the rational

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interpolation results presented here are distinct from other literature results as they refer to the matrix case and concentrate on fundamental representation questions. Other results in the literature attempt to characterize rational functions that satisfy certain interpolation constraints and are optimal in some sense and so they rather complement our results than compete with them.

In this paper, a polynomial matrix interpolation of the type $Q(s_j)a_j = b_j$, where $Q(s)$ is a matrix and a_j, b_j are vectors, is introduced as a generalization of the scalar polynomial interpolation of the form $q(s_j) = b_j$. This generalization appears to be well suited to studying and solving a variety of multivariable system and control problems. The original motivation for the development of the matrix interpolation theory was to be able to solve the polynomial matrix equations that appear in the theory of systems and control and, in particular, the diophantine equation; the results presented here however go well beyond solving that equation. It should be pointed out that the driving force, while developing the theory and the properties of matrix interpolation, has always been system and control needs. This explains why no attempt has been made to generalize more of the classical polynomial interpolation theory results to the matrix case. This was certainly not because it was felt that it would be impossible, quite the contrary. The emphasis on system and control properties in this paper simply reflects the main research interests of the authors.

Characteristic values and vectors of polynomial matrices are also discussed in this paper. Note that contrary to the polynomial case, the zeros of the determinant of a square polynomial matrix $Q(s)$ do not adequately characterize $Q(s)$; additional information is needed that is contained in the characteristic vectors of $Q(s)$, which must also be given together with the characteristic values, to characterize $Q(s)$.

The use of interpolation-type constraints in system and control theory is first discussed and a number of examples are presented.

Motivation: interpolation-type constraints in systems and control theory

Many control system constraints and properties that are expressed in terms of conditions on a polynomial or rational matrix $R(s)$, can be written in an easier to handle form in terms of $R(s_j)$, where $R(s_j)$ is $R(s)$ evaluated at certain (complex) values $s = s_j$, $j = 1, l$. We shall call such conditions in terms of $R(s_j)$, interpolation (type) conditions on $R(s)$. This is because, in order to understand the exact implications of these constraints on the structure and properties of $R(s)$, one needs to use results from polynomial interpolation theory. Next, a number of examples from systems and control theory where polynomial and polynomial matrix interpolation constraints are used, are outlined. This list is not complete, by far.

Eigenvalue/eigenvector conrtrollability tests: It is known that all the uncontrollable eigenvalues of $\dot{x} = Ax + Bu$ are given by the roots of the determinant of a greatest left divisor of the polynomial matrices $sI - A$ and B. An alternative, and perhaps easier to handle, form of this result is that s_j is an uncontrollable eigenvalue if and only if $\operatorname{rank}[s_jI - A, B] < n$ where A is $n \times n$ (PBH controllability test—see Kailath 1980). This is a more restrictive version of the previous result which involves left divisors, since it is not clear how to handle multiple eigenvalues when it is desirable to determine all uncontrollable eigenvalues. The results presented here can readily provide the solution to this problem.

Selecting $T(s)$: in the Model Matching Problem, the plant $H(s)$ and the desired transfer function matrix $T(s)$ are given and a proper and stable $M(s)$ is to be found so that $T(s) = H(s)M(s)$. The selection of $T(s)$ for such $M(s)$ to exist can be handled with matrix interpolation.

The *state feedback pole assignment problem* has a rather natural formulation in terms of interpolation-type constraints; similarly, so has the *output feedback pole assignment problem*.

More recently, stability constraints in the H^∞ *formulation of the optimal control problem* have been expressed in terms of interpolation-type constraints (Kimura 1987, Shaked 1989, Chang and Pearson 1984). It is rather interesting that Shaked (1989) and Chang and Pearson (1984) discussed a 'directional' approach which is in the same spirit of the approach taken here.

The above are just a few of the many examples of the strong presence of interpolation type conditions in the systems and control literature; this is because they represent a convenient way to handle certain types of constraints. However, a closer look reveals that the relationships between conditions on $R(s_i)$ and properties of the matrix $R(s)$ are not clear at all and this needs to be explained. Only in this way can one take full advantage of the method and develop new approaches to handle control problems. Our research on matrix interpolation and its applications addresses this need.

The main ideas of the polynomial matrix interpolation results can be found in earlier publications (Antsaklis 1980, 1983, Antsaklis and Lopez 1984, Antsaklis and Gao 1990, Lopez 1984), with state and static output feedback applications appearing in Antsaklis (1977) and Antsaklis and Wolovich (1984); some of the material on rational matrix interpolation has appeared before in Antsaklis and Gao (1990).

Here, a rather complete theory of polynomial and rational matrix interpolation with applications is presented. Note that all the algorithms in this paper have been successfully implemented in Matlab. In summary, the contents of the paper are as follows.

Summary

Section 2 presents the main results of polynomial matrix interpolation. In particular, Theorem 2.1 shows that a $p \times m$ polynomial matrix $Q(s)$ of column degrees d_i, $i = 1, m$, can be uniquely represented, under certain conditions, by $l = \sum d_i + m$ triplets (s_j, a_j, b_j), $j = 1, l$, where s_j is a complex scalar and a_j, b_j are vectors such that $Q(s_j)\ a_j = b_j$, $j = 1, l$. It is shown that this formulation is most general and it includes, as special cases, other interpolation constraints which have been used in the literature.

In § 3 equations involving polynomial matrices are studied using interpolation. All solutions of (highest) degree r are characterized and it is shown how to impose additional constraints on the solutions. The diophantine equation is an important special case and it is examined at length. The conditions under which a solution to the diophantine equation of degree r does exist are established and a method based on the interpolation results to find all such solutions is also given.

In § 4 the characteristic values of a polynomial matrix $Q(s)$ are discussed and all matrices with given characteristic values and vectors are characterized. Based on these results it is possible to impose restrictions on $Q(s)$ of the form $Q(s_j)a_j = 0$ that imply certain characteristic value locations with certain algebraic and geometric multiplicity. This problem is completely solved here. The cases when the desired multiplicities require the use of conditions involving derivatives of $Q(s)$ are handled in the Appendix.

In § 5, the results developed in the previous section on the characteristic values and vectors of a polynomial matrix $Q(s)$ are used to study several systems and control problems. The pole or eigenvalue assignment is a problem studied extensively in the literature. It is shown how this problem can be addressed using interpolation, in a way which is perhaps more natural and effective; dynamic (and static) output feedback as well as state feedback is used and assignment of both characteristic values and vectors is studied. Tests for controllability and observability and the control design problem with the interpolation-type of constraints are also discussed.

Section 6 introduces rational matrix interpolation. It is first shown that rational matrix interpolation can be seen as a special case of polynomial matrix interpolation and the conditions under which a rational matrix $H(s)$ is uniquely represented by interpolation triplets are derived in Theorem 6.1. It is also shown how additional constraints on $H(s)$ can be incorporated. These results are then applied to rational matrix equations and results analogous to the results on polynomial matrix equations derived in the previous sections are obtained.

The Appendix contains the general versions of the results in § 4, that are valid for repeated values of s_j, with multiplicities beyond those handled in that section. Smith forms are defined and the relation between Smith and Jordan canonical forms is shown.

2. Polynomial matrix interpolation

In this section the theory of polynomial matrix interpolation is introduced. The main result is given by Theorem 2.1 where it is shown that a $p \times m$ polynomial matrix $Q(s)$ of column degrees d_i, $i = 1, m$, can be uniquely represented, under certain conditions, by $l = \sum d_i + m$ triplets (s_j, a_j, b_j), $j = 1, l$, where s_j is a complex scalar and a_j, b_j are vectors such that $Q(s_j)a_j = b_j$, $j = 1, l$. One may have fewer than $\sum d_i + m$ interpolation points l in which case the matrix (with column degrees d_i) can satisfy additional constraints. This is very useful in applications and it is shown in (2.6); in Corollary 2.2 the leading coefficient is assigned. Connections to the eigenvalues and eigenvectors are established in Corollary 2.3. In Lemma 2.4 the choice of the interpolation points is discussed. In Theorem 2.1 the vector a_j postmultiplies $Q(s)$; in Corollary 2.5 premultiplication of $Q(s)$ by a vector is considered and similar (dual) results are derived. The theory of polynomial matrix interpolation presented here is a generalization of the interpolation theory of polynomial and there are, of course, alternative approaches which are discussed; they are shown to be special cases of the formulation in Theorem 2.1. In particular, $Q(s)$ is seen as a matrix polynomial and alternative expressions are derived in Corollary 2.6; in Corollary 2.7 interpolation constraints of the form $Q(z_k) = R_k$, $k = 1, q$, are considered, which may be seen as a direct generalization of polynomial constraints. Finally, in Theorem 2.8 derivatives of $Q(s)$ are used to generalize the main interpolation results.

The basic theorem of polynomial interpolation can be stated as follows. Given l distinct complex scalars s_j, $j = 1, l$, and l corresponding complex values b_j, there exists a unique polynomial $q(s)$ of degree $n = l - 1$ for which

$$q(s_j) = b_j \quad j = 1, l \tag{2.1}$$

That is, and nth degree polynomial $q(s)$ can be uniquely represented by the $l = n + 1$ interpolation (points or doublets or) pairs (s_j, b_j), $j = 1, l$. To see this, write the nth degree polynomial $q(s)$ as $q(s) = q[1, s, \ldots, s^n]'$ where q is the $(1 \times (n+1))$ row vector of the coefficients and $[\,]'$ denotes the transpose. The $l = n + 1$ equations in (2.1) can then be written as

$$qV = q\begin{bmatrix} 1 & \cdots & 1 \\ s_1 & & s_l \\ \vdots & & \vdots \\ s_1^{l-1} & \cdots & s_l^{l-1} \end{bmatrix} = [b_1, \ldots, b_l] = B_l$$

Note that the matrix $V(l \times l)$ is the well-known Vandermonde matrix which is non-singular if and only if the l scalars s_j, $j = 1, l$, are distinct. Here s_i are distinct and therefore V is non-singular. This implies that the above equation has a unique solution q; that is, there exists a unique polynomial $q(s)$ of degree n which satisfies (2.1). This proves the above-stated basic theorem of polynomial interpolation.

There are several approaches to generalize this result to the polynomial matrix case and a number of these are discussed later in this section. It is shown that they are special cases of the basic polynomial matrix interpolation theorem that follows.

Let $S(s) := \text{blk diag}\{[1, s, \ldots, s^{d_i}]'\}$ where d_i, $i = 1, m$, are non-negative integers; let $a_j \neq 0$ and b_j denote $(m \times 1)$ and $(p \times 1)$ complex vectors respectively and s_j complex scalars.

Theorem 2.1: *Given interpolation (points) triplets (s_j, a_j, b_j), $j = 1, l$, and non-negative integers d_i with $l = \sum d_i + m$ such that the $(\sum d_i + m) \times l$ matrix*

$$S_l := [S(s_1)a_1, \ldots, S(s_l)a_l] \tag{2.2}$$

has full rank, there exists a unique $(p \times m)$ polynomial matrix $Q(s)$, with ith column degree equal to d_i, $i = 1, m$ for which

$$Q(s_j)a_j = b_j \quad j = 1, l \tag{2.3}$$

Proof: Since the column degrees of $Q(s)$ are d_i, $Q(s)$ can be written as

$$Q(s) = QS(s) \tag{2.4}$$

where Q $(p \times (\sum d_i + m))$ contains the coefficients of the polynomial entries. Substituting in (2.3), Q must satisfy

$$QS_l = B_l \tag{2.5}$$

where $B_l := [b_1, \ldots, b_l]$. Since S_l is non-singular, Q and therefore $Q(s)$ are uniquely determined. □

It should be noted that when $p = m = 1$ and $d_1 = l - 1 = n$ this theorem reduces to the polynomial interpolation theorem. To see this, note that in this case the non-zero scalars a_j, $j = 1, l$, can be taken to be equal to 1, in which case $S_l = V$ the well known Vandermonde matrix; V is non-singular if and only if s_j, $j = 1, l$, are distinct.

Example 2.1: Let $Q(s)$ be a 1×2 ($= p \times m$) polynomial matrix and let $l = 3$ interpolation points or triplets be specified:

$$\{(s_j, a_j, b_j)\ j = 1, 2, 3\} = \{(-1, [1, 0]', 0), (0, [-1, 1]', 0), (1, [0, 1]', 1)\}$$

In view of Theorem 2.1, $Q(s)$ is uniquely specified when d_1 and d_2 are chosen so that $l\ (= 3) = \sum d_i + m = (d_1 + d_2) + 2$ or $d_1 + d_2 = 1$ assuming that S_3 has full rank. Clearly there is more than one choice for d_1 and d_2; the resulting $Q(s)$ depends on the particular choice for the column degrees d_i and different combinations of d_i will result in different matrices $Q(s)$. In particular:

(i) let $d_1 = 1$, and $d_2 = 0$. Then $S(s) = \text{blk diag}\{[1\ s]', 1\}$ and (2.5) becomes:

$$QS_3 = Q[S(s_1)a_1, S(s_2)a_2, S(s_3)a_3] = Q\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} = [0, 0, 1] = B_3$$

from which $Q = [1, 1, 1]$ and $Q(s) = QS(s) = [s + 1, 1]$;

(ii) let $d_1 = 0$, $d_2 = 1$. Then $S(s) = \text{blk diag}\{1, [1, s]'\}$ and (2.5) gives $Q = [0, 0, 1]$ from which $Q(s) = [0, s]$, is clearly different from (i) above. □

2.1. *Discussion of the interpolation theorem*

2.1.1. *Representations of Q(s).* Theorem 2.1 provides an alternative way to represent a polynomial matrix (or a polynomial) other than by its coefficients and degree of each entry. More specifically: a polynomial $q(s)$ is specified uniquely by its degree, say, n and its $n + 1$ ordered coefficients. Alternatively, in view of (2.1) the l pairs (s_j, b_j), $j = 1, l$, uniquely specify the nth degree polynomial $q(s)$ provided that $l = n + 1$ and the scalars s_j are distinct.

Similarly, a polynomial matrix $Q(s)$ is specified uniquely by its dimensions $p \times m$, its column degrees d_i, $i = 1, m$, and the $d_i + 1$ coefficients in each entry of column i. In view of Theorem 2.1, given the dimensions $p \times m$, the polynomial matrix $Q(s)$ is uniquely specified by its column degrees d_i, $i = 1, m$, and the l triplets (s_j, a_j, b_j), $j = 1, l$, provided that $l = \sum d_i + m$ and (s_j, a_j) are so that S_l in (2.2) has full rank. Notice that when $p = m = 1$ these conditions reduce to the well-known polynomial interpolation conditions described above, namely that s_j must be distinct.

2.1.2. *Number of interpolation points.* It is of interest to examine what happens when the number of interpolation points l, in Theorem 2.1, is different from the required number determined by the number of columns m and the desired column degrees of $Q(s)$, d_i, $i = 1, m$. That is, what happens when $l \neq \sum d_i + m$: the equation of interest is $QS_l = B_l$ in (2.5). A solution $Q(p \times (\sum d_i + m))$ of this equation exists if and only if

$$\text{rank}\begin{bmatrix} S_l \\ B_l \end{bmatrix} = \text{rank}\, S_l$$

This implies that there exists a solution Q for any B_l if and only if $\text{rank}(S_l) = l$; that is, if and only if S_l, a $(\sum d_i + m) \times l$ matrix has full column rank.

(i) When $l > \sum d_i + m$, the system of equations in (2.5) is over specified; there are more equations than unknowns as S_l is a $(\sum d_i + m) \times l$ matrix. If now the additional $(l - (\sum d_i + m))$ equations are linearly dependent upon the previous $(\sum d_i + m)$ ones, then a $Q(s)$ with column degrees d_i, $i = 1, m$, is uniquely determined provided that $(\sum d_i + m)$ interpolation triplets (s_j, a_j, b_j) satisfy the conditions of Theorem 2.1. Otherwise there is no matrix of column degrees d_i, $i = 1, m$, which satisfies these interpolation constraints. In this case these interpolation points represent a matrix of column degrees greater than d_i.

(ii) When $l < \sum d_i + m$, then $Q(s)$ with column degrees d_i, $i = 1, m$, is not uniquely specified, since there are more unknown than equations in (2.5). That is, in this case there are many $(p \times m)$ matrices $Q(s)$ with the same column degrees d_i which satisfy the l interpolation constraints (2.3) and therefore can be represented by these l interpolation triplets (s_j, a_j, b_j).

2.1.3. *Additional constraints.* This additional freedom (in (ii) above) can be exploited so that $Q(s)$ satisfies additional constraints. In particular, $k := (\sum d_i + m) - l$ additional linear constraints, expressed in terms of the coefficients of $Q(s)$ (in Q), can be satisfied in general. The equations describing the constraints can be used to augment the equations in (2.5). This is a very useful characteristic and it is used extensively in later sections. In this case the equations to be solved become

$$Q[S_l, C] = [B_l, D] \tag{2.6}$$

where $QC = D$ represent $k := (\sum d_i + m) - l$ linear constraints imposed on the coefficients Q; C and D are matrices (real or complex) with k columns each. The following examples illustrate the above.

Example 2.2. (i) Consider a 1×2 polynomial matrix $Q(s)$ and $l = 3$ interpolation points:

$$\{(s_j, a_j, b_j)\ j = 1, 2, 3\} = \{(-1, [1, 0]', 0), (0, [-1, 1]', 0), (1, [0, 1]', 1)\}$$

as in Example 2.1. Let $d_1 = 1$, $d_2 = 0$. It was shown in Example 2.1 (i) that the above uniquely represent $Q(s) = [s + 1, 1]$. Suppose now that an additional interpolation point $(s_4, a_4, b_4) = (1, [1, 0]', 2)$ is specified. Here $l = 4 > \sum d_i + m = 1 + 2 = 3$ and

$$QS_4 = Q\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = [0, 0, 1, 2] = B_4$$

Notice however that the last equation $Q[1\ 1\ 0]' = 2$ can be obtained from $QS_3 = B_3$, by a postmultiplication by $[-1\ -2\ 2]'$. Clearly, the additional interpolation point does not impose any additional constraints on $Q(s)$ as it does not contain any new information about $Q(s)$. If now the new interpolation point is taken to be $(s_4, a_4, b_4) = [1, [1, 0]', 3)$ then, as can be easily verified, there is no $Q(s)$ with $d_1 + d_2 = 1$ which satisfies all four interpolation constraints. In this case one should consider $Q(s)$ with higher column degrees, namely $d_1 + d_2 = 2$.

(ii) Consider again a 1×2 polynomial matrix $Q(s)$ but with $l = 2$ interpolation points:

$$\{(s_j, a_j, b_j)\ j = 1, 2\} = \{(-1, [1, 0]', 0), (0, [-1, 1]', 0)\}$$

from Example 2.1. Let $d_1 = 1$, $d_2 = 0$. Here $l = 2 < \sum d_i + m = 1 + 2 = 3$. In this case it is possible, in general, to satisfy $(\sum d_i + m) - l = 1$ additional (linear) constraint. In particular

$$Q[S_2, C] = Q\begin{bmatrix} 1 & -1 & c_1 \\ -1 & 0 & c_2 \\ 0 & 1 & c_3 \end{bmatrix} = [0, 0, d] = [B_2, D]$$

where $Q[c_1\ c_2\ c_3]' = d$ is the additional constraint on the coefficients Q of

$$Q(s) = QS(s) = [q_1\ q_2\ q_3]\begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix}$$

For example, if it is desired that the coefficient $q_1 = 2$, this can be enforced by taking $c_1 = c_3 = 0$ and $c_2 = 1$, $d = 2$. Then $Q = [2\ 2\ 2]$ and $Q(s) = [2s + 2\ 2]$ satisfies all requirements. □

The additional constraints on $Q(s)$ (or Q) do not, of course, have to be linear. They can be described, for example, by nonlinear algebraic equations or inequalities. However, in contrast to the linear constrains, it is difficult in this case to show general results.

2.1.4. *Determination of the leading coefficients.* It is well known that if the leading coefficient of an nth degree polynomial is given, then n, not $n + 1$, distinct points suffice to determine uniquely this polynomial. A corresponding result is true in the polynomial matrix case.

Let C_c denote the matrix with ith column entries the coefficients of s^{d_i}, in the ith column of $Q(s)$; that is, the leading matrix coefficient (with respect to columns) of $Q(s)$. Let also $S_1 := \operatorname{blk\,diag}\{[1, s, \ldots, s^{d_i - 1}]'\}$, $i = 1, m$, where the assumption that d_i is greater than zero is made for S_1 to be well defined. Note that this assumption is relaxed in the alternative expression of these results discussed immediately following the Corollary.

Corollary 2.2: *Given (s_j, a_j, b_j), $j = 1, l$, and non-negative integers d_i with $l = \sum d_i$ such that the $(\sum d_i) \times l$ matrix $S_{1l} := [S_1(s_1)a_1, \ldots, S_1(s_l)a_l]$ has full rank, there exists a unique $(p \times m)$ polynomial matrix $Q(s)$, with ith column degree d_i, and a given leading coefficient matrix C_c which satisfies (2.3).*

Proof: $Q(s) = C_c D(s) + Q_1 S_1(s)$ with $D(s) := \operatorname{diag}[s^{d_i}]$ for some coefficient $p \times (\sum d_i)$ matrix Q_1. Equation (2.3) implies

$$Q_1 S_{1l} = B_l - C_c[D(s_1)a_1, \ldots, D(s_l)a_l] \tag{2.7}$$

which has a unique solution Q_1 since S_{1l} is non-singular. $Q(s)$ is therefore uniquely determined. □

Note that here the given C_c provides the additional m constraints (for a total of $\sum d_i + m$) needed to determine uniquely $Q(s)$ in view of Theorem 2.1. It is also easy to see that when $p = m = 1$, the corollary reduces to the polynomial interpolation result mentioned above.

The results in Corollary 2.2 can be seen in view of our previous discussion for the case when only $l < \sum d_i + m$ interpolation points are given. In that case it was possible to satisfy, a general $k := (\sum d_i + m) - l$ additional constraints. Here, the requirement that the leading coefficients should be C_c can be written as

$$Q[S_l, C] = [B_l, C_c] \tag{2.8}$$

where C is chosen to extract the leading coefficients from Q. Since C_c has $k = m$ columns, $l = \sum d_i$ interpolation points will suffice to generate $\sum d_i + m$ equations with $\sum d_i + m$ unknowns, to determine uniquely $Q(s)$.

Example 2.3: Consider a 1×2 polynomial matrix $Q(s)$ with column degrees $d_1 = 1$, $d_2 = 0$. Assume that the interpolation point $(l = \sum d_i = 1)$ is $(s_1, a_1, b_1) = (-1, [1, 0]', 0)$ and the desired leading coefficient is $C_c = [c_1, c_2]$. Then

$$Q[S_1, C] = Q\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0\ c_1\ c_2] = [B_1, C_c]$$

from which $Q = [c_1, c_1, c_2]$ and $Q(s) = [c_1 + c_1 s, c_2]$. □

2.1.5. *Interpolation constraints with $B_l = 0$.* Often the interpolation constraints (2.3) are of the form

$$Q(s_j)\, a_j = 0 \quad j = 1, l \tag{2.9}$$

leading to a system of equations

$$QS_l = 0 \tag{2.10}$$

where S_l is a $(\sum d_i + m) \times l$ matrix; see Theorem 2.1. In this case, if the conditions of Theorem 2.1 are satisfied then the unique $Q(s)$ which is described by the $l = (\sum d_i + m)$ interpolation points is $Q(s) = 0$. It is perhaps instructive to point out what this result means in the polynomial case. In the polynomial case this result simply states that the only nth degree polynomial with $n + 1$ distinct roots s_j is the zero polynomial, a rather well-known fact. It is useful to determine non-zero solutions Q of (2.10). One way to achieve this is to use:

$$Q[S_l, C] = [0, D] \tag{2.11}$$

where again S_l is a $(\sum d_i + m) \times l$ matrix but the number of interpolation points l is taken to be $l < \sum d_i + m$. In this way $Q(s)$ is not necessarily equal to a zero matrix. The matrices C and D each have $k := (\sum d_i + m) - l$ columns, so that $Q(s)$ can satisfy, in general, k additional constraints; see Example 2.3.

2.1.6. *Eigenvalues and eigenvectors.* An interesting result is derived when Corollary 2.2 is applied to an $(n \times n)$ matrix $Q(s) = sI - A$. In this case $d_i = 1$, $i = 1, n$. $C_c = I$, $S_1(s) = I$ and $Q_1 = A$; also $l = n$, $S_{1n} = [a_1, \ldots, a_n]$ and (2.7) can be written as:

$$A\,[a_1, \ldots, a_n] = B_n - [a_1, \ldots, a_n]\operatorname{diag}[s_j] \tag{2.12}$$

When $[b_1, \ldots, b_n] = B_n = 0$ then, in view of (2.12) and Corollary 2.2, the following is true.

Corollary 2.3: *Given (s_j, a_j), $j = 1, n$, such that the $(n \times n)$ matrix $S_{1n} = [a_1, \ldots, a_n]$ has full rank, there exists a unique $n \times n$ polynomial matrix $Q(s)$*

with column degrees all equal to 1 and a leading coefficient matrix equal to I which satisfies (2.3) with all $b_j = 0$; *that is* $Q(s_j)a_j = (s_jI - A)a_j = 0$.

The above corollary simply says that A is uniquely determined by its n eigenvalues s_j and the n corresponding linearly independent eigenvectors a_j, a well-known result from matrix algebra. Here, this result was derived from our polynomial matrix interpolation theorem, thus pointing to a *strong connection between the polynomial matrix interpolation theory developed here and the classical eigenvalue eigenvector matrix algebra results.*

2.1.7. *Choice of interpolation points.* The main condition of Theorem 2.1 is that S_l, a $(\sum d_i + m) \times l$ matrix, has full (column) rank l. This guarantees that the solution Q in (2.5) exists for any B_l and it is unique. In the polynomial case S_l can be taken to be a Vandermonde matrix which has full rank if and only if s_j, $j = 1, l$, are distinct, and this has already been pointed out. In general however, in the matrix case, s_j, $j = 1, l$, do not have to be distinct; repeated values for s_j, coupled with appropriate choices for a_j will still produce full rank in S_l in many instances, as can be easily verified by example. This is a property unique to the matrix case.

Example 2.4: Consider a 1×2 polynomial matrix $Q(s)$ with $d_1 = 1$, $d_2 = 0$ (as in Example 2.1). Suppose that $l = 3$ interpolation points are given:

$$\{(s_j, a_j, b_j)\ j = 1, 2, 3\} = \{(0, [1, 0]', 1), (0, [0, 1]', 1), (1, [1, 0]', 2)\}.$$

Here $S(s) = \text{blk diag}\{[1, s]', 1\}$ and

$$QS_3 = Q\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = [1, 1, 2] = B_3$$

from which $Q(s) = QS(s) = [1\ 1\ 1]S(s) = [s + 1, 1]$. Note that the first two columns of S_3 are $S(0)[1, 0]'$ and $S(0)[0, 1]'$. They correspond to the same $s_j = 0$, $j = 1, 2$, and they are linearly independent. □

If s_j, $j = 1, l$, are taken to be distinct, then there always exist $a_j \neq 0$ such that S_l has full rank. An obvious choice is $a_j = e_1$ for $j = 1, d_1 + 1$, $a_j = e_2$ for $j = d_1 + 2, \ldots, d_1 + d_2 + 2$ etc, where the entries of column vector e_i are zero except for the ith entry which is 1; in this way, S_l is block diagonal with m Vandermonde matrices of dimensions $(d_i + 1) \times (d_i + 1)$, $i = 1, m$, on the diagonal, which has full rank since s_j are distinct (in fact we only need groups of $d_i + 1$ values of s_j to be distinct).

Example 2.5: In Example 2.4 ($Q(s)$ 1×2, $l = 3$, $d_1 = 1$, $d_2 = 0$) take s_1, s_2, and s_3 to be distinct and let $a_1 = a_2 = e_1 = [1\ 0]'$ and $a_3 = e_2 = [0\ 1]'$. Then in $QS_3 = B_3$,

$$S_3 = \begin{bmatrix} 1 & 1 & 0 \\ s_1 & s_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which has a block diagonal form with 2 $(= m)$ Vandermonde matrices on the diagonal. Clearly S_3 has full rank since s_1 and s_2 are distinct; so there is a unique solution Q for any B_3. □

It is also important to know, especially in applications, what happens to the rank of S_l for given a_j. It turns out that S_l has full rank for almost any choice of a_j when s_j are distinct. Consider the following.

Lemma 2.4: *Let* s_j, $j = 1, l$, *with* $l \leq \sum d_i + m$ *be distinct complex scalars. Then the* $(\sum d_i + m) \times l$ *matrix* S_l *in (2.2) has full column rank* l *for almost any set of non-zero* a_j, $j = 1, l$.

Proof: First note that S_l has at least as many rows $(\sum d_i + m)$ as columns (l). The structure of $S(s)$ together with the fact that $a_j \neq 0$ and s_j distinct imply that the lth order minors of S_l are non-zero multivariate polynomial in a_{ij}, the entries of a_j, $j = 1, l$. These minors become zero only for values of a_{ij} on a hypersurface in the parameter space. Furthermore, note that there always exists a set of a_j (see above) for which at least one lth order minor is non-zero. This implies that rank $S_l = l$ for almost any set of $a_j \neq 0$. □

Example 2.6: Let $S(s) = \text{blk diag}\{[1, s]', 1\}$ and take $s_1 = 0$, $s_2 = 1$ (distinct). Then

$$S_2 = [S(s_1)a_1, S(s_2)a_2] = \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

where $a_1 = [a_{11}, a_{21}]'$ and $a_2 = [a_{12}, a_{22}]'$ $(\neq 0)$. Rank S_2 will be less than 2 $(= l)$ for values of a_{ij} which make zero all the second-order minors: $a_{11}a_{12}$, $a_{11}a_{22} - a_{12}a_{21}$, $a_{12}a_{21}$. Such a case is, for example, when $a_{11} = a_{12} = 0$. □

2.1.8. *Alternative bases.* Note that alternative polynomial bases, other than $[1, s, s^2, \ldots]'$, which might offer computational advantages in determining $Q(s)$ from interpolation equations (2.5) can, of course, be used. Choices include Chebychev polynomials, among others, and they are discussed further later in this paper in relation to applications of the interpolation results.

2.2. *Alternative approaches to matrix interpolation*

(i) *Dual version.* In Theorem 2.1, a_j are column vectors which postmultiply $Q(s_j)$ in (2.3) to obtain the interpolation constraints $Q(s_j)a_j = b_j$; b_j are also column vectors. It is clear that one could also have interpolation constraints of the form

$$\underline{a}_j\, Q(s_j) = \underline{b}_j \quad j = 1, l \tag{2.13}$$

where $\underline{a}_j$ and $\underline{b}_j$ are row vectors. Equation (2.13) gives rise to an alternative ('dual') matrix interpolation result which we include here for completeness.

Let $\underline{S}(s) = \text{blk diag}\{[1, s, \ldots, s^{\underline{d}_i}]\}$ where $\underline{d}_i$, $i = 1, p$, are non-negative integers; let $\underline{a}_j \neq 0$ and $\underline{b}_j$ denote $(1 \times p)$ and $(1 \times m)$ complex vectors respectively and s_j complex scalars.

Corollary 2.5: *Given* $(s_j, \underline{a}_j, \underline{b}_j)$, $j = 1, l$, *and non-negative integers* $\underline{d}_i$ *with* $l = \sum \underline{d}_i + p$ *such that the* $l \times (\sum \underline{d}_i + p)$ *matrix*

$$\underline{S}_l := \begin{bmatrix} \underline{a}_1 \underline{S}(s_1) \\ \vdots \\ \underline{a}_l \underline{S}(s_l) \end{bmatrix} \tag{2.14}$$

has full rank, there exists a unique $(p \times m)$ *polynomial matrix* $Q(s)$, *with ith row degree equal to* $\underline{d}_i$, $i = 1, p$, *for which (2.13) is true.*

Proof: The proof is similar to the proof of Theorem 2.1: $Q(s)$ can be written as

$$Q(s) = \underline{S}(s)\underline{Q} \tag{2.15}$$

where $\underline{Q}((\sum \underline{d}_i + p) \times m)$ contains the coefficients of the polynomial entries of $Q(s)$. Substituting in (2.8) where $B_l = [\underline{b}_1', \ldots, \underline{b}_l']'$, $\underline{Q}$ must satisfy

$$\underline{S}_l \underline{Q} = \underline{B}_l \tag{2.16}$$

Since $\underline{S}_l$ is non-singular, $\underline{Q}$ and therefore $Q(s)$ are uniquely determined. □

Example 2.7: Let $Q(s)$ be a 1×2 $(= p \times m)$ polynomial matrix and let $l = 2$ interpolation points be specified: $\{(s_j, \underline{a}_j, \underline{b}_j)\ j = 1, 2\} = \{(-1, 1, [0\ 1]), (0, 1, [1\ 1])\}$. Here $l = 2 = \sum \underline{d}_i + p$ from which $\underline{d}_1 = 1$; that is, a matrix of row degree 1 may be uniquely determined. Note that $\underline{S}(s) = [1, s]$. Then

$$\underline{S}_l\ \underline{Q} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix} \underline{Q} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

from which

$$\underline{Q} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

and $Q(s) = \underline{S}(s)\ \underline{Q} = [s + 1, 1]$ □

(ii) *$Q(s)$ as a matrix polynomial.* The relation between representation (2.4) used in Theorem 2.1 and an alternative, also commonly used, representation of $Q(s)$ is now shown, namely:

$$Q(s) = \bar{Q}S_d(s) = Q_0 + \cdots + Q_d s^d \tag{2.17}$$

where $S_d(s) := [I, \ldots, Is^d]'$ a $m(d + 1) \times m$ matrix with $d = \max(d_i)$, $i = 1, m$, and $\bar{Q} = [Q_0, \ldots, Q_d]$ the $(p \times m(d + 1))$ coefficient matrix. Notice that $S(s) = KS_d(s)$ where $K((\sum d_i + m) \times m(d + 1))$ describes the appropriate interchange of rows in $S_d(s)$ needed to extract $S(s)$ (of Theorem 2.1). Representation (2.17) can be used in matrix interpolation as the following corollary shows.

Corollary 2.6: *Given* (s_j, a_j, b_j), $j = 1, l$, *and non-negative integer d with* $l = m(d + 1)$ *such that the* $m(d + 1) \times l$ *matrix*

$$S_{dl} = [S_d(s_1)a_1, \ldots, S_d(s_l)a_l] \tag{2.18}$$

has full rank, there exists a unique $(p \times m)$ *polynomial matrix* $Q(s)$ *with highest degree d which satisfies (2.3).*

Proof: Consider Theorem 2.1 with $d_i = d$; then

$$\bar{Q}S_{dl} = B_l \tag{2.19}$$

is to be solved. The result immediately follows in view of $S(s) = KS_d(s)$ which implies that S_{dl} is non-singular, since here K is non-singular. □

Notice that, in order to represent uniquely a matrix $Q(s)$ with column degrees d_i, $i = 1, m$, Corollary 2.6 requires more interpolation points (s_j, a_j, b_j) than Theorem 2.1 since $md \geq \sum d_i$. This is, however, to be expected as, in this case, less information about the matrix $Q(s)$ is used (only the highest degree d), than in the case of the theorem where the individual column degrees are supposed to be known $(d_i, i = 1, m)$.

Example 2.8: Let $Q(s)$ be 1×2 $(= p \times m)$, $d = 1$ and let the $l = m(d + 1) = 4$ interpolation points (s_j, a_j, b_j) be as follows: let the first three be the same as in Example 2.1 and the fourth be $(2, [0, 1]', 1)$. The equation (2.19) now becomes

$$\bar{Q}S_{d4} = \bar{Q}\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} = [0, 0, 1, 1] = B_4$$

from which $\bar{Q} = [1, 1, 1, 0]$ and $Q(s) = \bar{Q}S_d(s) = Q_1 s + Q_0 = [s + 1, 1]$ as in Example 2.1 (i). If the fourth interpolation point is taken to be equal to $(2, [0, 1]', 2)$ then $B_l = [0, 0, 1, 2]$ while S_{dl} remains the same. Then $\bar{Q} = [0, 0, 0, 1]$ and $Q(s) = \bar{Q}S_d(s) = [0, s]$ as in Example 2.1(ii) □

Similarly to the case of Theorem 2.1, if the number of interpolation points $l < m(d + 1)$ then $Q(s)$ of degree d is not uniquely specified. In this case one could, in general, satisfy $k := m(d + 1) - l$ additional linearly constraints by solving

$$\bar{Q}[S_{dl}, C] = [B_l, D] \tag{2.20}$$

where $\bar{Q}C = D$ represent the k linear constraints imposed on the coefficients Q. Constraints on Q other than linear can, of course, be imposed in the same way as in the case of Theorem 2.1.

(iii) *Constraints of the form* (z_k, R_k), $k = 1, q$. Interpolation constraints of the form

$$Q(z_k) = R_k \quad k = 1, q \tag{2.21}$$

have also appeared in the literature. These conditions are but a special case of (2.3). In fact, for each k, (2.21) represents m special conditions of the form $Q(s_j)a_j = b_j$, $j = 1, l$, in (2.3). To see this, consider (2.3) and blocks of m interpolation points where $s_i = z_1$, $i = 1, m$ with $a_i = e_i$, $s_{m+i} = z_2$, $i = 1, m$, with $a_{m+i} = e_i$ and so on, where the entries of e_i are zero except the ith entry which is 1; then R_1 of (2.21) above is $R_1 = [b_1, \ldots, b_m]$, $R_2 = [b_{m+1}, \ldots, b_{2m}]$ and so on. In this case s_j are not distinct but they are m-multiple. This is illustrated in Example 2.9 below where: $m = 2$ and $s_1 = s_2 = 0$ with $a_1 = [1, 0]'$, $a_2 = [0, 1]'$ and $R_1 = [b_1, b_2] = [1, 1]$; also $s_3 = s_4 = 1$ with $a_3 = [1, 0]'$, $a_4 = [0, 1]'$ and $R_2 = [b_3, b_4] = [2, 1]$.

A simple comparison of the constraints (2.21) to the polynomial constraints (2.1) seems to suggest that this is an attempt to generalize directly the scalar results to the matrix case. As in the polynomial case, z_k, $k = 1, q$, therefore should perhaps be distinct for $Q(s)$ to be uniquely determined. Indeed this is the case, as is shown in the proof of the following corollary.

Corollary 2.7: *Given* (z_k, R_k), $k = 1, q$ *with* $q = d + 1$, *and* R_k $p \times m$, *such that the* $m(d + 1) \times mq$ *matrix*

$$S_{dk} := [S_d(z_1), \ldots, S_d(z_k)] \tag{2.22}$$

has full rank, there exists a unique $(p \times m)$ polynomial matrix $Q(s)$ with highest degree d which satisfies (2.21).

Proof: The proof is direct in view of Corollary 2.6; there are $l = mq$ interpolation points. Notice that here S_{dk} (after some reordering of rows and columns) is a block diagonal Vandermonde type matrix, and it is non-singular if and only if z_k are distinct. □

Example 2.9: Let $Q(s)$ be 1×2 $(= p \times m)$, $d = 1$ and let the $q = d + 1 = 2$ interpolation points be $\{(z_k, R_k), k = 1, 2\} = \{(0, [1, 1]), (1, [2, 1])\}$. In view of $Q(s) = \bar{Q}S_d(s) = (Q_1 s + Q_0)$

$$\bar{Q}[S_d(z_1), S_d(z_2)] = [R_1, R_2] \text{ or}$$

$$\bar{Q}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [1, 1, 2, 1]$$

from which $\bar{Q}$ $(= (Q_0, Q_1)) = [1, 1, 1, 0]$ and $Q(s) = \bar{Q}S_d(s) = [s + 1, 1]$ as in Examples 2.1 and 2.8. □

Note that if, instead of degree d, the column degrees d_i, $i = 1, m$, of $Q(s)$ are known, then a result similar to Corollary 2.7, but based directly on Theorem 2.1, can be derived and used to determine $Q(s)$ which satisfies (2.21) given (z_k, R_k), $k = 1, q$. In this case, for a unique solution, q is selected so that $mq \geqslant (\sum d_i + m)$.

In Corollaries 2.6 and 2.7 above, it is clear that the dual interpolation results of Corollary 2.5, instead of Theorem 2.1, could have been used to derive dual versions. These dual versions involve the row dimension p instead of m and they could lead in certain cases to requirements for fewer interpolation points, depending on the relative size of p and m. These alternative versions of the corollaries can be easily derived and they are not presented here.

(iv) *Using derivatives.* In the polynomial case, they are interpolation constraints which involve derivatives of $q(s)$ with respect to s. In this way, one could use repeated values s_j and still have linearly independent equations to work with. In the matrix case it is not necessary to have derivatives to allow some repeated values for s_j, since the key condition in Theorem 2.1 is S_l of (2.2) to be of full rank which, in general, does not imply that s_j must be distinct; see Example 2.4 and Corollary 2.7 above. Nevertheless, it is quite easy to introduce derivatives of $Q(s)$ in interpolation constraints and this is now done for generality and completeness.

Notice that the kth derivative of $S(s) := \text{blk diag}\{[1, s, \ldots, s^{d_i}]'\}$, $i = 1, m$, with respect to s, denoted by $S^{(k)}(s)$, is easily determined using the formula $(s^{d_i})^{(k)} = d_i(d_i - 1)\ldots(d_i - k + 1)s^{d_i - k}$ for k less or equal to d_i and $(s^{d_i})^{(k)} = 0$ for k larger than d_i. The interpolation constraints $Q(s_j)a_j = b_j$ in (2.3) now have a more general form

$$Q^{(k)}(s_j)a_{kj} = b_{kj} \quad k = 0, 1, \ldots \tag{2.23}$$

for each distinct value s_j. Clearly, $Q(s) = QS(s)$ implies $Q^{(k)}(s) = QS^{(k)}(s)$ and

$$QS^{(k)}(s_j)a_{kj} = b_{kj} \tag{2.24}$$

in view of (2.23). There is a total of l relations of this type which can be written as $QS_l = B_l$, as in (2.5). To be able to determine uniquely $Q(s)$, the new matrix S_l, which now contains columns of the form $S^{(k)}(s_j)a_{kj}$, must have full (column) rank. In particular, the following result can be shown.

Theorem 2.8: *Consider interpolation triplets (s_j, a_{kj}, b_{kj}) where s_j, $j = 1, \sigma$, distinct complex scalars and $a_{kj} \neq 0$ $(m \times 1)$, b_{kj} $(p \times 1)$ complex vectors. If $k = 0, l_j - 1$, let the total number of interpolation points be*

$$l = \sum_1^{\sigma} l_j.$$

For non-negative integers d_i, $i = 1, m$, and $l = \sum d_i + m$ assume that the $(\sum d_i + m) \times l$ matrix S_l with columns of the form $S^{(k)}(s_j)a_{kj}$ $j = 1, \sigma$, $k = 0, l_j - 1$ namely

$$S_l := [S^{(0)}(s_1)a_{01}, \ldots, S^{(l_1 - 1)}(s_1)a_{l_1 - 1, 1}, \ldots, S^{(0)}(s_\sigma)a_{0\sigma}, \ldots] \tag{2.25}$$

has full column rank. Then there exists a unique $p \times m$ polynomial matrix $Q(s)$ which satisfies (2.23).

Proof: The proof is similar to Theorem 2.1. Solve $QS_l = B_l$ to derive the unique Q and $Q(s) = QS(s)$. □

Example 2.10: Consider a 1×2 polynomial matrix Q with $d_1 = 1$, $d_2 = 0$ and let the $l = \sum d_i + m = 3$ interpolation points $\{(s_1, a_{01}, b_{01}), (s_1, a_{11}, b_{11}), (s_2, a_{02}, b_{03})\} = \{(-1, [1\ 0]', 0), (-1, [1\ 0]', 1), (0, [0\ 1]', 1)\}$ satisfy $Q(s_1)a_{01} = b_{01}$, $Q^{(1)}(s_1)a_{11} = b_{11}$ and $Q(s_2)a_{02} = b_{02}$. Here, $\sigma = 2$, $l_1 = 2$, $l_2 = 1$ and $l = \sum_1^{\sigma} l_j = 3$. Now

$$QS_3 = Q[S^{(0)}(s_1)a_{01}, S^{(1)}(s_1)a_{11}, S^{(0)}(s_2)a_{02}] = Q\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0\ 1\ 1]$$

$$= [b_{01}, b_{11}, b_{02}] = B_3$$

from which $Q = [1\ 1\ 1]$ and $Q(s) = QS(s) = [s + 1, 1]$. □

3. Solution of polynomial matrix equations

In this section equations of the form $M(s)L(s) = Q(s)$ are studied. The main result is Theorem 3.1 where it is shown that all solutions $M(s)$ of (highest) degree r can be derived by solving an equation $MS_{rl} = B_l$ derived using interpolation. In this way, all solutions of degree r of the polynomial equation, if they exist, are characterized. The existence and uniqueness of solutions is discussed, as well as methods to impose constraints on the solutions. Alternative bases are examined in numerical considerations. The diophantine equation is an important special case and it is examined at length. Lemma 3.2 and Corollary 3.3 establish some technical results necessary to prove the main result in Theorem 3.4 which shows the conditions under which a solution to the diophantine equation of degree r does exist; a method based on the interpolation results to find all such solutions is also given. Using this method, it is quite easy to impose additional constraints the solutions must satisfy and this is shown.

Consider the equation

$$M(s)L(s) = Q(s) \tag{3.1}$$

where $L(s)$ $(t \times m)$ and $Q(s)$ $(k \times m)$ are given polynomial matrices. The polynomial matrix interpolation theory developed above will now be used to solve this equation and determine the polynomial matrix solutions $M(s)$ $(k \times t)$ when one exists.

First consider the left-hand side of equation (3.1). Let

$$M(s) := M_0 + \cdots + M_r s^r \tag{3.2}$$

where r is a non-negative integer, and let $d_i := \deg_{ci}[L(s)]$, $i = 1, m$. If

$$\hat{Q}(s) := M(s)L(s) \tag{3.3}$$

then $\deg_{ci}[\hat{Q}(s)] = d_i + r$ for $i = 1, m$. According to the basic polynomial matrix interpolation Theorem 2.1, the matrix $\hat{Q}(s)$ can be uniquely specified using $\sum(d_i + r) + m = \sum d_i + m(r+1)$ interpolation points. Therefore consider l interpolation points (s_j, a_j, b_j), $j = 1, l$, where

$$l = \sum d_i + m(r+1) \tag{3.4}$$

Let $S_r(s) := \operatorname{blk\,diag}\{[1, s, \ldots, s^{d_i + r}]'\}$ and assume that the $(\sum d_i + m(r+1)) \times l$ matrix

$$S_{rl} := [S_r(s_1)\, a_1, \ldots, S_r(s_l) a_l] \tag{3.5}$$

has full rank; that is, the assumptions in Theorem 2.1 are satisfied. Note that in view of Lemma 2.2, for distinct s_j, S_{rl} will have full column rank for almost any set of non-zero a_j. Now, in view of Theorem 2.1, $\hat{Q}(s)$ which satisfies

$$\hat{Q}(s_j)a_j = b_j \quad j = 1, l \tag{3.6}$$

is uniquely specified given these l interpolation points (s_j, a_j, b_j). To solve (3.1), these interpolation points must be appropriately chosen so that the equation $\hat{Q}(s)$ $(= M(s)L(s)) = Q(s)$ is satisfied:

Write (3.1) as

$$ML_r(s) = Q(s) \tag{3.7}$$

where

$$M := [M_0, \ldots, M_r]\ (k \times t(r+1))$$

$$L_r(s) := [L(s)', \ldots, s^r L(s)']'\ (t(r+1) \times m)$$

Let $s = s_j$ and postmultiply (3.7) by a_j $j = 1, l$; note that s_j and a_j, $j = 1, l$, must be so that S_{rl} above has full rank. Define

$$b_j := Q(s_j)a_j \quad j = 1, l \tag{3.8}$$

and combine the equations to obtain

$$ML_{rl} = B_l \tag{3.9}$$

where

$$L_{rl} := [L_r(s_1)\, a_1, \ldots, L_r(s_l)a_l]\ (t(r+1) \times l)$$

$$B_l := [b_1, \ldots, b_l]\ (k \times l)$$

Theorem 3.1: *Given $L(s)$, $Q(s)$ in (3.1), let $d_i := \deg_{ci}[L(s)]$ $i = 1, m$ and select r to satisfy*

$$\deg_{ci}[Q(s)] \le d_i + r \quad i = 1, m \tag{3.10}$$

Then, a solution $M(s)$ of (highest) degree r exists if and only if a solution M of (3.9) does exist; furthermore, $M(s) = M[I, sI, \ldots, s^r I]'$.

Proof: First note that (3.10) is a necessary condition for a solution $M(s)$ in (3.1) of degree r to exist, since $\deg_{ci}[M(s)L(s)] = d_i + r$. Assume that such a solution does exist; clearly (3.9) also has a solution M. That is, all solutions of (3.1) of degree r map into solutions of (3.9). Suppose now that a solution to (3.9) does exist. Notice that the left-hand side of (3.9) $ML_{rl} = \hat{Q}S_{rl}$ where $\hat{Q}(s) = M(s)L(s) = \hat{Q}S_r(s)$. Furthermore, the right-hand side of (3.9) $B_l = QS_{rl}$, in view of (3.8); also note that $Q(s)$ is uniquely represented by the l interpolation points (s_j, a_j, b_j) in view of (3.10) and the interpolation theorem. Therefore, (3.9) implies that $\hat{Q}S_{rl} = QS_{rl}$ or $\hat{Q} = Q$, since S_{rl} is non-singular, or that $M(s)L(s) = \hat{Q}(s) = Q(s)$; that is $M(s) = M_0 + \cdots + M_r s^r = M[I, sI, \ldots, s^r I]'$ is a solution of (3.1). □

Alternative expression

It is not difficult to show that solving (3.9) is equivalent to solving

$$M(s_j)c_j = b_j \quad j = 1, l \tag{3.11}$$

where

$$c_j := L(s_j)a_j,\ b_j := Q(s_j)a_j \quad j = 1, l \tag{3.12}$$

In view now of Corollary 2.6, the matrices $M(s)$ which satisfy (3.11) are obtained by solving

$$MS_{rl} = B_l \tag{3.13}$$

where $S_{rl} := [S_r(s_1)c_1, \ldots, S_r(s_l)c_l]$ $(t(r+1) \times l)$, with $S_r(s) := [I, sI, \ldots, s^r I]'$ $(t(r+1) \times t)$ and $B_l := [b_1, \ldots, b_l]$ $(k \times l)$; $M(s)$ is then $M(s) = M[I, sI, \ldots, s^r I]'$ where M $(k \times t(r+1))$ satisfies (3.13). Solving (3.13) is an alternative to solving (3.9).

Discussion

Theorem 3.1 shows that there is a one-to-one mapping between the solutions of degree r of the polynomial matrix equation (3.1) and the solutions of the linear system of equations (3.9) (or of (3.13)). In other words, using (3.9) (or (3.13)), we can characterize all solutions of degree r of (3.1). Note that the conditions (3.10) of the theorem are not restrictive as they are necessary conditions for a solution $M(s)$ in (3.1) of degree r to exist; that is, all solutions of $M(s)L(s) = Q(s)$ of any degree can be found using Theorem 3.1. Also note that no assumptions were made regarding the polynomial matrices in (3.1); that is, Theorem 3.1 is valid for any matrices $L(s)$, $Q(s)$ of appropriate dimensions.

To solve (3.1), first determine the column degrees d_i, $i = 1, m$, of $L(s)$ and select r to satisfy (3.10). Choose (s_j, a_j), $j = 1, l$, with $l = \sum d_i + m(r+1)$, so that $S_{rl} := [S_r(s_1)a_1, \ldots, S_r(s_l)a_l]$ has full rank; note that, in view of Lemma 2.2, for s_j distinct S_{rl} will have full rank for almost any a_j. Calculate $b_j := Q(s_j)a_j(B_l)$ and L_{rl} in (3.9), or S_l in (3.13). Solving (3.9) (or (3.13)) is equivalent to solving (3.1) for solutions $M(s)$ of degree $\leqslant r$; $M(s) = M[I, sI, \ldots, s^rI]'$. When applying this approach, it is not necessary to determine in advance a lower bound for r; it suffices to use a large enough r. Theorem 3.1 provides the theoretical guarantee that, in this way, all solutions of (3.1) can be obtained. Searching for solutions is straightforward in view of the availability of computer software packages to solve a linear system of equations. Even when an exact solution does not exist, it can be approximated using, for example, least squares approximation.

Existence and uniqueness of solutions

A solution $M(s)$ of degree $\leqslant r$ might not exist or, if it exists, might not be unique. A solution M to (3.9) exists if and only if

$$\operatorname{rank}\begin{bmatrix} L_{rl} \\ B_l \end{bmatrix} = \operatorname{rank} L_{rl} \tag{3.14}$$

If rank $L_{rl} = l$, full column rank, (3.14) is satisfied for any B_l, which implies that the polynomial equation (3.1) has a solution for any $Q(s)$ such that (3.10) is satisfied. Such would be the case, for example, when $L(s)$ is unimodular (real or complex scalar in the polynomial case). In the case when L_{rl} does not have full column rank, a solution M exists only when there is a similar column dependence in B_l (see (3.14)), which implies a certain relationship between $L(s)$ and $Q(s)$ for a solution to exist. Such would be the case, for example, when $L(s)$ is a (right) factor of $Q(s)$. A necessary condition for L_{rl} to have full column rank is that it must have at least as many rows $t(r+1)$, as columns $l = \sum d_i + m(r+1)$. It can be easily seen that if $t \leqslant m$, this is impossible to happen. This implies that if $L(s)$ has more columns than rows, solutions $M(s)$ exist only under certain conditions on $L(s)$ and $Q(s)$, a known fact. For example, when $|L(s)| \neq 0$ $(t = m)$, a solution exists if and only if $L(s)$ is unimodular. When $t > m$, more rows than columns in $L(s)$, a necessary condition for L_{rl} to have full column rank is:

$$r \geqslant \frac{1}{t-m}\sum d_i - 1 \tag{3.15}$$

In this case, if (3.9) has a solution, then it has more than one solution. Similar results can be derived if (3.13) is considered. This is the case in solving the diophantine equation, which is considered in detail later in this section.

Example 3.1: Consider the polynomial equation

$$M(s)L(s) = M(s)(s+1) = Q(s)$$

Here $m = 1$ and $d_1 = \deg L(s) = 1$. Then $l = \sum d_i + m(r+1) = 2 + r$ interpolation points will be taken where r is to be decided upon. Note that since $m = 1$, $a_j = 1$ and S_{rl} will have full rank if s_j are taken to be distinct. Suppose $Q(s) = s^2 + 3s + 2$, a second degree polynomial. In view of Theorem 3.1, $\deg Q(s) = 2 \leqslant d_1 + r = 1 + r$ from which $r = 1, 2, \ldots$. Let $r = 1$, and take $\{s_j, j = 1, 2, 3\} = \{0, 1, 2\}$. Then from (3.9)

$$\begin{aligned} ML_{rl} &= [M_0, M_1]\,[L_r(0), L_r(1), L_r(2)] \\ &= [M_0, M_1]\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \end{bmatrix} \\ &= [Q(0), Q(1), Q(2)] = [2, 6, 12] = B_l \end{aligned}$$

Here $\operatorname{rank}[L_{rl}', B_l']' = \operatorname{rank} L_{rl} = 2$ so a solution exists. It is also unique: $[M_0, M_1] = [2, 1]$. That is $M(s) = (s+2)$ is the unique solution of $M(s)(s+1) = s^2 + 3s + 2$.

It is perhaps of interest at this point to demonstrate the conditions for existence of solutions in the polynomial equation $M(s)(s+1) = Q(s)$ via (3.9) and the discussion above; note that the polynomial equation has a solution if and only if $Q(s)/(s+1)$ is a polynomial or equivalently $Q(-1) = 0$. From the above system of equations $(r = 1)$, for a solution to exist $Q(2) = -3Q(0) + 3Q(1)$ or $\bar{d}_1 = \bar{d}_2 + \bar{d}_0$ if $Q(s) = \bar{d}_2 s^2 + \bar{d}_1 s + \bar{d}_0$. But this is exactly the condition for $Q(-1) = 0$ as it should be. Similarly, it can be shown that $Q(-1) = 0$ must be true for $r = 2, 3, \ldots$

If now $\deg Q(s) = 0$ or 1 then $r = 0$ satisfies $\deg Q(s) \leqslant d_1 + r$ and $l = 2$ interpolation points are needed. Let $\{s_j\ j = 1, 2\} = \{0, 1\}$. Then

$$\begin{aligned} ML_{rl} &= [M_0, M_1]\,[L_r(0), L_r(1)] \\ &= [M_0, M_1]\,[1, 2] = [Q(0), Q(1)] = B_l \end{aligned}$$

Clearly, a solution exists only when $Q(1) = 2Q(0)$. That is, for $\deg Q(s) = 1$, and $Q(s) = \bar{d}_1 s + \bar{d}_0$ a solution exists only when $\bar{d}_1 + \bar{d}_0 = 2\bar{d}_0$ or $\bar{d}_1 = \bar{d}_0$ or when $Q(s) = \bar{d}_0(s+1)$ in which case $M(s) = \bar{d}_0$. For $\deg Q(s) = 0$ and $Q(s) = \bar{d}_0$, a constant, it is impossible to satisfy $Q(1) = 2Q(0)$; that is, a solution does not exist in this case. □

It was demonstrated in the example that, using the interpolation results in Theorem 3.1, one can derive the conditions for existence of solutions in polynomial equations. However, the main use of Theorem 3.1 is in finding all solutions of polynomial matrix equation of certain degree when they exist.

Example 3.2: Consider

$$M(s)L(s) = M(s)\begin{bmatrix} s & 1 \\ s-1 & 1 \end{bmatrix} = [s+1, 1] = Q(s)$$

Here, $m = 2$, $d_1 = 1$ and $d_2 = 0$; $l = \sum d_i + m(r+1) = 1 + 2(r+1) = 3 + 2r$. To select r, consider the conditions of Theorem 3.1:

$$\deg_1 Q(s) = 1 \leqslant d_1 + r = 1 + r$$

$$\deg_2 Q(s) = 0 \leqslant d_2 + r = 0 + r$$

so $r = 0, 1, \ldots$ satisfy the conditions. Let $r = 0$, then $l = 3$; take $\{(s_j, a_j), j = 1, 2, 3\} = \{(0, [1, 0]'), (0, [0, 1]'), (1, [1, 0]')\}$ and note that S_{rl} does have full rank. Then

$$\begin{aligned} ML_{rl} &= M_0[L(0)a_1, L(0)a_2, L(1)a_3] = M_0\begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \\ &= [Q(0)a_1, Q(1)a_2, Q(2)a_3] \\ &= [1, 1, 2] = B_l \end{aligned}$$

This has a unique solution $M(s) = M_0 = [2, -1]$. Note that here $L(s)$ is unimodular and, in fact, the equation has a unique solution for any (1×2) $Q(s)$. □

Constraints on solutions

When there are more unknowns $(t(r+1))$ than equations $(l = \sum d_i + m(r+1))$ in (3.9) or (3.13), this additional freedom can be exploited so that $M(s)$ satisfies additional constraints. In particular, $k := t(r+1) - l$ additional linear constraints, expressed in terms of the coefficients of $M(s)$ (in M), can be satisfied in general. The equations describing the constraints can be used to augment the equations in (3.9). In this case the equations to be solved become

$$M[L_{rl}, C] = [B_l, D] \tag{3.16}$$

where $MC = D$ represents $k := t(r+1) - l$ linear constraints imposed on the coefficients M; C and D are matrices (real or complex) with k columns each. Similarly, if (3.13) is the equation to be solved, then to satisfy additional linear constraints one solves

$$M[S_l, C] = [B_l, D] \tag{3.17}$$

This formulation for additional constraints is used extensively in the following to obtain solutions of the diophantine equation which have certain properties. It should also be noted that additional constraints on solutions which cannot be expressed as linear algebraic equations on the coefficients M can, of course, be handled directly. One could, for example, impose the condition that coefficients in M must minimize some suitable performance index.

Numerical considerations

In $ML_{rl} = B_l$ (3.9), the matrix L_{rl} $(t(r+1) \times l)$ is constructed from $L_r(s) = [L(s)', \ldots, s^r L(s)']'$ and (s_j, a_j), $j = 1, l$. The choice of the interpolation points (s_j, a_j) certainly affects the condition number of L_{rl}. Typically, a random choice suffices to guarantee an adequate condition number. This condition number can be improved many times by using an alternative (other than $[1, s, \ldots]$) polynomial basis such as Chebychev polynomials. Similar comments apply to equation $MS_l = B_l$ (3.13). It is shown below how (3.9) and (3.13) change in this case.

Let $[p_0, \ldots, p_r]' = T[1, s, \ldots, s^r]'$ where $p_i(s)$ are the elements of the new polynomial basis and $T = [t_{ij}]$, $i, j = 1, r+1$ is the transformation matrix. Then $M(s) = M[I, sI, \ldots, s^r I]' = \hat{M}[p_0 I, \ldots, p_r I]'$ from which

$$M = \hat{M}[T \otimes I_t] \tag{3.18}$$

where $\otimes$ denotes the Kronecker product. M and $\hat{M}$ are of course the representation of $M(s)$ with respect to the different bases. Equation (3.9) now becomes

$$\hat{M}\hat{L}_{rl} = B_l \tag{3.19}$$

where $\hat{L}_{rl}$ involves $\hat{L}_r(s) = [p_0 L(s)', \ldots, p_r L(s)']'$ instead of $L_r(s)$. Here

$$\hat{L}_{rl} = [T \otimes I_t]L_{rl} \tag{3.20}$$

where $\hat{L}_{rl}$ will have improved condition number over L_{rl} for appropriate choices of $p_i(s)$ or T. Similarly, (3.13) becomes in this case

$$\hat{M}\hat{L}_l = B_l \tag{3.21}$$

where

$$\hat{S}_l = [T \otimes I_t]S_l \tag{3.22}$$

3.1. *The diophantine equation*

An important case of (3.1) is the diophantine equation:

$$X(s)D(s) + Y(s)N(s) = Q(s) \tag{3.23}$$

where the polynomial matrices $D(s)$, $N(s)$ and $Q(s)$ are given and $X(s)$, $Y(s)$ are to be found. Rewrite as

$$[X(s), Y(s)]\begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = M(s)L(s) = Q(s) \tag{3.24}$$

from which it is immediately clear that the diophantine equation is a polynomial equation of the form (3.1) with

$$M(s) = [X(s), Y(s)], \quad L(s) = \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} \tag{3.25}$$

and all the previous results do apply. That is, Theorem 3.1 guarantees that all solution of (3.24) of degree r are found by solving (3.9) (or (3.13)). In the theory of systems and control the diophantine equation used involves a matrix $L(s) = [D'(s), N'(s)]'$ which has rather specific properties. These will now be exploited to solve the diophantine equation and to derive results beyond the results of Theorem 3.1. In particular, conditions are derived which, if satisfied, a solution to (3.24) of degree r does exist.

Consider $N(s)$ $(p \times m)$ and $D(s)$ $(m \times m)$ with $|D(s)| \neq 0$; $N(s)D^{-1}(s) = H(s)$ a proper transfer matrix, that is

$$\lim_{S \to \infty} H(s) < \infty$$

Then $L(s)$ $((p+m) \times m)$ in (3.25) has full column rank and, as is known, the diophantine equation (3.24) has a solution if and only if a greatest right divisor (g.r.d.) of the columns of $L(s)$ is a right divisor (r.d.) of $Q(s)$. Let (N, D) be right coprime (r.c.), a typical case. This implies that a solution $M = [X, Y]$ of some degree r always exists. We shall now establish lower bounds for r, in addition to (3.10), for the system of linear equations (3.9) (or equivalently (3.13)) to have a solution for any B_l; that is, in view of (3.14) we are interested in the conditions under which L_{rl} $((p+m)(r+1) \times l)$ has full column rank l. Clearly these equations can be used to search for solutions for lower degree than r, if desirable. Such solutions $M(s)$ may exist for particular

$L(s)$ and $Q(s)$, as discussed above; approximate solutions of certain degree may also be obtained using this approach.

$L_r(s)$ in (3.7) has column degrees $d_i + r$, $i = 1, m$, and it can be written as

$$L_r(s) = L_r S_r(s) \tag{3.26}$$

where $S_r(s) := \text{blk diag}\,[1, s, \ldots, s^{d_i+r}]'$. It will be shown that under certain conditions L_r $((p+m)(r+1) \times [\sum_1^m d_i + m(r+1)])$ has full column rank. Then, in view of

$$\begin{aligned} L_{rl} &:= [L_r(s_1)a_1, \ldots, L_r(s_l)a_l] \\ &= L_r[S_r(s_1)a_1, \ldots, S_r(s_l)a_l] = L_r S_{rl} \end{aligned} \tag{3.27}$$

and Sylvester's inequality it will be shown that L_{rl} $((p+m)(r+1) \times l)$ also has full column rank, thus guaranteeing that a solution to $ML_{rl} = B_l$ (3.9) does exist.

$N(s)$, $D(s)$ are right coprime with $N(s)D^{-1}(s) = H(s)$ a proper transfer matrix. Let v be the observability index and $n := \deg|D(s)|$, the order of this system. Assume that $D(s)$ is column reduced (column proper); note that $\deg_{ci}(L(s)) = d_i = \deg_{ci} D(s)$ since the transfer matrix is proper. Then $n = \sum d_i$.

Lemma 3.2: *Rank* $L_r = n + m(r+1)$ *for* $r \geqslant v - 1$.

Proof: First, note that L_r has more rows than columns when $r \geqslant n/p - 1$. It is now known that the observability index satisfies $v \geqslant n/p$. Therefore, for $r \geqslant v - 1$ L_r has more rows than columns and full column rank is possible. For $r = v - 1$, rank $L_r = n + mv = n + m(r+1)$, since L_r in this case is the eliminant matrix in Wolovich (1974) which has full rank when N, D are coprime. Let now $r = v$ and consider the system defined by $N_e(s) := L_{v-1}(s)$, $D_e(s) := s^v D(s)$ with $H_e(s) = N_e(s)D_e(s)^{-1}$. It can be quite easily shown that N_e and D_e are right coprime and D_e is column reduced; furthermore, the observability index of this system is $v_e = 1$. This is because there are $n + mv$ non-zero observability indices $\geqslant 1$ since L_{r-1}; the output map of a state-space realization of $H(s)$ has $n + mv$ independent rows; in view of the fact that the order of the system is $\deg|s^v D(s)| = n + mv$, all these indices must be equal to 1. Now

$$\begin{bmatrix} N_e(s) \\ D_e(s) \end{bmatrix} = \begin{bmatrix} L_{v-1}(s) \\ s^v D(s) \end{bmatrix} = L_e S_v(s)$$

and rank $L_e = n + mv + m$ since N_e, D_e satisfy all the requirements of the eliminant matrix theorem (Wolovich 1974). This implies that for $r = v$, rank $L_r = n + mv + m$, since $L_r(s) = [N_e(s)', D_e(s)', s^v N(s)']'$ and the addition of rows to L_e, to obtain L_v, does not affect its full column rank. A similar proof can be used to show, in general, that if rank $L_r = n + m(r+1)$ for some $r = r_1 > v - 1$ then it is also true for $r = r_1 + 1$. In view of the fact that it is also true for $r = v - 1$ (also $r = v$), the statement of the lemma is true, by induction. □

The following corollary of the Lemma is now obtained. Assume that (s_j, a_j) are selected to satisfy the assumptions of Theorem 3.1, S_{rl} full column rank, and let $D(s)$ be column reduced.

Corollary 3.3: *Rank* L_{rl} = *rank* $S_{rl} = l \leqslant \sum d_i + m(r+1)$ *for* $r \geqslant v - 1$.

Proof: In (3.27), $L_{rl} = L_r S_{rl}$ where L_{rl} $((p+m)(r+1) \times l)$, L_r $((p+m)(r+1) \times [\sum d_i + m(r+1)])$. Applying Sylvester's inequality,

$$\text{rank } L_r + \text{rank } S_{rl} - [\textstyle\sum d_i + m(r+1)] \leqslant \text{rank } L_{rl} \leqslant \min[\text{rank } L_r, \text{rank } S_{rl}]$$

For $r \geqslant v - 1$, rank $L_r = n + m(r+1)$ with $n = \sum d_i$ ($D(s)$ column reduced) in view of Lemma 3.2. Therefore rank L_{rl} = rank S_{rl} which equals the number of columns l, as is assumed in Theorem 3.1. □

The main result of this section can now be stated. Consider the diophantine equation of (3.24) where $N(s)(p \times m)$, $D(s)(m \times m)$ right coprime and $H(s) = N(s)D^{-1}(s)$ a proper transfer matrix. Let v be the observability index of the system and let $D(s)$ be column reduced with $d_i := \deg_{ci} D(s)$. Let $l = \sum d_i + m(r+1)$ interpolation points (s_j, a_j, b_j) $j = 1, l$ be taken such that S_{rl} has full rank (condition of Theorem 3.1). Then the following theorem holds.

Theorem 3.4: *Let r satisfy*

$$\deg_{ci}[Q(s)] \leqslant d_i + r \quad i = 1, m \text{ and } r \geqslant v - 1 \tag{3.28}$$

Then the diophantine equation (3.23) has solutions of degree r, which can be found by solving $ML_{rl} = B_l$ *(3.9) (or (3.13)).*

Proof: In view of Theorem 3.1 all solutions of degree r, if such solutions exist, can be found by solving (3.9). If, in addition, $r \geqslant v - 1$, in view of Corollary 3.3, L_{rl} has full column rank which implies that a solution exists for any B_l, or that a solution of the diophantine of degree $\leqslant r$ exists for any $Q(s)$. □

The theorem basically says that if the degree r of a solution to be found is taken large enough, in particular $r \geqslant v - 1$, then such a solution to the diophantine does exist. All such solutions of degree r can be found by using the polynomial matrix interpolation results in Theorem 3.1 and solving (3.9) (or (3.13)). The fact that a solution of degree $r \geqslant v - 1$ exists when $D(s)$ is column reduced and certain constraints are on the degrees of $Q(s)$, has been known (see for example Chapter 6 in Callier and Desoer 1982 and Theorem 9.17 in Chen 1984). This result was derived here using a novel formulation and a proof based on interpolation results.

Example 3.3: Let

$$D(s) = \begin{bmatrix} s^2 & 0 \\ 1 & -s+1 \end{bmatrix}, \quad N(s) = \begin{bmatrix} s+1 & 0 \\ 1 & 1 \end{bmatrix}, \text{ and}$$

$$Q(s) = \begin{bmatrix} s^3 + 2s^2 - 3s - 5 & -5s - 5 \\ -2s^2 - 5s - 4 & -s^2 - 3s - 2 \end{bmatrix}$$

Here, $D(s)$ is column reduced with $d_1 = 2$, $d_2 = 1$ and $v = 2$. According to Theorem 3.1, $\deg_{ci}[Q(s)] \leqslant d_i + r$, $i = 1, 2$, implies $3 \leqslant 2 + r$ and $2 \leqslant 1 + r$ from which $r \geqslant 1$; $l = \sum_1^m d_i + m(r+1) = 5 + 2r$ interpolation points. For such r, all solutions of degree r are given by (3.9) or (3.13). Here $r \geqslant v - 1 = 1$, therefore in view of Theorem 3.4 a solution of degree $r = 1$ does exist. All such solutions are found using $ML_{rl} = B_l$ (3.9) or (3.13). For $r = 1$, $s_j = -3, -2, -1, 0, 1, 2, 3$ and

$$a_j = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

a solution is

$$M(s) = [X(s), Y(s)] = \begin{bmatrix} s+5 & 5 & -3s & -10 \\ 2 & s+4 & -4s-2 & -6 \end{bmatrix} \qquad \square$$

If, in $D(s)$, the column reduced assumption is relaxed then the following corollary applies.

Corollary 3.5: *Rank* $L_{rl} =$ *rank* $L_r = n + m(r+1)$ *for* $l = \sum d_i + m(r+1)$ *and* $r \geq v - 1$.

Proof: First, note that S_{rl} in this case is square and non-singular which, in view of (3.27), implies that rank $L_{rl} =$ rank L_r. Since $D(s)$ is not column reduced then $n < \sum d_i$. In general, in this case, for $r \geq v - 1$ rank $L_r = n + m(r+1) \leq \sum d_i + m(r+1)$ (with equality holding when $D(s)$ is column reduced); that is, $n + m(r+1)$ is the highest rank L_r can achieve. This can be shown by reducing $D(s)$ to a column reduced matrix by unimodular multiplication and using Sylvester's inequality together with Lemma 3.2. $\square$

When $D(s)$ is not column reduced, then, in view of Corollary 3.5, L_{rl} in $ML_{rl} = B_l$ (3.9) will not have full column rank l but rank $L_{rl} = n + m(r+1) < \sum d_i + m(r+1) = l$. In view of (3.14), a solution will exist in this case if $Q(s)$ is such that the rank condition in (3.14) is satisfied; this will happen when only $n + m(r+1)$ equations in (3.9), out of l, are independent. If r is chosen larger in this case, that is if it is selected to satisfy $\sum \deg_{ci} Q + m < n + m(r+1)$ or $\sum \deg_{ci} Q < n + mr$, instead of $\sum \deg_{ci} Q \leq \sum d_i + mr$ as required by Theorem 3.4, then in view of Theorem 2.1, there are $l - (\sum \deg_{ci} Q + m)$ more interpolation equations than needed to specify uniquely $Q(s)$ and these additional columns in B_l will be linearly dependent on the previous ones. If similar dependence exists between the corresponding columns of L_{rl} then (3.14) is satisfied and a solution exists. In other words, if r is taken to be large enough, then the conditions of Theorem 3.4 on r will always be satisfied in this case (after $D(s)$ is reduced to column reduced form by a multiplication of the diophantine equation by an appropriate unimodular matrix). It should also be stressed at this point that numerically it is straightforward to try different values for r in solving $ML_{rl} = B_l$ (3.9).

Constraints on solutions

In the equation $ML_{rl} = B_l$ (3.9) there are, at each row, $t(r+1) = (p+m)(r+1)$ unknowns (number of columns of $M = [M_0, \ldots, M_r] = [(X_0, Y_0), \ldots, (X_r, Y_r)]$) and $l = \sum d_i + m(r+1)$ linearly independent equations (number of columns of L_{rl}). Therefore, for r sufficiently large, there are $p(r+1) - \sum d_i$ more unknowns than equations and it is possible to satisfy, in general, an equal number of additional constraints on the coefficients M of $M(s) = [X(s), Y(s)]$. These constraints can be accommodated by selecting larger values for r and they are exceptionally easy to handle in this setting when they are linear. Then, the equation to be solved becomes

$$M[L_{rl}, C] = [B_l, D] \tag{3.29}$$

where $MC = D$ are the, say, k_d desired constraints on the coefficients of the solution; the matrices C and D have k_d columns each. The degree of the solution r should then be chosen so that

$$p(r+1) - \sum d_i \geq k_d \tag{3.30}$$

in addition to satisfying the conditions of Theorem 3.4.

Typically, we want solutions of the diophantine with $|X(s)| \neq 0$. This can be satisfied by requiring for example that $X_r = I$ (or any other non-singular matrix) which, in addition guarantees that $X^{-1}(s)Y(s)$ will be proper. Note that to guarantee that $X_r = I$ one needs to use m linear equations, that is in this case the number of columns of C and D will be at least m.

To gain some insight into this important technique, consider the scalar case which has been studied by a variety of methods. In particular, consider the polynomial diophantine where $p = m = 1$. Let $d_i = \deg D(s) = n$, $n_q = \deg Q(s)$ and note that $v = n$. Therefore r, according to Theorem 3.4, must be chosen to satisfy $r \geq n_q - n$ and $r \geq n - 1$. Select $Q(s)$ so that $n_q = 2n - 1$ then $r \geq n - 1$ satisfies all conditions, as is well known. In view of the above, to guarantee that $X^{-1}Y$ will be proper, one needs to set an additional constraint such as $X_r = 1$ ($m = 1$) which, in view of (3.30), implies that $X^{-1}(s)Y(s)$ proper can be guaranteed if r is chosen to satisfy $r \geq n$. In the case when $N(s)D^{-1}(s)$ is strictly proper (instead of proper), however, this additional constraint is not needed and $X^{-1}(s)Y(s)$ proper can be obtained for $r \geq n - 1$. This is because, in this case, a solution with $X_r = 0$ leading perhaps to a non-proper $X^{-1}(s)Y(s)$ is not possible. Notice that for $r = n - 1$ the solution is unique.

Example 3.4: Consider Example 3.3, $p(r+1) - \sum d_i = 2(1+1) - (2+1) = 1$. From (3.30), one can add one extra constraint on the solution in the form of (3.16) or (3.17). Assume that in addition to solving for $[X(s), Y(s)]$ in Example 3.3, it is desirable that $X(s)$ has a zero at $s = -10$ and $X(-10)[1\ 2]' = [0\ 0]'$. This can be easily incorporated as an extra interpolation triplet using (3.17). The solution obtained is

$$M(s) = [X(s), Y(s)] = \begin{bmatrix} s-10 & 10 & 12s & -15(s+1) \\ 16 & -s-18 & -18s-2 & 16(s+1) \end{bmatrix}$$

Note that $X(s)$ has a zero at -10 and $[X(s), Y(s)]$ is a solution of the diophantine equation (3.23) with the $D(s)$, $N(s)$ and $Q(s)$ given in Example 3.3. $\square$

Example 3.5: Let

$$D(s) = \begin{bmatrix} s-2 & 0 \\ 0 & s+1 \end{bmatrix}, \quad N(s) = \begin{bmatrix} s-1 & 0 \\ 1 & 1 \end{bmatrix}, \text{ and } Q(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

From which $d_1 = d_2 = 1$; $\deg_{ci} Q(s) = 0$, $i = 1, 2$; and $l = 2 + 2(r+1)$. For $r = 1$, $s_j = -2, -1, 0, 1, 2, 3$ and

$$a_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

a solution is

$$M(s) = [X(s), Y(s)] = \begin{bmatrix} s & -1 & -s & s+1 \\ 1/3 & 1/3 & 0 & -1/3s + 2/3 \end{bmatrix} \qquad \square$$

Example 3.6: Let

$$D(s) = \begin{bmatrix} s-2 & 0 \\ 0 & s+1 \end{bmatrix}, \; N(s) = \begin{bmatrix} s-1 & 0 \\ 1 & 1 \end{bmatrix}, \text{ and } Q(s) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

From which $d_1 = d_2 = 1$; $\deg_{ci} Q(s) = 0$, $i = 1, 2$; and $l = 2 + 2(r+1)$
For $r = 1$, $s_j = -2, -1, 0, 1, 2, 3$ and

$$a_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

a solution is

$$M(s) = [X(s), Y(s)]$$

$$= \begin{bmatrix} -0{\cdot}4665s - 0{\cdot}2954 & 0{\cdot}3805 & 0{\cdot}4665s + 0{\cdot}2085 & -0{\cdot}3805(s+1) \\ 0{\cdot}3401s - 0{\cdot}4040 & 0{\cdot}0320 & -0{\cdot}3401s + 0.7761 & -0{\cdot}0320(s+1) \end{bmatrix}$$

Note that, in this example, the rows of $[M_0, M_1]$ form the basis for the left null space of S_{dl}. □

Note that in Example 3.5 and 3.6 we solved the problem

$$\begin{bmatrix} X(s) & Y(s) \\ -\tilde{N}(s) & \tilde{D}(s) \end{bmatrix} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$$

separately, where $X(s)$ and $Y(s)$ are the solution of the Bezout identity and $\tilde{D}^{-1}(s)\tilde{N}(s) = N(s)D^{-1}(s)$ gives the left coprime factorization.

4. Characteristic values and vectors

When all the n zeros of an nth degree polynomial $q(s)$ are given, then $q(s)$ is specified within a non-zero constant. In contrast, the zeros of the determinant of a polynomial matrix $Q(s)$ do not adequately characterize $Q(s)$; information about the structure of $Q(s)$ is also necessary. This additional information is contained in the characteristic vectors of $Q(s)$, which must also be given, together with the characteristic values, to characterize $Q(s)$. The characteristic values and vectors of $Q(s)$ are studied in this section.

We are interested in cases where the complex numbers s_j, $j = 1, l$, used in interpolation, have special meaning. In particular, we are interested in cases where s_j are the roots of the non-trivial polynomial entries of the Smith form of the polynomial matrix $Q(s)$ or, equivalently, roots of the minors of $Q(s)$, or roots of the invariant polynomials of $Q(s)$ (see the Appendix). Such results are useful in addressing a variety of control problems, as is shown later in this and the following sections. Here, we first specialize certain interpolation results from § 2 to the case when b_j, in the interpolation constraints (2.3), are zero and we derive Corollary 4.1. This corollary characterizes the structure of all non-singular $Q(s)$ if all of the roots of $|Q(s)|$ together with their associated directions, i.e. (s_j, a_j), are given. We then concentrate on the characteristic values and vectors of $Q(s)$, and in Theorems 4.2, 4.3, 4.7 and in the Appendix, we completely characterize all matrices with such given characteristic values and vectors.

Note that here only the right characteristic vectors are discussed ($Q(s_j)a_j = 0$); similar results are, of course, valid for left characteristic vectors ($\underline{a}_j Q(s_j) = 0$; see also Corollary 2.5) and they can be easily derived in a manner completely analogous to the derivations of the results presented in this and the following sections. These dual results are omitted here.

Consider the interpolation constraints (2.3) with $b_j = 0$; that is

$$Q(s_j)a_j = 0 \quad j = 1, l \tag{4.1}$$

In this case one solves (2.5) with $B_l = 0$; that is

$$QS_l = B_l = 0 \tag{4.2}$$

where $S_l = [S(s_1)a_1, \ldots, S(s_l)a_l](\sum d_i + m) \times l$ and $Q(p \times (\sum d_i + m))$. This case, $B_l = 0$, was briefly discussed in § 2, see (2.11); see also the discussion on eigenvalues and eigenvectors. We shall now start with the case when $Q(s)$ is non-singular. The following corollary is a direct consequence of Corollary 2.2:

Corollary 4.1: *Let $Q(s)$ be $(m \times m)$ and non-singular with $n = \deg|Q(s)|$. Let d_i, $i = 1, m$, be its column degrees and let $\sum d_i = n$. If (s_j, a_j), $j = 1, l$, with $l = n$, are given and they are such that S_{1l} has full rank, then a $Q(s)$ which satisfies (4.1) is uniquely specified within a premultiplication by an $(m \times m)$ non-singular leading coefficient matrix C_c.*

Proof: Since $\deg|Q(s)| = n = \sum d_i$ the leading coefficient matrix C_c of $Q(s)$ must be non-singular. The rest follows directly from (2.7). □

This corollary says that if all the n zeros s_j of the determinant of $Q(s)$ are given together with the corresponding vectors a_j which satisfy (4.1) then, under certain assumptions (S_{1l} full rank), $Q(s)$ is uniquely determined within a non-singular leading coefficient matrix C_c provided that its column degrees d_i (given) satisfy $\sum d_i = n$. If d_i are not specified, there are many such matrices. One could relax some of the assumptions (S_{1l} full rank) and further extend some of the results of § 2 by using derivatives of $Q(s)$ and Theorem 2.8. Instead, we start a new line of inquiry which concentrates on the meaning of (s_j, a_j) when they satisfy relations such as (4.1). We return to Corollary 4.1 later on in this section.

If a complex scalar z and vector a satisfy $Q(z)a = 0$, where $Q(s)$ is a $p \times m$ matrix and the vector $a \neq 0$, then under certain conditions z and a are called the *characteristic value and vector of* $Q(s)$ respectively. This is, of course, an extension of the well-known concepts in the special case when $Q(s) = sI - A$; then z and a are an eigenvalue and the corresponding eigenvector of A respectively. Note that in the general matrix case, the fact that z and a satisfy $Q(z)\,a = 0$ does not necessarily imply that they do have special meaning; for example, for $Q(s) = [1, 0]$ and $a = [0, 1]'$, $Q(z)\,a = 0$ for any scalar z. On the other hand if $Q(s)$ is square and non-singular, $Q(z)\,a = 0$ would imply that z is a root of the determinant of $Q(s)$; in fact in this case z and a are indeed a characteristic value and vector of $Q(s)$. Conditions of the form $Q(z)\,a = 0$ are imposed to force $Q(s)$ to have certain characteristic values and vectors which are very important in applications. The definitions of characteristic values and vectors are given below.

Given a $p \times m$ polynomial matrix $Q(s)$, its Smith form is uniquely defined;

see the Appendix. The characteristic values (or zeros) of $Q(s)$ are defined using the invariant polynomials $\varepsilon_i(s)$ of $Q(s)$.

Definition 4.1: The *characteristic values of* $Q(s)$ are the roots of the invariant polynomials of $Q(s)$ taken all together. If a complex scalar s_j is a characteristic value of $Q(s)$, the $m \times 1$ complex non-zero vector a_j which satisfies

$$Q(s_j)a_j = 0 \tag{4.3}$$

is the corresponding *characteristic vector* of $Q(s)$. □

$Q(s)$ may have repeated characteristic values and the algebraic and a geometric multiplicity of s_j are defined below for $Q(s)$ square and non-singular; it is straightforward to extend these definitions to a $p \times m$ $Q(s)$. In the case of a real matrix A, if some of the eigenvalues are repeated one may have to use generalized eigenvectors. Here generalized characteristic vectors of $Q(s)$ are also defined. The general definition involves derivatives of $Q(s)$ and it is treated in the Appendix. In the results below, only characteristic vectors that satisfy relation (4.1), which does not contain derivatives of $Q(s)$, are considered for reasons of simplicity and clarity; a general version of these results can be found in the Appendix.

Let $Q(s)$ be an $(m \times m)$ non-singular matrix. If s_j is a zero of $|Q(s)|$ repeated n_j times, define n_j to be the *algebraic multiplicity* of s_j; define also the *geometric multiplicity* of s_j as the quantity $(m - \operatorname{rank} Q(s_j))$.

Theorem 4.2: *There exist complex scalar s_j and l_j non-zero linearly independent $(m \times 1)$ vectors a_{ij}, $i = 1, l_j$, which satisfy*

$$Q(s_j)a_{ij} = 0 \tag{4.4}$$

if and only if s_j is a zero of $|Q(s)|$ with algebraic multiplicity $(= n_j) \geq l_j$ and geometric multiplicity $(= m - \operatorname{rank} Q(s_j)) \geq l_j$.

Proof: This is a special case of the Theorem A.1 of the Appendix for $k_{ij} = 1$, $i = 1, l_j$. □

The complex values s_j and vectors a_{ij} are characteristic values and vectors of $Q(s)$. In the case when $l_j = 1$, the theorem simply states that s_j is a zero of $|Q(s)|$ if and only if $\operatorname{rank} Q(s_j) < m$, an obvious and well-known result. The conditions of Theorem 4.2 imply a certain structure for the Smith form of $Q(s)$, as is shown in Corollary A.3 in the Appendix. In particular, if the conditions of Theorem 4.2 are satisfied then the Smith form of $Q(s)$ contains the factor $(s - s_j)$ in l_j separate locations on the diagonal.

In the following it is assumed that $n = \deg|Q(s)|$ is known and the matrices $Q(s)$ with given characteristic values and vectors s_j and a_{ij} are characterized.

Theorem 4.3: *Let $n = \deg|Q(s)|$. There exist σ distinct complex scalars s_j and $(m \times 1)$ non-zero vectors a_{ij}, $i = 1, l_j$, $j = 1, \sigma$ with $\sum_1^{\sigma} l_j = n$ and a_{ij}, $i = 1, l_j$, linearly independent which satisfy (4.4) if and only if the zeros of $|Q(s)|$ have σ distinct values s_j, $j = 1, \sigma$, each with algebraic multiplicity $(= n_j) = l_j$ and geometric multiplicity $(= m - \operatorname{rank} Q(s_j)) = l_j$.*

Proof: This is a special case of the Theorem A.4 in the Appendix. □

Note that the independence condition on the $m \times 1$ vectors $a_{1j}, a_{2j}, \ldots, a_{l_j j}$ implies that $l_j \leq m$; that is, no characteristic value is repeated more than m times. One should use the general Theorem A.4 if this is not sufficient.

The following corollary of Theorem 4.3 formalizes the most familiar case.

Corollary 4.4: *Let $n = \deg|Q(s)|$. There exist n distinct complex scalars s_j and $(m \times 1)$ non-zero vectors a_j, $j = 1, n$, which satisfy (4.1) if and only if the zeros of $|Q(s)|$ have n distinct values s_j.*

If a matrix $Q(s)$ satisfies the conditions of Theorem 4.3, its Smith form contains the factor $(s - s_j)$ in exactly l_j different locations on the diagonal; see Corollary A.5 and (A.4). This is true for each distinct value s_j, $j = 1, \sigma$. In view of the divisibility properties of the diagonal entries of the Smith form, this information specifies uniquely the Smith form.

Corollary 4.5: *All $Q(s)$ which satisfy the conditions of Theorem 4.3 have the same Smith form.*

If a Smith form with factors $(s - s_j)^{k_{ij}}$, $k_{ij} \neq 1$, in certain location is desired, one must then use Theorem A.4 and Corollary A.5 which utilize the derivatives of $Q(s)$.

Example 4.1: Suppose for some $Q(s)$, $\deg|Q(s)| = n = 2$ and, $Q(s_j)a_{ij} = 0$ is satisfied for $s_1 = 1$ and $a_{11} = [1, 0]'$ and $a_{21} = [0, 1]'$. Here $l_j = l_1 = 2$. Since $l_1 = 2 = n$, Theorem 4.3 implies that $\sigma = 1$, or that $s_1 = 1$ is the only distinct root of $|Q(s)|$ and it has an algebraic multiplicity $(=n) = 2 = l_1$ and geometric multiplicity $= 2 = l_1$. Its Smith form has $s - 1$ in $l_1 = 2$ locations on the diagonal and it is uniquely determined. It is

$$E(s) = \begin{bmatrix} s-1 & 0 \\ 0 & s-1 \end{bmatrix}$$

(See also Example A.1). □

Additional structural information about matrices $Q(s)$, which satisfy the conditions of Theorem 4.3 is given by applying Corollary 4.1. Corollary 4.1 has the condition that S_{1l} must have full (column) rank. Notice that the repeated values s_j give rise to l_j linearly independent columns $S(s_j)a_{ij}$, $i = 1, l_j$, in S_{1l} because a_{ij}, $i = 1, l_j$, are linearly independent; therefore S_{1l} has full rank for almost any set of (s_j, a_{ij}) of Theorem 4.3. Corollary 4.1 then implies that the matrices $Q(s)$ which satisfy the conditions of Theorem 4.3 are uniquely specified within a premuitiplication by a non-singular matrix C_c if the column degrees d_i are given and they satisfy $\Sigma d_i = n$; note that it is not possible to have $\Sigma d_i < n$ since $n = \deg|Q(s)|$. It should be pointed out that this result does not contradict the fact that if the eigenvalues and the eigenvectors of a matrix A are known, then $sI - A = Q(s)$ is uniquely determined since, in this case, the additional facts that $d_i = 1$, $i = 1, n$, and $C_c = I$ are being used; see Corollary 2.3. If $\Sigma d_i > n$ then $Q(s)$ is underdefined and there are many such matrices $Q(s)$ (note that C_c is singular in this case). To obtain such matrices in this case $(\Sigma d_i > n)$ one could select a $Q(s)$ with $\Sigma d_i = n$ and then premultiply $Q(s)$ by an arbitrary unimodular matrix $U(s)$; note that $|Q(s)|$ and $|U(s)Q(s)|$ have exactly the same zeros. Therefore, the conditions of Theorem 4.3 specify $Q(s)$ within a unimodular premultiplication.

Lemma 4.6: *Theorem 4.3 is satisfied by a matrix $Q(s)$ if and only if it is satisfied by $U(s)Q(s)$ where $U(s)$ is any unimodular matrix.*

Proof: The proof is straightforward. Note that (4.3) is satisfied if and only if it is satisfied for $U(s)Q(s)$ with the same s_j and a_{ij}; this is because $U(s_j)$ is non-singular. □

It is of interest at this point to summarize briefly the results so far: Assume that, for an $(m \times m)$ polynomial matrix $Q(s)$ yet to be chosen, we have decided upon the degree of $|Q(s)|$ as well as its zero locations—that is, about n, s_j and the algebraic multiplicities n_j. Clearly there are many matrices that satisfy these requirements; consider for example all the diagonal matrices that satisfy these requirements. If we specify the geometric multiplicities l_j as well, then this implies that the matrices $Q(s)$ must satisfy certain structural requirements so that $m - \operatorname{rank} Q(s_j) = l_j$ is satisfied; in our example the diagonal matrix, the factors $(s - s_j)$ must be appropriately distributed on the diagonal. If k_{ij} are also chosen to be equal to 1 as it is the case studied here (see Appendix for $k_{ij} \neq 1$), then the Smith form of $Q(s)$ is completely defined, that is $Q(s)$ is defined within pre and post unimodular matrix multiplications. Note that this is equivalent to imposing the restriction that $Q(s)$ must satisfy n relations of type (4.4), as in Theorem 4.3, without fixing the vectors a_{ij}. If, in addition, a_{ij} are completely specified then $Q(s)$ is determined within a unimodular premultiplication; see Lemma 4.6.

If an $(m \times m)$ non-singular polynomial matrix $Q(s)$ satisfies all conditions of Theorem 4.3 with the exception that $\deg|Q(s)|$ is not specified, then, in view of Theorem 3.2, the following can be shown.

Corollary 4.7: *Let $|Q(s)| \neq 0$. There exist σ distinct complex scalars s_j and $(m \times 1)$ non-zero vectors a_{ij}, $i = 1, l_j$, $j = 1, \sigma$, with $\sum_1^\sigma l_j = n$ and a_{ij}, $i = 1, l_j$, linearly independent which satisfy (4.4) if and only if $\tilde{n} := \deg|Q(s)| \geq n$ with s_j, $j = 1, \sigma$, roots of $|Q(s)|$, and with algebraic and geometric multiplicity of s_j in $Q(s) \geq l_j$.*

In view of this corollary, it can now be shown that the conditions of Theorem 4.3, with the exception that the $\deg|Q(s)|$ is not given, specify $Q(s)$ within a premultiplication by a polynomial matrix.

Corollary 4.8: *Let $|Q(s)| \neq 0$ and let (4.4) be satisfied for (s_j, a_{ij}), $i = 1, l_j$, $j = 1, \sigma$, with $\sum_1^\sigma l_j = n$ with a_{ij}, $i = 1, l_j$, linearly independent and s_j, $j = 1, \sigma$, distinct. Then $Q(s)$ is specified within a premultiplication by a polynomial matrix. This polynomial matrix is unimodular if $\deg|Q(s)| = n$.*

Note that if $\tilde{n} = n$, then the conditions of Corollary 4.7 are the same as the ones in Theorem 4.3 and the fact that $Q(s)$ is specified within a premultiplication by a unimodular matrix in Corollary 4.8 agrees with Lemma 4.6. Corollary 4.8 also agrees with Corollary 4.1 when it is applied with $\Sigma d_i > n$ (see the discussion following Example 4.1).

The above Theorems and Corollaries show that the existence of appropriate (s_j, a_{ij}) which satisfy (4.4) implies (and is implied by) the occurrence of certain roots in $|Q(s)|$ and certain directions associated with these roots. How does one go about selecting such a_{ij} and how does one go about finding an appropriate $Q(s)$? This can of course be done by Corollary 4.1. (s_j, a_{ij}) are chosen so that S_{ll} has full rank, as was discussed following Example 4.1. Note that in view of Lemma 2.4, if s_j are distinct the corresponding (non-zero) a_j can be chosen almost arbitrarily as, in this case, S_{ll} will have full rank for almost any set of non-zero a_j. Therefore, if one is interested in determining a polynomial matrix $Q(s)$ with $|Q(s)|$ having n distinct zeros, one could (almost) arbitrarily choose n non-zero vectors a_j and apply Corollary 4.1 to determine such $Q(s)$. If additional requirements are imposed, such as certain algebraic and geometric multiplicities for the zeros, then the results in this section and in the Appendix should be utilized.

In the following, the results in Corollaries 4.7 and 4.8 derived for $Q(s)$ square and non-singular are extended to the non-square case.

Given $(m \times m)$ $Q(s)$ let $n = \deg|Q(s)|$ and assume that

$$Q(s_j)a_{ij} = 0 \tag{4.5}$$

is satisfied for σ distinct s_j, $j = 1, \sigma$, with a_{ij}, $i = 1, l_j$, linearly independent and $\sum l_j = n$. That assumes that s_j, a_{ij} and $Q(s)$ satisfy Theorem 4.3.

Theorem 4.9: *$Q(s)$ is a right divisor* (r.d.) *of an $(r \times m)$ polynomial matrix $M(s)$ if and only if $M(s)$ satisfies*

$$M(s_j)a_{ij} = 0 \tag{4.6}$$

with the same (s_j, a_{ij}) as in (4.5) above.

Proof: Necessity: if Q is a r.d. of M, $M = \hat{M}Q$. Premultiply (4.5) by $\hat{M}(s_j)$ to obtain (4.6). Sufficiency: let $M(s)$ satisfy (4.6) and let $G(s)$ be a greatest r.d. of M and Q: then there exists a unimodular matrix U such that $U\left[\begin{smallmatrix} Q \\ M \end{smallmatrix}\right] = \left[\begin{smallmatrix} G \\ 0 \end{smallmatrix}\right]$. This implies that G satisfies the same n relations as $Q(s)$ and $M(s)$ in (4.5) and (4.6) respectively. Therefore $\deg|G(s)| \geq n$ in view of Corollary 4.6. Since G is a r.d. of Q, $Q = \hat{Q}G$ which implies that $\hat{Q}$ is unimodular since $\deg|Q| = n$. Therefore $M = \tilde{M}G = (\tilde{M}\hat{Q}^{-1})Q$, that is Q is a r.d. of M. □

Theorem 4.9 is very important; a more general version is given in Theorem A.7 in the Appendix. From the theoretical point of view, it generalizes the characteristic value and vector results to the non-square, non-full rank case. In addition, from the practical point of view it provides a convenient way to impose the restriction on a $r \times m$ $M(s)$ that can be written as

$$M = WQ \tag{4.7}$$

where the square and non-singular Q has specific characteristic values and vectors and W is a 'do not care' polynomial matrix.

In the polynomial case, Theorem 4.9 states that the polynomial $m(s)$ has a factor $q(s)$ if the (distinct) roots of $q(s)$ are also roots of $m(s)$. For repeated roots one should use Theorem A.7 in the Appendix.

In view of the above, it should now be clear that n relations of the form $M(s_j)a_j = 0$, $j = 1, n$, with s_j distinct, and a_j non-zero $(m \times 1)$ vectors will guarantee that the $(r \times m)$ $M(s)$ has a r.d. $Q(s)$ which has n distinct zeros of $|Q(s)|$ equal to s_j. Such $M(s)$ can be determined using Corollary 4.1.

Corollary 4.10: *An $r \times m$ polynomial matrix $M(s)$ has a* r.d. *$Q(s)$ with the property that the zeros of $|Q(s)|$ are equal to the n distinct values s_j, $j = 1, n$, if and only if there exist non-zero vectors a_j such that*

$$M(s_j)a_j = 0 \quad j = 1, n \tag{4.8}$$

Proof: There exists an $m \times m$ $Q(s)$ with $\deg|Q(s)| = n$ which satisfies $Q(s_j)a_j = 0$. Then, in view of Theorem 4.9, the result follows. □

5. Pole placement and other applications

The results developed in the previous section on the characteristic values and vectors of a polynomial matrix $Q(s)$ are useful in a wide range of problems in systems and control. Several of these problems and their solutions using interpolation are discussed in this section. The pole placement or pole assignment problem is discussed first.

Pole or eigenvalue assignment is a problem studied extensively in the literature. In the following it is shown how this problem can be addressed using interpolation, in a way that is perhaps more natural and effective. Dynamic (and static) output feedback is used first to shift arbitrarily the closed loop eigenvalues (also known as the poles of the system). Then state feedback is studied.

5.1. *Output feedback—diophantine equation*

5.1.1. *Dynamic output feedback.* Here all proper output controllers of (highest) degree r (of order mr) that assign all the closed loop eigenvalues to arbitrary locations are characterized in a convenient way. This has not been done before.

We are interested in solutions $[X(s), Y(s)]$ $(m \times (p + m))$ of the diophantine equation

$$X(s)D(s) + Y(s)N(s) = Q(s) \tag{5.1}$$

where only the roots of $|Q(s)|$ are specified, furthermore $X^{-1}(s)Y(s)$ should exist and be proper. This problem is known as the pole placement (eigenvalue assignment) problem where $N(s)D^{-1}(s)$ $(p \times m)$ is a description of the plant to be controlled and $C = X^{-1}(s)Y(s)$ $(m \times p)$ is the desired controller which assigns the closed-loop poles (eigenvalues) at desired locations.

Note the difference between the problem studied in § 3, where $Q(s)$ is known, and the problem studied here where only the roots of $|Q(s)|$ (or $|Q(s)|$ within multiplication by some non-zero real scalar) are given. It is clear, especially in view of § 4, that there are many (in fact an infinite number) of $Q(s)$ with the desired roots in $|Q(s)|$. So, if one selects in advance a $Q(s)$ with desired roots in $|Q(s)|$ that does not satisfy any other design criteria (and there are usually additional control goals to be accomplished) as is typically done, then one really solves a more restrictive problem than the eigenvalues assignment problem. In fact, in this case one solves a problem where the methods of § 3 are appropriate, as in this case $Q(s)$ is given; note that this approach to the problem is closer to the characteristic value and vector assignment problem (eigenvalue/eigenvector problem) discussed below, than just the pole assignment problem. In the scalar polynomial case if $Q(s)$ is selected so that the roots of $|Q(s)|$ are the desired ones then one really arbitrarily selects, in addition, only the leading coefficient of $Q(s)$, which is not really restrictive. This perhaps explains the tendency to do something analogous in the multivariable case; this however clearly changes and restricts the original problem. It is shown here that one does not have to select $Q(s)$ in advance. For the pole placement problem it is more natural to use the interpolation approach of § 4, where the flexibility in selecting $Q(s)$ is expressed in terms of selecting the characteristic vectors of $Q(s)$; in general, for almost any choice for the characteristic vectors, subject to some rather mild rank conditions (see § 4) the pole assignment is accomplished. These vectors can then be seen as design parameters and they can be selected to satisfy requirements in addition to pole assignment. Note that this design approach is rather well known in the state feedback case, as is discussed later in this section.

Consider now the diophantine equation (5.1). The results of § 3 and 4 will be used to solve the pole assignment problem.

The diophantine equation (5.1) has been studied at length in § 3 and the notation developed there will also be used in this section. In particular, let $M(s) := [X(s), Y(s)]$ and $L(s) := [D'(s), N'(s)]'$ then (5.1) becomes $M(s)L(s) = Q(s)$. This equation can be written as $ML_r(s) = Q(s)$ (3.7) where $M := [M_0, \ldots, M_r]$ a real matrix with $M(s) := M_0 + \cdots + M_r s^r$ and $L_r(s) := [L(s)', \ldots, s^r L(s)']'$. If now $b_j := Q(s_j)a_j$, $j = 1, l$, and $B_l := [b_1, \ldots, b_l]$ then the equation to be solved, (see (3.9)) is

$$ML_{rl} = B_l = 0 \tag{5.2}$$

where $L_{rl} := [L_r(s_1)a_1, \ldots, L_r(s_l)a_l]$ $(p + m)(r + 1) \times l$); the unknown matrix M is $m \times (p + m)(r + 1)$.

If the column degrees of $L(s) = [D'(s), N'(s)]'$ are d_i and the highest degree of $M(s) = [X(s), Y(s)]$ is r, then $\deg|X(s)D(s) + Y(s)N(s)| = \deg|M(s)L(s)| \leq \sum d_i + mr$; the equality is satisfied when $X(s)D(s) + Y(s)N(s)$ is column reduced. In Corollary 3.3 the conditions under which L_{rl} has full column rank were derived: if (s_j, a_j) are selected to satisfy the assumptions of Theorem 3.1, that is S_{rl} to have full column rank, then $2(r + 1)$ for $r \geq v - 1$, where v is the observability index of the system; note that $L_{rl} := [L_r(s_1)a_1, \ldots, L_r(s_l)a_l] = L_r[S_r(s_1)a_1, \ldots, S_r(s_l)a_l] = L_r S_{rl}$ where $S_r(s) := \text{blk diag}[1, s, \ldots, s^{d_i + r}]'$. That is, under mild conditions on (s_j, a_j) and for $r \geq v - 1$, L_{rl} in (5.2) has full column rank l.

Suppose now that $X(s)D(s) + Y(s)N(s)$ is forced to satisfy

$$M[L_{rl}, C] = [0, D] \tag{5.3}$$

where $l = \sum d_i + mr$. Note that $ML_{rl} = 0$ imposes the condition that

$$(X(s_j)D(s_j) + Y(s_j)N(s_j))a_j = 0 \quad j = 1, l$$

$(= \sum d_i + mr)$; that is the $\sum d_i + mr$ roots of $|X(s)D(s) + Y(s)N(s)|$ are to take on the values s_j, $j = 1, l$, (see Corollary 4.8 and Theorem 4.9 for the proof of this claim). Here (s_j, a_j) must be such that S_{rl} above has full column rank l (see Corollaries 3.3, 3.5 and the discussion above); note that this is true for almost any a_j when s_j are distinct (Lemma 2.4). For L_{rl} also to have full column rank l, we need $r \geq v - 1$ as was shown in Corollary 3.3.

In the case when $N(s)D^{-1}(s)$ is proper with $|D(s)| = n$, n instead of $\sum d_i$ may be used in which case $l = n + mr$ poles are assigned. Note that n must be used when $D(s)$ is not column reduced, as in this case $\deg|X(s)D(s) + Y(s)N(s)| = \deg|X(s)D(s)| \leq n + mr < \sum d_i + mr$ since $X^{-1}(s)Y(s)$ is also proper; Corollary 3.5 shows that rank $L_{rl} = n + mr$ in this case and Corollary 4.8 shows that $|X(s)D(s) + Y(s)N(s)|$ will have the desired roots.

The equations $MC = D$ can guarantee that the leading coefficient of $X(s)$ is non-singular so that $X^{-1}(s)$ exists and $X^{-1}(s)Y(s)$ is proper. This will add m more equations (or columns of C and D) for a total of $\Sigma d_i + m(r+1)$ equations. Thus, the following theorem has been shown.

Let $N(s)D^{-1}(s)$ be proper with N, D right coprime and $|D(s)| = n$.

Theorem 5.1: *Let $r \geq v - 1$. Then $(X(s), Y(s))$ exists such that all the $n + mr$ zeros of $|X(s)D(s) + Y(s)N(s)|$ are arbitrarily assigned and $X^{-1}(s)Y(s)$ is proper. It is obtained by solving (5.3).* □

In (5.3) there are (in each row) $(p+m)(r+1)$ unknowns and $n + m(r+1)$ equations; the fact that $r \geq v - 1$ implies that there are more unknowns than independent equations as $pv \geq n$. Note that the Theorem was proved for the case when s_j are distinct, or more generally the case when (s_j, a_j) exist, so that S_{rl} has full rank. The general case, where the desired values s_j and their multiplicities are not considered in § 4, can be studied using the results in the Appendix which involve derivatives of the polynomial matrices and similar results can be derived.

Notice that the order of the compensator $C(s) = X^{-1}(s)Y(s)$ is mr with minimum order $m(v-1)$. By reducing the system to a single input controllable system and by using, if necessary, dual results it can be shown that the minimum order of the pole assigning compensator $C(s)$ using this method is $\min(\mu - 1, v - 1)$, where μ and v are the controllability and observability indices of the system respectively. This agrees with the well-known results of Brasch and Pearson (1970). Furthermore, in certain cases lower-order compensators which assign the desired poles can be determined. Our method makes it possible to search easily for such lower-order compensators.

Example 5.1: Let $D(s) = s^2 - 1$, $N(s) = s + 2$ and $|Q(s)| = (s+1)(s-1+\mathrm{j}1)(s-1-\mathrm{j}1)$, from which $n = v = 2$; $r \geq 1$ and $\deg|Q(s)| = 2 + r$. For $r = 1$, $s_i = -1, 1 \pm j1$ and $a_1 = a_2 = a_3 = 1$. Here

$$L(s) = \begin{bmatrix} s^2 - 1 \\ s + 2 \end{bmatrix}, \quad L_r(s) = \begin{bmatrix} s^2 - 1 \\ s + 2 \\ s(s^2 - 1) \\ s(s+2) \end{bmatrix}, \quad L_{rl} = \begin{bmatrix} 0 & -1+\mathrm{j}2 & -1-\mathrm{j}2 \\ 1 & 3+\mathrm{j}1 & 3-\mathrm{j}1 \\ 0 & -3+\mathrm{j}1 & -3-\mathrm{j}1 \\ -1 & 2+\mathrm{j}4 & 2-\mathrm{j}4 \end{bmatrix}$$

Notice that L_{rl} is a complex matrix. To solve (5.2) only the real part of L_{rl} needs to be considered. A solution is $M = [4 \ -1 \ -3 \ -1]$, that is $X(s) = -3s + 4$ and $Y(s) = -(s+1)$, where $X^{-1}(s)Y(s)$ is proper. □

Example 5.2: Let

$$D(s) = \begin{bmatrix} s - 2 & 0 \\ 0 & s + 1 \end{bmatrix}, \quad N(s) = \begin{bmatrix} s - 1 & 0 \\ 1 & 1 \end{bmatrix}$$

with $n = \deg|D(s)| = 2$. Here, there are $\deg|X(s)D(s) + Y(s)N(s)| = n + mr = 2 + 2r$ number of closed-loop poles to be assigned. Note that $r \geq v - 1 = 1 - 1 = 0$.

(i) For $r = 0$ and $\{(s_j, a_j), j = 1, 2)\} = \{(-1, [1 \ 0]'), (-2, [0 \ 1]')\}$,

$$L(s) = L_r(s) = \begin{bmatrix} s - 2 & 0 \\ 0 & s + 1 \\ s - 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and } L_{rl} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \\ -2 & 0 \\ 1 & 1 \end{bmatrix}$$

and a solution of (5.2) is

$$M = \begin{bmatrix} 2 & 0 & -3 & 0 \\ 0 & 2 & 1 & 2 \end{bmatrix}$$

For this case, $M = M(s) = [X(s), Y(s)]$.

(ii) For $r = 1$, and

$$\{(s_j, a_j), j = 1, 4\} = \{(-1, [1 \ 0]'), (-2, [0 \ 1]'), (-3, [-1 \ 0]'), (-4, [0 \ -1]')\}$$

$$L_r(s) = \begin{bmatrix} s - 2 & 0 \\ 0 & s + 1 \\ s - 1 & 0 \\ 1 & 1 \\ s(s-2) & 0 \\ 0 & s(s+1) \\ s(s-1) & 0 \\ s & s \end{bmatrix}, \quad L_{rl} = \begin{bmatrix} -3 & 0 & 5 & 0 \\ 0 & -1 & 0 & 3 \\ -2 & 0 & 4 & 0 \\ 1 & 1 & -1 & -1 \\ 3 & 0 & -15 & 0 \\ 0 & 2 & 0 & -12 \\ 2 & 0 & -12 & 0 \\ -1 & -2 & 3 & 4 \end{bmatrix}$$

a solution of (5.2) yields

$$[X(s), Y(s)] = \begin{bmatrix} s - 7 & -1 & 12 & s + 1 \\ 5 & s + 4 & -6 & s + 4 \end{bmatrix}$$

Note that $X(s)^{-1}Y(s)$ exists and is proper. □

Example 5.3: Consider the same problem as in Example 5.2. Now we would like to add the following two constraints. First, that the leading coefficient matrix of $X(s)$ must be an identity matrix; second, that the first column of $Y(s)$ must be zero; that is, only the second output is used in the feedback loop.

For $r = 1$, let $X(s) = X_0 + X_1 s$ and $Y(s) = Y_0 + Y_1 s$. From the above constraints, $X_1 = I$ and the first columns of Y_0 and Y_1 are zero vectors. Here $M = [X_0, Y_0, X_1, Y_1]$ and (5.2) is again

$$ML_{rl} = [X_0, Y_0, X_1, Y_1] \begin{bmatrix} -3 & 0 & 5 & 0 \\ 0 & -1 & 0 & 3 \\ -2 & 0 & 4 & 0 \\ 1 & 1 & -1 & -1 \\ 3 & 0 & -15 & 0 \\ 0 & 2 & 0 & -12 \\ 2 & 0 & -12 & 0 \\ -1 & -2 & 3 & 4 \end{bmatrix} = [0]$$

To find the solution M that satisfies the two extra constraints, L_{rl} is first partitioned as

$$L_{rl} = \begin{bmatrix} L_{rl1} \\ L_{rl2} \\ L_{rl3} \end{bmatrix}, \quad \text{where } L_{rl1} = \begin{bmatrix} -3 & 0 & 5 & 0 \\ 0 & -1 & 0 & 3 \\ -2 & 0 & 4 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

$$L_{rl2} = \begin{bmatrix} 3 & 0 & -15 & 0 \\ 0 & 2 & 0 & -12 \end{bmatrix}, \quad L_{rl3} = \begin{bmatrix} 2 & 0 & -12 & 0 \\ -1 & -2 & 3 & 4 \end{bmatrix}$$

Since $X_1 = I$, the above equation can be rewritten as

$$[X_0, Y_0, Y_1]\begin{bmatrix} L_{rl1} \\ L_{rl3} \end{bmatrix} = L_{rl2}$$

To zero the first columns of Y_0 and Y_1, two additional columns are added to the equation

$$[X_0, Y_0, Y_1]\,[L_{rl13}, C] = [L_{rl2}, D]$$

where

$$L_{rl13} = \begin{bmatrix} L_{rl1} \\ L_{rl3} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving the last equation yields

$$M = \begin{bmatrix} 1 & -5 & 0 & 5 & 1 & 0 & 0 & 5 \\ 1 & 6 & 0 & 2 & 0 & 1 & 0 & -1 \end{bmatrix}$$

or,

$$X(s) = \begin{bmatrix} s+1 & -5 \\ 1 & s+6 \end{bmatrix}, \quad \text{and} \quad Y(s) = \begin{bmatrix} 0 & 5(s+1) \\ 0 & -(s-2) \end{bmatrix}$$

Clearly $X^{-1}(s)Y(s)$ is proper and satisfies the constraints. □

5.1.2. *$Q(s) = W(s)R(s)$.* There are cases when the equation to be solved has the form

$$X(s)D(s) + Y(s)N(s) = W(s)R(s) \qquad (5.4)$$

where $R(s)$ is a given $m \times m$ non-singular matrix and $W(s)$ is not specified; $D(s)$, $N(s)$ are right coprime. It is necessary to preserve the freedom in $W(s)$ since $X(s)$, $Y(s)$ must satisfy additional constraints. An instance where this type of equation appears is the regulator problem with internal stability when the measured plant outputs may be different from the regulated outputs; in that case $X(s)$, $Y(s)$ must also satisfy another diophantine equation (5.1) for pole assignment. The problem here in (5.4) is to select $X(s)$, $Y(s)$ so that $R(s)$ is a right divisor of $X(s)D(s) + Y(s)N(s)$. This problem can be easily solved using the approach presented here. The approach is based on Corollary 4.8 (Theorem 4.9 for the non-square case) and it is illustrated below.

Example 5.4: Let

$$D(s) = \begin{bmatrix} s^2 & 0 \\ 1 & -s+1 \end{bmatrix}, \quad N(s) = \begin{bmatrix} s+1 & 0 \\ 1 & 1 \end{bmatrix}$$

Solve (5.4) with

$$R(s) = \begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix}$$

To solve (5.4), determine first the appropriate (s_j, a_{ij}). In this case, $\deg|R(s)| = 2$ and $s_1 = -1$, $a_{11} = [1\ 0]'$, $a_{21} = [0\ 1]'$. Note that $R(s_j)a_{ij} = 0$ and the problem is reduced to solving (5.2) with $l = 2$ and $r = 1$. A solution can be found as

$$X(s) = \begin{bmatrix} s+3/2 & 1/2 \\ s+1/2 & s+1/2 \end{bmatrix}, \quad Y(s) = \begin{bmatrix} s+1 & s \\ s & 1 \end{bmatrix}$$

where $X^{-1}(s)Y(s)$ is proper and

$$W(s) = 1/2\begin{bmatrix} 2s^2+3s+3 & 1 \\ 2s^2+s+3 & -2s+3 \end{bmatrix}$$ □

5.1.3. *$H(s) = N(s)D^{-1}(s)$.* In the pole assignment problem, if the desired closed loop poles are different than the open loop poles (that is the poles of $H(s) = N(s)D^{-1}(s)$) then it is not necessary to use a coprime factorization $D(s)$, $N(s)$ as the transfer function matrix can be used directly. In particular, (5.1) can be written as $X(s) + Y(s)N(s)D^{-1}(s) = Q(s)D^{-1}(s)$. Substituting s_j and postmultiplying by a_j one obtains the equation to be solved

$$(X(s_j) + Y(s_j)H(s_j))a_j = 0 \quad j = 1, l \qquad (5.5)$$

Notice that the characteristic vector corresponding to s_j is in this case $D^{-1}(s_j)a_j$.

Example 5.5: Let the open loop transfer function be

$$H(s) = \frac{s+2}{s^2-1}$$

and $|Q(s)| = s(s+2)(s+3)(s+4)$. If $s_i = -2, -3, -4, 0$ and $a_i = 1$, $i = 1, 4$, then a solution of (5.5) is

$$X(s) = s^2 + 9s + 14 \text{ and } Y(s) = 13s + 7$$ □

Example 5.6: Let the open loop transfer function matrix be

$$H(s) = \begin{bmatrix} \dfrac{s-1}{s-2} & 0 \\ \dfrac{1}{s-2} & \dfrac{1}{s+1} \end{bmatrix} \quad \text{and} \quad |Q(s)| = s(s+2)(s+3)(s+4)(s+5)$$

If $\{(s_j, a_j), j = 1, 4\} = \{(-2, [1\ 0]'), (-3, [0\ 1]'), (-4, [-1\ 0]'), (-5, [0\ -1]'), (0, [1\ -1]')\}$, then a solution is

$$X(s) = \begin{bmatrix} 77{\cdot}25s+1 & s \\ 76{\cdot}25s & s+1 \end{bmatrix}, \quad Y(s) = \begin{bmatrix} -81s+43 & 7s+15 \\ -80s+44 & 6s+14 \end{bmatrix}$$

Note that $X^{-1}(s)Y(s)$ is proper. □

5.1.4. *Static output feedback.* This is a special case of the dynamic output feedback discussed above. Interpolation was first used to assign closed loop poles using static output feedback by Antsaklis (1977) and Antsaklis and Wolovich (1977). It offers a convenient way to assign at least some of the poles arbitrarily and study the locations of the remaining poles. The equations to be solved here are

$$(D(s_j) + KN(s_j))a_j = 0, \quad j = 1, l \tag{5.6}$$

where K is a real matrix, the static output feedback gain matrix. Equivalently, it can also be written as

$$(I + KH(s_j))a_j = 0 \quad j = 1, l \tag{5.6 a}$$

The example below illustrates the approach.

Example 5.7: Let the open loop transfer matrix be

$$H(s) = \begin{bmatrix} \dfrac{s+1}{s^2} & \dfrac{s+2}{s^2+1} \\ \dfrac{2}{s} & \dfrac{2s+3}{s^2+2} \end{bmatrix}$$

and the desired poles are $s_1 = -1, -2$ with $a_i = [-26{\cdot}456\ \ 92{\cdot}16]', [-0{\cdot}4432\ \ 1]$. From (5.6 a), $KH(s_i)a_i = -a_i$. That is,

$$K[H(s_1)a_1, H(s_2)a_2] = -[a_1, a_2]$$

The solution is

$$K = \begin{bmatrix} -157{\cdot}08 & 73{\cdot}39 \\ 321{\cdot}30 & -150{\cdot}49 \end{bmatrix}$$

Note that by choosing a_i appropriately other poles can be assigned as well. The above solution places the other two poles at -3 and -4. For details, see Antsaklis and Wolovich (1977).

5.2. *State feedback*

Given a state space description $\dot{x} = Ax + Bu$, the linear state feedback control law is defined by $u = Fx$. It is now known that if (A, B) is controllable then there exists F such that all the closed loop eigenvalues, that is, the zeros of $|sI - (A + BF)|$ are arbitrarily assigned. It will now be shown that F which arbitrarily assigns all closed loop eigenvalues can be determined using interpolation.

Let A, B, F be $n \times n$, $n \times m$, $m \times n$ real matrices respectively. Note that $|sI - (A + BF)| = |sI - A| \cdot |I_n - (sI - A)^{-1}BF| = |sI - A| \cdot |I_m - F(sI - A)^{-1}B|$. If now the desired closed-loop eigenvalues s_j are different from the eigenvalues of A, then F will assign all n desired closed loop eigenvalues s_j if

$$F[(s_jI - A)^{-1}Ba_j] = a_j \quad j = 1, n \tag{5.7}$$

The $m \times 1$ vectors a_j are selected so that $(s_jI - A)^{-1}Ba_j$, $j = 1, n$, are linearly independent vectors.

Alternatively one could approach the problem as follows: let $M(s)$ $(n \times m)$ $D(s)$ $(m \times m)$ be right coprime polynomial matrices such that

$$[sI - A, B]\begin{bmatrix} M(s) \\ -D(s) \end{bmatrix} = 0 \tag{5.8}$$

That is $(sI - A)^{-1}B = M(s)D^{-1}(s)$. An internal representation equivalent to $\dot{x} = Ax + Bu$ in polynomial matrix form is $Dz = u$ with $x = Mz$. The eigenvalue assignment problem is then to assign all the roots of $|D(s) - FM(s)|$; or to determine F so that

$$FM(s_j)a_j = D(s_j)a_j \quad j = 1, n \tag{5.9}$$

Relation (5.9) was originally used in Antsaklis (1977) to determine F. Note that this formulation does not require that s_j be different from the eigenvalues of A as in (5.7). The $m \times 1$ vectors a_j are selected so that $M(s_j)a_j$, $j = 1, n$, are independent. Note that $M(s_j)$ has the same column rank as $S(s_j) = \text{block diag}\{[1, s, \ldots, s^{d_i - 1}]'\}$ where d_i are the controllability indices of (A, B) (Wolovich 1974, Kailath 1980). Therefore, it is possible to select a_j so that $M(s_j)a_j$, $j = 1, n$, are independent even when s_j are repeated (see § 2; choice of interpolation points).

In general, there is great flexibility in selecting the non-zero vectors a_j. Note for example that when s_j are distinct, a very common case, a_j can almost be arbitrarily selected in view of Lemma 2.4. For all the appropriate choices of a_j ($M(s_j)a_j$, $j = 1, n$, linearly independent), the n eigenvalues of the closed-loop system will be at the desired locations s_j $j = 1, n$. Different a_j correspond to different F (via (5.9)) that produce, in general, different system behaviour; this is a phenomenon unique to the multivariable case. This can be explained by the fact that the vectors a_j one selects in (5.9) are related to the eigenvectors of the closed-loop system and although the closed-loop eigenvalues are at s_j, for different a_j one assigns different eigenvectors, which lead to different behaviour in the closed-loop system.

The exact relation of the eigenvectors to the a_j can be found as follows:

$$\begin{aligned}[s_jI - (A + BF)]M(s_j)a_j &= (s_jI - A)M(s_j)a_j - BFM(s_j)a_j \\ &= BD(s_j)a_j - BD(s_j)a_j = 0\end{aligned}$$

where (5.8) and (5.9) were used. Therefore, $M(s_j)a_j = v_j$ are the closed-loop eigenvectors corresponding to s_j.

It is not difficult to see that the results in the Appendix can be used to assign generalized closed-loop eigenvectors (that correspond to Jordan forms of a certain type) using this approach. This is, of course, related to the assignment of invariant polynomials of $sI - (A + BF)$ using state feedback, a problem originally studied by Rosenbrock. One may select a_j in (5.9) to impose constraints on the gain f_{ij} in F. For example, one may select a_j so that a column of F is zero (take the corresponding row of all a_j to be non-zero), or an element of F, $f_{ij} = 0$. This point is not elaborated further here.

In § 5.3 the problem of selecting a_j to achieve additional objectives, beyond pole assignment is discussed. Now, the relation to a similar approach for eigenvalue assignment via state feedback (Moore 1976) is shown; note that this approach was developed in parallel, but independently of, the interpolation method described above.

Consider $s_jI - (A + BF)$ and postmultiply by the corresponding right eigenvector v_j to obtain

$$[s_jI - A, B]\begin{bmatrix} v_j \\ -Fv_j \end{bmatrix} = 0 \tag{5.10}$$

In view of this, determine a basis for the right kernel of $[sI - A, B]$ (Moore 1976), namely

$$[s_jI - A, B]\begin{bmatrix} M_j \\ -D_j \end{bmatrix} = 0 \tag{5.11}$$

where the basis has m (independent) columns; note that $\operatorname{rank}[sI - A, B] = n$ since (A, B) is controllable. Since it is a basis, there exists $m \times 1$ vectors a_j so that $M_ja_j = v_j$ and $D_ja_j = Fv_j$. Combining, we obtain

$$FM_ja_j = D_ja_j \tag{5.12}$$

which, for $j = 1, n$ determines F (for appropriate a_j). Note the similarity with (5.9); they are exactly the same, in fact if we take $M(s_j) = M_j$ in (5.8) and (5.11). The difference between the two approaches in Antsaklis (1977) and Moore (1976) is that in Antsaklis (1977) a polynomial basis for the kernel of $[sI - A, B]$ is found first and then it is evaluated at $s = s_j$, while in Moore (1976) a basis for the kernel of $[s_jI - A, B]$ is determined without involving polynomial bases and right factorizations.

Example 5.8: Consider

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -4 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and let the desired eigenvalues be $s_i = -0{\cdot}1, -0{\cdot}2, -2, -1 \pm \mathrm{j}1$. Take

$$a_i = \begin{bmatrix} 1{\cdot}2648 \\ -0{\cdot}3391 \end{bmatrix}, \begin{bmatrix} 1{\cdot}67744 \\ -0{\cdot}15072 \end{bmatrix}, \begin{bmatrix} 101 \\ -60 \end{bmatrix}, \begin{bmatrix} -7 - \mathrm{j}16 \\ 8 + \mathrm{j}10 \end{bmatrix}, \begin{bmatrix} -7 + \mathrm{j}16 \\ 8 - \mathrm{j}10 \end{bmatrix}$$

Then, the state feedback matrix that assigns the eigenvalues of $(sI - (A + BF))$ to the desired locations is obtained by solving (5.7)

$$F = \begin{bmatrix} 1{\cdot}16 & 0{\cdot}64 & 17{\cdot}76 & 9{\cdot}44 & 6{\cdot}6 \\ -0{\cdot}08 & -1{\cdot}32 & -8{\cdot}88 & -3{\cdot}22 & -3{\cdot}3 \end{bmatrix} \qquad \square$$

5.3. *Assignment of characteristic values and vectors*

In view of the discussion above on state feedback, the characteristic vectors a_j of $(D(s) - FM(s))$ or the eigenvectors $v_j = M(s_j)a_j$ of $sI - (A + BF)$ can be assigned so that additional design goals are attained, beyond the pole assignment at s_j, $j = 1, n$. Two examples of such assignments follow.

5.3.1. *Optimal control.* It is possible to select (s_j, a_j) so that the closed-loop system satisfies some optimality criteria. In fact it is straightforward to select (s_j, a_j) so that the resulting F calculated using the above interpolation method, is the unique solution of a Linear Quadratic Regulator (LQR) problem; see for example Kailath (1980).

5.3.2. *Unobservable eigenvalues.* It is possible, under certain conditions, to select (s_j, a_j) so that s_j become an unobservable eigenvalue in the closed loop system. Suppose $\dot{x} = Ax + Bu$, $y = Cx$ is equivalent to $D(q)z = u$, $y = N(q)z$; $H(s) = C(sI - A)^{-1}B = N(s)D^{-1}(s)$. Let $M(s)$ be such that

$$(sI - A)M(s) = BD(s)$$

is satisfied, or,

$$M(s)D^{-1}(s) = (sI - A)^{-1}B$$

Assume that it is possible to select (s_j, a_j) so that $CM(s_j)a_j = N(s_j)a_j = 0$. Now if (s_j, a_j) is used in (5.9) or (5.12) to determine F, then s_j will be an unobservable closed-loop eigenvalue. This is because of the fact that its eigenvectors $M(s_j)a_j$ satisfy $CM(s_j)a_j = 0$; see the PBH test below. This can be used to derive solutions for problems such as diagonal decoupling and disturbance decoupling, among others.

Example 5.9: Let $H(s) = N(s)D^{-1}(s) = (s + 1)/(s^2 + 2s + 2)$ with a corresponding state space model

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \quad B = [0\ 1]', \quad \text{and } C = [1\ 1]$$

Here, $CM(s) = N(s) = s + 1$ and $CM(-1) = 0$. Obviously, if a desired closed-loop pole is chosen at -1, it will be unobservable. Indeed, if the desired closed-loop poles are -1 and -2, a solution of (5.7) or (5.9) is $F = [0\ -1]$, which makes the eigenvalues of $(A + BF) = \{-1, -2\}$. The closed-loop transfer function is $1/(s + 2)$. Clearly, the eigenvalue at -1 is unobservable. $\square$

5.4. *Characteristic value/vector tests for controllability and observability—PBH test*

It is known that s_j is an uncontrollable eigenvalue if and only if $\operatorname{rank}[s_jI - A, B] < \operatorname{rank}[sI - A, B]$ or if and only if there exists a non-zero row vector v_j such that $v_j[s_jI - A, B] = 0$ (PBH controllability test, see Kailath 1980). The dual result is also true, namely that s_j is an unobservable eigenvalue if and only if $\operatorname{rank}[(s_jI - A)', C'] < \operatorname{rank}[(sI - A)', C']$ or if and only if there exists a non-zero column vector v_j such that $[(s_jI - A)', C']'\ v_j = 0$ (PBH observability test). These tests can be rather confusing when there are multiple eigenvalues in A; as it is not really clear which one of the multiple eigenvalues is the one that is uncontrollable or unobservable. So instead, many times the uncontrollable eigenvalues are defined by the roots of the determinant of a greatest left divisor of the polynomial matrices $sI - A$ and B; this definition is applicable to polynomial matrix descriptions as well (Rosenbrock 1970, Wolovich 1974). The exact relation between these two different approaches can now be derived. In particular, in view of the results in § 4, (s_j, v_j) that satisfy $[(s_jI - A)', C']'\ v_j = 0$ define a square and non-singular polynomial matrix, that is a right divisor of the columns in $[(sI - A)', C']'$ (see Theorem 4.9); one may have to use the results in the Appendix when the multiplicities of the eigenvalues in question cannot be handled by the results in § 4. Based on this one can handle now cases of multiple eigenvalues using eigenvalue/eigenvector tests (characteristic value/vector tests) $[(s_jI - A)', C']' v_j = 0$ without confusion.

5.5. *Choosing an appropriate closed loop transfer function matrix*

One of the challenging problems in practical control design is to choose an appropriate closed loop transfer function matrix that satisfies all the control specifications such as disturbance rejection, command following, etc which can be obtained from the given plant by applying an internally stable feedback loop. For example, in the SISO system control design, if the plant has a RHP zero, then the desired close loop transfer function must have the same RHP zero, otherwise, the closed loop system will be internally unstable. Selecting appropriate closed loop transfer matrices is even more difficult for MIMO systems; note that in this case it is possible to have both a pole and a zero at the same location without cancelling each other. To prevent cancelling of the RHP zeros and to guarantee the internal stability of feedback control systems, both locations and directions of the RHP zeros must be considered. This can be best explained in the context of the Stable Model Matching Problem (Gao and Antsaklis 1989):

Given proper rational matrices $H(s)$ $(p \times m)$ and $T(s)$ $(p \times q)$, find a proper and stable rational matrix $M(s)$ such that the equation

$$H(s)M(s) = T(s) \tag{5.13}$$

holds. It is known that a stable solution for (5.13) exists if and only if $T(s)$ has, as it zeros, all the RHP zeros of $H(s)$ together with their directions. Let the coprime fraction representations of $H(s)$ and $T(s)$ be $H(s) = N(s)D^{-1}(s)$ and $T(s) = N_T(s)D_T^{-1}(s)$. The direction associated with a zero of $H(s)$, s_j, is given by the vector a_j which satisfies

$$a_jN(s_j) = 0 \tag{5.14}$$

Furthermore, $T(s)$ will have the same zero, s_j, together with its direction if $T(s)$ satisfies

$$a_jN_T(s_j) = 0 \tag{5.15}$$

Thus, (5.15) must be taken into consideration when $T(s)$ is selected.

Example 5.10: Consider a diagonal $T(s)$; that is, the control specifications demand diagonal decoupling of the system. Let

$$H(s) = \frac{1}{s+1}\begin{bmatrix} s-1 & 0 \\ 1 & 1 \end{bmatrix}$$

with a zero at $s = 1$. Then $aH(1) = 0$ gives $a = [1\ 0]$ and $T(s)$ must satisfy $aT(1) = [1\ 0]T(1) = 0$. Since $T(s)$ must be diagonal, $t_{11}(1) = 0$; that is the RHP zero of the plant should appear in the (1, 1) entry of $T(s)$. Certainly $T(s)$ can be chosen to have 1 as a zero in both diagonal entries. However, the RHP zeros are undesirable in control and the minimum possible number should be included in T. □

6. Rational matrix interpolation — theory and applications

In this section the results on polynomial matrix interpolation derived in previous sections are used to study rational matrix interpolation. In the first part, on theory, it is shown that rational matrix interpolation can be seen as a special case of polynomial matrix interpolation. This result is shown in Theorem 6.1, where the conditions under which a rational matrix $H(s)$ is uniquely represented by interpolation triplets are derived. Theorem 6.1 is the rational interpolation theorem that corresponds to the main interpolation, Theorem 2.1. Constraints are incorporated in (6.5) and an alternative form of the theorem is presented in Corollary 6.2. Theorem 6.3 shows the conditions under which the denominator of $H(s)$ can be specified arbitrarily. These results are applied to rational matrix equations and results analogous to the results on polynomial matrix equations derived in the previous sections are obtained.

6.1. *Theory*

Similarly to the polynomial matrix case, the problem here is to represent a $(p \times m)$ rational matrix $H(s)$ by interpolation triplets or points (s_j, a_j, b_j), $j = 1, l$, which satisfy

$$H(s_j)a_j = b_j \quad j = 1, l \tag{6.1}$$

where s_j are complex scalars and $a_j \neq 0$, b_j complex $(m \times 1)$, $(p \times 1)$ vectors respectively.

It is now shown that interpolation of rational matrices can be studied via the polynomial matrix interpolation results developed above. In fact it is shown below that the rational matrix interpolation problem reduces to a special case of polynomial matrix interpolation.

Write $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ where $\tilde{D}(s)$ and $\tilde{N}(s)$ are $(p \times p)$ and $(p \times m)$ polynomial matrices respectively. Then (6.1) can be written as $\tilde{N}(s_j)a_j = \tilde{D}(s_j)b_j$ or as

$$[\tilde{N}(s_j), -\tilde{D}(s_j)]\begin{bmatrix} a_j \\ b_j \end{bmatrix} = Q(s_j)c_j = 0, \quad j = 1, l \tag{6.2}$$

That is the rational matrix interpolation problem for a $p \times m$ rational matrix $H(s)$ can be seen as a polynomial interpolation problem for a $p \times (p+m)$ polynomial matrix $Q(s) := [\tilde{N}(s), -\tilde{D}(s)]$ with interpolation points $(s_j, c_j, 0) = (s_j, [a_j', b_j']', 0)$, $j = 1, l$. There is also the additional constraint that $\tilde{D}^{-1}(s)$ exists. It should be pointed out here that this is a problem similar to the pole assignment problem studied in § 5, where the characteristic values and vectors of $Q(s)$ defined in § 4 were used; the difference here is that $Q(s)$ is not square and non-singular, however results appropriate for such $Q(s)$ have also been developed above, in § 4. We shall now apply polynomial interpolation results to (6.2).

Let the column degrees of $Q(s) = [\tilde{N}(s), -\tilde{D}(s)]$ be d_i, $i = 1, p+m$. By Corollary 2.2 $l = \Sigma d_i$ interpolation points $(s_j, [a_j', b_j']', 0)$, $j = 1, l$, together with a given $p \times (p+m)$ leading coefficient matrix C_c uniquely specify $Q(s)$. It is assumed here, (see Corollary 2.2) that the matrix S_{1l} has full rank. Since C_c is chosen, the columns which correspond to $\tilde{D}(s)$ can, of course, be arbitrarily selected; for example, they could be taken to be any $p \times p$ non-singular matrix or simply the identity I_p thus guaranteeing that $\tilde{D}^{-1}(s)$ exists.

Alternatively, as it was done in (2.11) ($B_l = 0$ case) the additional constraints to be satisfied can be expressed as

$$[\tilde{N}, -\tilde{D}][S_l, C] = [0, D] \tag{6.3}$$

where $[\tilde{N}(s), -\tilde{D}(s)] = [\tilde{N}, -\tilde{D}]\ S(s)$ with $S(s) = \text{blk diag}\{[1, s, \ldots, s^{d_i}]'\}$ $i = 1, p+m$

$$S_l := [S(s_1)c_1, \ldots, S(s_l)c_l] \tag{6.4}$$

Here, $c_j = [a_j', b_j']'$ and (s_j, c_j) are so that S_l $(\Sigma d_i + (p+m)) \times l$ has full rank l (see Theorem 2.1). Equations $[\tilde{N}, -\tilde{D}]C = D$ express the k additional constraints on the coefficients; k is the number of columns of C or D and it is taken to be $k = (\Sigma d_i + (p+m)) - l$. Furthermore, C is selected so that $\operatorname{rank}[S_l, C] = l$; in this way a unique solution exists for any D. Since $\tilde{D}(s)$ is a $p \times p$ matrix, it is possible to guarantee that the leading coefficient matrix of $\tilde{D}(s)$ is, say, I_p by using p equations (p columns of C). So the number l of interpolation points can be $l = \sum d_i + m$. These l interpolation points, together with the p constraints, guarantee that $\tilde{D}^{-1}(s)$ exists and uniquely define $[\tilde{N}(s), -\tilde{D}(s)]$ and therefore $H(s)$, assuming that $[S_l, C]$ has full rank; note that full rank can always be attained if S_l has full column rank. The following theorem has been shown.

Theorem 6.1: *Assume that interpolation triplets (s_j, a_j, b_j), $j = 1, l$, and non-negative integers d_i, $i = 1, p+m$, with $l = \sum d_i + m$ are given such that S_l $(\sum d_i + (p+m)) \times l$ in (6.4) has full column rank. There exists a unique $(p \times m)$ rational matrix $H(s)$ of the form $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ where the column degrees of the polynomial matrix $[\tilde{N}(s), -\tilde{D}(s)]$ are d_i $i = 1, p+m$, with the leading coefficient matrix of $\tilde{D}(s)$ being I_p (non-singular), which satisfies (6.1).*

When the number of interpolation constraints l on $H(s)$ is less than $\sum d_i + m$, additional constraints can be used to impose other properties on $H(s)$. For example, additional linear equations of the form $D(s_j)\alpha_j = 0$ can be added in (6.3) so that $H(s)$ has poles in certain locations. Similarly, for zeros of $H(s)$ (see Example 6.2 below). In view of Corollary 2.6 an alternative form for (6.3) is

$$\bar{Q}[S_{dl}, C_d] = [0, D_d] \tag{6.5}$$

where d is the degree of $[\tilde{N}(s), -\tilde{D}(s)]$; see Corollary 2.6 and the related discussion for details. Here S_{dl} is a $((p+m)(d+1) \times l)$ matrix. Similar to the above, it is possible with p equations (p columns in C_d or D_d) to guarantee that $\tilde{D}^{-1}(s)$ exists. Therefore, one could have $l = (p+m)d + m$ interpolation constraints together with the p additional equations to determine uniquely $\bar{Q}$ in (6.5) and therefore $H(s)$. So, the following Corollary has been shown.

Corollary 6.2: *Assume that interpolation triplets (s_j, a_j, b_j), $j = 1, l$, and non-negative integer d with $l = (p+m)d + m$ are given such that $S_{dl}((p+m)(d+1) \times l)$ in (6.5) has full column rank. There exists a unique $(p \times m)$ rational matrix $H(s)$ of the form $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ where the highest degree of the polynomial matrix $[\tilde{N}(s), -\tilde{D}(s)]$ is d, with the leading coefficient matrix of $\tilde{D}(s)$ being I_p (non-singular), which satisfies (6.1).*

Example 6.1: Consider a scalar rational $H(s)$ ($p = m = 1$) with first degree

numerator and denominator ($d = 1$). Here, we can have up to $l = (p+m)d + m = 2d + 1 = 3$ interpolation constraints and still guarantee that the denominator exists and is of degree 1. Let

$$\{(s_j, a_j, b_j) j = 1, 2, 3\} = \{(0, 1, b_1), (1, 1, b_2), (-1, 1, b_3)\}$$

Also let $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s) = (\alpha_1 s + \alpha_0)^{-1}(\beta_1 s + \beta_0)$. Here

$$[\tilde{N}(s), -\tilde{D}(s)] = [\tilde{N}, -\tilde{D}]S(s) = [\beta_0, \beta_1, -\alpha_0, -\alpha_1]\begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \\ 0 & s \end{bmatrix}$$

$$c_1 = \begin{bmatrix} 1 \\ b_1 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 \\ b_2 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 1 \\ b_3 \end{bmatrix}$$

and

$$[N, -D]S_l = [\beta_0, \beta_1, -\alpha_0, -\alpha_1]\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ b_1 & b_2 & b_3 \\ 0 & b_2 & -b_3 \end{bmatrix} = [0\ 0\ 0]$$

A fourth equation representing additional constraints can be added (see (6.5)) to guarantee, say, $\alpha_1 = 1$. This is equivalent to solving

$$[\beta_0, \beta_1, -\alpha_0]\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ b_1 & b_2 & b_3 \end{bmatrix} = [0\ b_2\ -b_3] \text{ from which}$$

$$[\beta_0, \beta_1, -\alpha_0] = \frac{-1}{2b_1 - b_2 - b_3}[b_1(b_3 - b_2), 2b_2 b_3 - b_1(b_2 + b_3), b_2 - b_3]$$

□

Example 6.2: Consider only the first two interpolation constraints of the previous example and require that $\alpha_1(-3) + \alpha_0 =$ or that $H(s)$ has a pole at -3 and $\alpha_1 = 1$. Then

$$[\beta_0, \beta_1, -\alpha_0, -\alpha_1]\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ b_1 & b_2 & 1 & 0 \\ 0 & b_2 & -3 & -1 \end{bmatrix} = [0\ 0\ 0\ 1]$$

from which

$$[\beta_0, \beta_1, -\alpha_0] = [3b_1, -3b_1 + 4b_2, -3]$$

That is

$$H(s) = \frac{(-3b_1 + 4b_2)s + 3b_1}{s + 3}$$

satisfies all constraints. Namely, $H(0) = b_1$, $H(1) = b_2$ and the denominator of $H(s)$ has a zero at -3 (pole of $H(s)$) with leading coefficient equal to 1. □

Example 6.3: Consider a 2×2 rational matrix $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$. Let $Q(s) = [\tilde{N}(s), -\tilde{D}(s)]$ and $\deg_c Q(s) = \{1\ 0\ 1\ 1\}$. For a solution $Q(s)$ to exist, one needs $l \leqslant \sum d_i + p + m = 3 + 4 = 7$ interpolation triplets (s_j, a_j, b_j), $j = 1$,

l. Suppose that two interpolation triplets of the form in (6.2) are given as: $\{(1, [0\ 1\ 1\ 0]', [0\ 0]'), (2, [1\ 1\ 4/3\ -1/12]', [0\ 0]')\}$. In addition, it is required that $H(s)$ has a zero at $s = 0$ and poles at $s = -1$ and $s = -2$ with their directions specified as $\tilde{N}(0)[1\ 0]' = [0\ 0]'$, $\tilde{D}(-1)[1\ -1]' = [0\ 0]'$ and $\tilde{D}(-2)[0\ 1]' = [0\ 0]'$. These constraints can be equivalently expressed as interpolation triplets: $\{(0, [1\ 0\ 0\ 0]', [0\ 0]'), (-1, [0\ 0\ 1\ -1]', [0\ 0]'), (-2, [0\ 0\ 0\ 1]', [0\ 0]'\}$. Now the problem becomes a standard polynomial interpolation problem, i.e. to determine $Q(s)$ subject to $Q(s_j)c_j = b_j = [0\ 0]'$ for $j = 1, 5$. Let $SJ = \{s_1, \ldots, s_l\}$, $C_l = [c_1, \ldots, c_l]$, $B_l = [b_1, \ldots, b_l]$. Then $QS_5 = B_5$ (2.5) is to be solved where

$$SJ = \{-2\ -1\ 0\ 1\ 2\ 3\ 4\}, \quad B_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C_5 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 4/3 \\ 1 & -1 & 0 & 0 & -1/12 \end{bmatrix}$$

The orthonormal basis of the left null space of S_5 is found to be

$$N_{Sl} = \begin{bmatrix} 0{\cdot}0000 & 0{\cdot}3173 & 0{\cdot}2522 & 0{\cdot}0650 & -0{\cdot}3173 & 0{\cdot}7646 & 0{\cdot}3823 \\ 0{\cdot}0000 & -0{\cdot}2726 & -0{\cdot}7151 & 0{\cdot}4425 & 0{\cdot}2726 & 0{\cdot}3396 & 0{\cdot}1698 \end{bmatrix}$$

Note that, in general, N_{Sl} is a $((\sum d_i + m)\text{-rank}\{S_l\}) \times (\sum d_i + m)$ matrix and all solutions of (2.5) with $B_l = 0$ can be characterized as $Q = MN_{Sl}$ where M is any $p \times ((\sum d_i + m)\text{-rank}\{S_l\})$ real matrix. In this example M can be simply chosen as the identity matrix, that is $Q = N_{Sl}$, since $((\sum d_i + m)\text{-rank}\{S_l\}) = 7 - 5 = 2 = p$. Therefore,

$$\begin{aligned} Q(s) &= QS(s) \\ &= \begin{bmatrix} 0{\cdot}3173s & 0{\cdot}2522 & -0{\cdot}3173s + 0{\cdot}0650 & 0{\cdot}3823s + 0{\cdot}7647 \\ -0{\cdot}2726s & -0{\cdot}7151 & 0{\cdot}2726s + 0{\cdot}4425 & 0{\cdot}1698s + 0{\cdot}3396 \end{bmatrix} \\ &= [\tilde{N}(s), -\tilde{D}(s)] \end{aligned}$$

It can be easily verified that the resulting transfer matrix $H(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ has a zero at $s = 0$ and poles at $s = -1, -2$.

To determine uniquely $Q(s)$ in this example, two additional constraints in the form of (6.3): $\{(3, [0\ 1\ 0\ 0]', [2\ 1]'), (4, [0\ 1\ 1\ 1]', [-3\ -6]')\}$ are imposed which lead to

$$SJ = \{-2\ -1\ 0\ 1\ 2\ 3\ 4\}, \quad B_7 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

$$C_7 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 4/3 & 0 & 1 \\ 1 & -1 & 0 & 0 & -1/12 & 0 & 1 \end{bmatrix}$$

by solving (2.5),

$$Q = \begin{bmatrix} 0 & 1 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -2 & -1 \end{bmatrix}$$

and

$$Q(s) = QS(s) = \begin{bmatrix} s & 2 & -(s+1) & 0 \\ 0 & 1 & -1 & -(s+2) \end{bmatrix}$$

therefore,

$$H(s) = \begin{bmatrix} s+1 & 0 \\ 1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} s & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dfrac{s}{s+1} & \dfrac{2}{s+1} \\ \dfrac{-s}{(s+1)(s+2)} & \dfrac{s-1}{(s+1)(s+2)} \end{bmatrix}$$

□

If it is desired that the denominator of $H(s)$ be completely determined in advance, then this can be expressed in terms of equations (6.3) or (6.5). It is also possible to show this result directly based on Theorem 2.1.

Theorem 6.3: *Assume that interpolation triplets (s_j, c_j, b_j), $j = 1, l$, $c_j \neq 0$ and m non-negative integers d_i, $i = 1, m$, with $l = \sum d_i + m$ are given together with an $(m \times m)$ polynomial matrix $D(s)$, $|D(s_j)| \neq 0$, such that the S_l matrix in (2.2) with $a_j := [D(s_j)]^{-1}c_j$ has full rank. Then there exists a unique $(p \times m)$ rational matrix $H(s)$ of the form $H(s) = N(s)D(s)^{-1}$, where the polynomial matrix $N(s)$ has column degrees $\deg_{ci}[N(s)] = d_i$, $i = 1, m$, for which*

$$H(s_j)c_j = b_j \quad j = 1, l \tag{6.6}$$

Proof: Let $N(s) = NS(s)$ as in Theorem 2.1. The proof is similar also. Notice that (6.6) implies $NS_l = B_l$ with $a_j = [D(s_j)]^{-1}c_j$ in S_l of (2.2). □

The $m \times m$ denominator matrix $D(s)$ is arbitrarily chosen subject only to $|D(s_j)| \neq 0$. This offers great flexibility in rational interpolation. It should be pointed out that the matrix denominator $D(s)$ is much more general than the commonly used scalar one $d(s)$, since $D(s) = d(s)I$ is clearly a special case of matrices $D(s)$ with desired zeros of determinant; note that in this case $|D(s)| = d(s)^m$ that is, the zeros of $|D(s)|$ are all the zeros of $d(s)$ each repeated m times.

Example 6.4: Consider the scalar rational example discussed above. Here $l = \sum d_i + m = 1 + 1 = 2$ and $S(s) = [1\ s]'$. Consider interpolation points $(0, 1, b_1)$ and $(1, 1, b_2)$ as above and let the desired denominator be $D(s) = s + 3$. Then $c_1 = D^{-1}(0)a_1 = 1/3$, $c_2 = D^{-1}(1)a_2 = 1/4$ and

$$NS_l = [\beta_0, \beta_1][S(0)c_1, S(1)c_2] = [\beta_0, \beta_1]\begin{bmatrix} 1/3 & 1/4 \\ 0 & 1/4 \end{bmatrix} = [b_1, b_2] = B_2$$

from which $[\beta_0, \beta_1] = [3b_1, -3b_1 + 4b_2]$. That is

$$H(s) = \frac{(-3b_1 + 4b_2)s + 3b_1}{s+3}$$

satisfies all the constraints. Note that it is the same $H(s)$ as in Example 6.2 even though the constraints were imposed via different approaches. □

As was shown above, rational matrix interpolation results are directly derived from corresponding polynomial matrix interpolation results and all resutls of § 2 (§ 3–5) can therefore be extended to the rational matrix case. One could, of course, use the results of Corollaries 2.5 to 2.7 and 2.8 to obtain alternative approaches to rational matrix interpolation.

6.2. *Applications—rational matrix equations*

Now consider the rational matrix equation:

$$M(s)L(s) = Q(s) \tag{6.7}$$

where $L(s)$ $(t \times m)$ and $Q(s)$ $(k \times m)$ are given rational matrices. The polynomial matrix interpolation theory developed above will now be used to solve this equation and determine the rational matrix solutions $M(s)$ $(k \times t)$. Let $M(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$, a polynomial fraction form of $M(s)$ to be determined. Then equation (6.7) can be written as:

$$[\tilde{N}(s) \;\; -\tilde{D}(s)]\begin{bmatrix} L(s) \\ Q(s) \end{bmatrix} = 0 \tag{6.8}$$

Note that instead of solving (6.8) one could equivalently solve

$$[\tilde{N}(s) \;\; -\tilde{D}(s)]\begin{bmatrix} L_p(s) \\ Q_p(s) \end{bmatrix} = 0 \tag{6.9}$$

where $[L_p(s)'Q_p(s)']' = [L(s)'Q(s)']'\phi(s)$ a polynomial matrix with $\phi(s)$ the least common denominator of all entries of $L(s)$ and $Q(s)$; in general, $\phi(s)$ could be any denominator in a right fractional representation of $[L(s)', Q(s)']'$. The problem to be solved is now (3.1), a polynomial matrix equation, where $L(s) = [L_p(s)'Q_p(s)']'$ and $Q(s) = 0$. Therefore, Theorem 3.1 applies and all solutions $[\tilde{N}(s) - \tilde{D}(s)]$ of degree r can be determined by solving (3.9) or (3.13). Let $s = s_j$ and postmultiply (6.9) by a_j, $j = 1, l$, with a_j and l chosen properly (see below). Define

$$c_j := \begin{bmatrix} L_p(s) \\ Q_p(s) \end{bmatrix} a_j \quad j = 1, l \tag{6.10}$$

The problem now is to find a polynomial matrix $[\tilde{N}(s) \;\; -\tilde{D}(s)]$ which satisfies

$$[\tilde{N}(s_j) \;\; -\tilde{D}(s_j)]c_j = 0 \quad j = 1, l \tag{6.11}$$

as in (6.2). In fact (6.11) is of the form of (3.11) with $b_j = 0$.

Note that restrictions on the solutions can be easily imposed to guarantee that $\tilde{D}^{-1}(s)$ exists and/or that $M(s) = \tilde{D}^{-1}(s)\tilde{N}(s)$ is proper; see also above in this section, as well as § 4 and 5. The existence of solutions of (6.7) and their causality depends on the given rational matrices $L(s)$ and $Q(s)$ (see for example Chen 1984, Gao and Antsaklis 1989 and references therein). Our approach here will find a proper rational matrix of order kr in general when such a solution exists. Additional interpolation type constraints can be added so the solution satisfies additional specifications.

Example 6.5: This is an example of solving the Model Matching Problem (Gao and Antsaklis 1989) using matrix interpolation techniques. Here $L(s)$ and $Q(s)$ are given as:

$$L(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s} \\ 0 & -2 \\ \dfrac{-s}{s+1} & -1 \end{bmatrix} \quad Q(s) = \begin{bmatrix} \dfrac{s}{s+3} & \dfrac{s+1}{s+3} \\ -\dfrac{s}{s+3} & -\dfrac{3s+7}{s+3} \end{bmatrix}$$

The monic least common denominator of all entries is $\phi(s) = s(s+1)(s+3)$ and therefore

$$\begin{bmatrix} L_p(s) \\ Q_p(s) \end{bmatrix} = \begin{bmatrix} s(s+3) & (s+1)(s+3) \\ 0 & -2s(s+1)(s+3) \\ -s^2(s+3) & -s(s+1)(s+3) \\ s^2(s+1) & s(s+1)^2 \\ -s^2(s+1) & -(3s+7)(s+1)s \end{bmatrix}$$

Let

$$\{d_i = \deg_{ci} Q(s)\} = \{0, 0, 1, 1, 0\}$$

$$l = \Sigma d_i + t + k = 2 + 5 = 7,$$

$$\{s_j, j = 1, 5\} = \{-4, -2, 1, 2, 3\},$$

$$\{a_j, j = 1, 5\} = \{[0, 1]', [1, 0]', [1, 1]', [0, -1]', [-1, 0]'\}$$

$$\{b_j = [0, 0]', j = 1, 5\}$$

from which c_j, $j = 1, 5$, are obtained

$$[c_1, \ldots, c_5] = \begin{bmatrix} 3 & -2 & 12 & -15 & -18 \\ 24 & 0 & -16 & 60 & 0 \\ 12 & -4 & -12 & 30 & 54 \\ -36 & -4 & 6 & -18 & -36 \\ 60 & 4 & -22 & 78 & 36 \end{bmatrix}$$

Assume that two additional constraints are introduced in the form of: $\{s_6, s_7\} = \{4, 5\}$, $\{c_6, c_7\} = \{[0\ 1\ 0\ 0\ 0]', [0\ 0\ 0\ 1\ 0]'\}$ and $\{b_6, b_7\} = \{[1\ 0]', [-1, -8]'\}$. Now, solving the polynomial matrix interpolation problem: $[\tilde{N}(s_j) - \tilde{D}(s_j)]c_j = b_j$, $j = 1, 7$, we obtained

$$[\tilde{N}(s) \;\; -\tilde{D}(s)] = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & -(s+1) & -(s+3) & 0 \end{bmatrix}$$

which gives

$$M(s) = \begin{bmatrix} 1 & 1 \\ s+1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -(s+1) \end{bmatrix} \qquad \square$$

7. Concluding remarks

Some of the concepts and ideas presented here have appeared elsewhere. It is the first time however that the theory of polynomial and rational matrix interpolation in its complete form has appeared in the literature. The algorithms have been implemented in Matlab and are available upon request.

Interpolation is a very general and flexible way to deal with problems involving polynomial and rational matrices and the results presented here provide an appropriate theoretical setting and algorithms to deal effectively with such problems. At the same time it is also felt that the results presented here have only opened the way, as there are many more results that can and need to be developed to handle the wide range of problems it is possible to study, via polynomial and rational matrix interpolation theory.

Finally, it should be noted that the rational interpolation results presented here complement results that have appeared in the literature. The exact relationship is under investigation and new insights into the theory are certainly possible.

Appendix

In this Appendix, the general versions of the results in § 4 that are valid for repeated values of s_j, with multiplicities beyond those handled in § 4, are stated. Detailed proofs of these results can be found in the main reference for characteristic values and vectors (Antsaklis 1980).

Let $Q(s)$ be an $(m \times m)$ non-singular matrix and let $Q^{(k)}(s_j)$ denote the kth derivative of $Q(s)$ evaluated at $s = s_j$. If s_j is a zero of $|Q(s)|$ repeated n_j times, define n_j to be the algebraic multiplicity of s_j; define also the geometric multiplicity of s_j as the quantity (m-rank $Q(s_j)$).

Theorem A.1 (Antsaklis 1980, Theorem 1): *There exist complex scalar s_j and $\sum_{i=1}^{l_j} k_{ij}$ $m \times 1$ non-zero vectors $a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^{k_{ij}}$, $i = 1, l_j$, which satisfy*

$$\left.\begin{aligned} Q(s_j)a_{ij}^1 &= 0 \\ Q(s_j)a_{ij}^2 &= -Q^{(1)}(s_j)a_{ij}^1 \\ &\vdots \\ Q(s_j)a_{ij}^{k_{ij}} &= -\left[Q^{(1)}(s_j)a_{ij}^{k_{ij}-1} + \cdots + \frac{1}{(k_{ij}-1)!}Q^{(k_{ij}-1)}(s_j)a_{ij}^1\right] \end{aligned}\right\} \quad \text{(A 1)}$$

with $a_{1j}^1, a_{2j}^1, \ldots, a_{l_jj}^1$ linearly independent if and only if s_j is a zero of $|Q(s)|$ with algebraic multiplicity $(=n_j) \geq \sum_{i=1}^{l_j} k_{ij}$ and geometric multiplicity ($=$(m-rank $Q(s_j)$)) $\geq l_j$.

It is of interest to note that there are l_j chains of (generalized) characteristic vectors corresponding to s_j, each of length k_{ij}. Notice that Theorem 4.2 is a special case of this theorem; it involves only the top equation in (A 1) and it does not involve derivatives of $Q(s)$. The proof of Theorem A 1 is based on the following lemma.

Lemma A.2 (Antsaklis 1980, Lemma 2): *Theorem A.1 is satisfied for given $Q(s)$, s_j and a_{ij}^k if and only if it is satisfied for $U(s)Q(s)$, s_j and a_{ij}^k where $U(s)$ is any unimodular matrix (that is $|U(s)| = \alpha$, a non-zero scalar).*

This lemma allows one to carry on the proof of Theorem A.1 with a matrix $Q(s)$ which is column proper (reduced). The proof of Theorem A.1 is rather involved and it involves the generalized eigenvectors of a real matrix associated with $Q(s)$; it can of course be found in (Antsaklis 1980).

Given $Q(s)$, if s_j and a_{ij}^k satisfy the conditions of Theorem A.1, then this implies a certain structure for the Smith form of $Q(s)$. First, let us define the (unique) Smith form of a polynomial matrix.

Smith Form of $M(s)$ (Rosenbrock 1970, Kailath 1980)

Given a $p \times m$ polynomial matrix $M(s)$ with rank $M(s) = r$, there exist unimodular matrices U_1, U_2 such that $U_1(s)M(s)U_2(s) = E(s)$ where

$$E(s) = \begin{bmatrix} \Lambda(s) & 0 \\ 0 & 0 \end{bmatrix} \quad \Lambda(s) = \operatorname{diag}[\varepsilon_1(s), \varepsilon_2(s), \ldots, \varepsilon_r(s)] \qquad \text{(A 2)}$$

Each ε_i, $i = 1, r$, is a unique monic polynomial satisfying $\varepsilon_i(s)|\varepsilon_{i+1}(s)$, $i = 1$, $r-1$, where $p_2|p_1$ means that there exists polynomial p_3 such that $p_1 = p_2p_3$; that is ε_i divides ε_{i+1}. $E(s)$ is the *Smith form* of $M(s)$ and $\varepsilon_i(s)$ are the *invariant polynomials* of $M(s)$. It can be shown that

$$\varepsilon_i(s) = D_i(s)/D_{i-1}(s) \quad i = 1, r \qquad \text{(A 3)}$$

where $D_i(s)$ is the monic greatest common divisor of all the ith order minors of $M(s)$; note that $D_0(s) = 0$. $D_i(s)$ are the *determinantal divisors* of $M(s)$.

Corollary A.3 (Antsaklis 1980, Corollary 3): *Given $Q(s)$, there exist a scalar s_j and non-zero vectors $a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^{k_{ij}}$ $i = 1, l_j$ which satisfy the conditions of Theorem A.1 if and only if the Smith form of $Q(s)$ contains the factors $(s - s_j)^{k_{ij}}$ $i = 1, l_j$ in l_j separate locations on the diagonal; that is $(s - s_j)^{k_{ij}}$ is a factor in l_j distinct invariant polynomials of $Q(s)$.*

Theorem A.1 and Corollary A.3 refer to the value s_j, a root of $|Q(s)|$ which is repeated at least $\Sigma_{i=1}^{l_j} k_{ij}$ times. If σ distinct values s_j are given, then the following result is derived. Note that the deg$|Q(s)|$ is assumed to be known.

Theorem A.4 (Antsaklis 1980, Theorem 4): *Let $n = \deg|Q(s)|$. There exist σ distinct complex scalars s_j and n non-zero vectors $a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^{k_{ij}}$, $i = 1, l_j$, $j = 1, \sigma$ with $\sum_{j=1}^{\sigma}\sum_{i=1}^{l_j} k_{ij} = n$ with each of the σ sets $\{a_{1j}^1, a_{2j}^1, \ldots, a_{l_jj}^1\}$ linearly independent for $j = 1, \sigma$ that satisfy* (A 1) *if and only if the zeros of $|Q(s)|$ have σ distinct values s_j, $j = 1, \sigma$ each with algebraic multiplicity $(=n_j) = \sum_{i=1}^{l_j} k_{ij}$ and geometric multiplicity ($=$ m-rank $Q(s_j)$) $= l_j$.*

Note that to each distinct characteristic value s_j there correspond $\{a_{1j}^1, a_{1j}^2, \ldots, a_{1j}^{k_{1j}}\} \ldots, \{a_{l_jj}^1, a_{l_jj}^2, \ldots, a_{l_jj}^{k_{l_jj}}\}$ characteristic vectors; there are l_j ($= m -$ rank $Q(s_j)$ $=$ geometric multiplicity) chains of length $k_{1j}, k_{2j}, \ldots, k_{l_jj}$ for a total of $\sum_{i=1}^{l_j} k_{ij}$ characteristic vectors equal to the algebraic multiplicity n_j.

Corollary A.5 (Antsaklis 1980, Corollary 5): *Given $Q(s)$ with $n = \deg|Q(s)|$, there exist σ distinct complex scalars s_j and vectors a_{ij}^k, $i = 1, l_j$, $k = 1, k_{ij}$, $j = 1$, σ, which satisfy the conditions of Theorem A.4 if and only if the Smith form of $Q(s)$ consists of factors $(s - s_j)^{k_{ij}}$ $i = 1, l_j$ in l_j separate locations on the diagonal ($j = 1, \sigma$).*

Note that in view of the divisibility property of the invariant factors of $Q(s)$, if the conditions of Corollary A.5 or similarly of Theorem A.4 are satisfied, the Smith form of $Q(s)$ is uniquely determined. In particular, for $k_{1j} \leq k_{2j} \leq \ldots \leq k_{l_jj}$, the Smith form of $Q(s)$ in this case has the form

$$E(s) = \operatorname{diag}(\varepsilon_1(s), \ldots, \varepsilon_m(s))$$

$$\varepsilon_m(s) = (s - s_j)^{k_{l_jj}}(\cdot),\ \varepsilon_{m-1}(s) = (s - s_j)^{k_{l_j-1\,j}}(\cdot), \ldots,$$

$$\varepsilon_{m-(l_j-1)}(s) = (s - s_j)^{k_{1j}}(\cdot) \qquad \text{(A 4)}$$

with $\varepsilon_{m-l_j}(s) = \ldots = \varepsilon_1(s) = 1$. This is repeated for each distinct value of s_j, $j = 1, \sigma$, until the Smith form is completely determined.

Example A.1: To illustrate the above results consider

$$Q(s)=\begin{bmatrix} s^2 & -1 \\ 0 & s \end{bmatrix}$$

Notice that

$$Q^{(1)}(s)=\begin{bmatrix} 2s & 0 \\ 0 & 1 \end{bmatrix},\quad Q^{(2)}(s)=\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},\quad Q^{(k)}(s)=0 \text{ for } k>2$$

For $s_i = 0$ ($j = 1$), relations (A 1) become:

Let $i = 1$. $Q(0)a_{11}^1 = 0$ implies

$$a_{11}^1=\begin{bmatrix} \alpha \\ 0 \end{bmatrix}\quad (\alpha \neq 0)$$

Note that no other linearly independent a_{i1}^1 exists, so $l_j = 1$.

$$Q(0)a_{11}^2 = -Q^{(1)}(0)a_{11}^1 \quad \text{implies } a_{11}^2=\begin{bmatrix} \beta \\ 0 \end{bmatrix}\quad (\beta \neq 0)$$

$$Q(0)a_{11}^3 = -[Q^{(1)}(0)a_{11}^2 + \frac{1}{(2)!}Q^{(2)}(0)a_{11}^1] \quad \text{implies } a_{11}^3=\begin{bmatrix} \gamma \\ \alpha \end{bmatrix}\quad (\alpha\gamma \neq 0)$$

It can be verified that a_{11}^4 etc are zero. So $k_{11} = 3$. Note that $m - \operatorname{rank} Q(0) = 2 - 1 = 1 = l_1$; that is, the geometric multiplicity of $s_1 = 0$ is 1 and so no other chain of characteristic vectors associated with $s_1 = 0$ exists.

Assume that $Q(s)$ is not known and it is given that $s_1 = 0$ and a_{11}^k, $k = 1, 2, 3$, satisfy (A 1). Then, according to Theorem A.1, the algebraic multiplicity of $s_1 = 0$ is at least 3 ($= k_{11}$) and the geometric multiplicity is at least 1 ($= l_1$). Furthermore, in view of Corollary A.3 the factor s^3 ($= (s - s_1)^{k_{11}}$) appears in one ($= l_1$) location in the Smith form of $Q(s)$.

Assume now that $n = \deg|Q(s)| = 3$ is also given together with $s_1 = 0$ and a_{11}^k, $k = 1, 2, 3$, which satisfy (A 1). Notice that here $l_1 = 1$, $k_{11} = 3$ (see above) so $k_{11} = 3 = n$ which implies that $\sigma = 1$, or $s_1 = 0$ is the only distinct root of $|Q(s)|$. Theorem A.4 can now be applied to show that $s_1 = 0$ has algebraic multiplicity exactly equal to $k_{11} = 3$ and geometric multiplicity exactly equal to $l_1 = 1$. These can be easily verified from the given $Q(s)$. In view of Corollary A.5 and (A 4) the Smith form of $Q(s)$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix}$$

which can also be derived from $Q(s)$ via pre and post-multiplication by unimodular matrices. □

The following lemma highlights the fact that the conditions of Theorem A.4 specify $Q(s)$ within a unimodular premultiplication; see also Lemma 4.6.

Lemma A.6: *Theorem A.4 is satisfied by a matrix $Q(s)$ if and only if it is satisfied by $U(s)Q(s)$ where $U(s)$ is any unimodular matrix.*

It is important at this point to discuss briefly and illustrate the results so far. Assume that, for an $(m \times m)$ polynomial matrix $Q(s)$ yet to be chosen, we have decided upon the degree of $|Q(s)|$ as well as its zero locations—that is about n, s_j and the algebraic multiplicities n_j. Clearly there are many matrices that satisfy these requirements; consider, for example, all the diagonal matrices that satisfy these requirements. If we specify the geometric multiplicities l_j as well, then this implies that the matrices $Q(s)$ must satisfy certain structural requirements so that $m - \operatorname{rank} Q(s_j) = l_j$ is satisfied; in our example the diagonal matrix, the factors $(s - s_j)$ must be appropriately distributed on the diagonal. If k_{ij} are also chosen, then the Smith form of $Q(s)$ is completely defined, that is $Q(s)$ is defined with pre and post-unimodular matrix multiplications. Note that this is equivalent to imposing the restriction that $Q(s)$ must satisfy n relations of type (A 1), as in Theorem A.4, without fixing the vectors a_{ij}^k (see Example A.1). If, in addition, a_{ij}^k are completely specified then $Q(s)$ is determined within a unimodular premultiplication; see Lemma A.6.

Given $(m \times m)$ $Q(s)$, let $n = \deg|Q(s)|$ and assume that $Q(s)$ and s_j, a_{ij}^k satisfy the conditions of Theorem A.4; that is they satisfy (A 1) for σ distinct s_j, $j = 1, \sigma$.

Theorem A.7 (Antsaklis 1980, Theorem 6): *$Q(s)$ is a right divisor (r.d.) of an $(r \times m)$ polynomial matrix $M(s)$ if and only if $M(s)$ satisfies the conditions of Theorem A.4 with the same s_j and a_{ij}^k; that is $M(s)$ also satisfies the conditions (A 1) with the same s_j, a_{ij}^k for σ distinct s_j, $j = 1, \sigma$.*

Proof:

Necessity. If Q is a r.d. of M, $M = \hat{M}Q$, then is can be shown directly that (A 1) are also satisfied by $M(s)$ with the same s_j and a_{ij}^k.

Sufficiency. This is the same as the sufficiency proof of Theorem 4.9. □

In the proof of Theorem A.1 (Antsaklis 1980), the Jordan form of a real matrix A derived from $Q(s)$ was used. Later in the Appendix, results concerning the Smith form of $Q(s)$ were described. It is of interest to outline here the exact relations between the Jordan form of A and the Smith form of $sI - A$ and of $Q(s)$. This is done in the following.

Relations between the Smith and Jordan forms

Given an $m \times m$ non-singular polynomial matrix $Q(s)$ and a real $n \times n$ matrix A, assume that there exist matrices B $(n \times m)$ and $S(s)$ $(n \times m)$ so that

$$(sI - A)S(s) = BQ(s) \tag{A 5}$$

where $(sI - A)$, B are left and $S(s)$, $Q(s)$ right coprime. Then there is a direct relation between the Smith forms of $(sI - A)$ and $Q(s)$, as will be shown. First the relation between the Jordan form of A and the Smith form of $(sI - A)$ is described.

Let A $(n \times n)$ have σ distinct eigenvalues s_j each repeated n_j times ($\sum n_j = n$); n_j is the algebraic multiplicity of s_j. The geometric multiplicity of s_j, l_j, is defined as $l_j = n - \operatorname{rank}(s_jI - A)$; that is, the reduction in rank in $sI - A$ when $s = s_j$. There exists a similarity transformation matrix P such that $PA = JP$ where J is the Jordan canonical form of A.

$$J = \operatorname{diag}[J_j],\ J_j = \operatorname{diag}[J_{ij}] \tag{A 6}$$

where $J_j(n_j \times n_j)$, $j = 1, \sigma$ is the block diagonal matrix associated with s_j; J_j has

l_j ($\leq n_j$) matrices $J_{ij}(k_{ij} \times k_{ij})$, $i = 1, l_j$, on the diagonal each of the form

$$J_{ij} = \begin{bmatrix} s_j & 1 & 0 & \dots & 0 \\ 0 & s_j & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & 0 & & \dots & s_j \end{bmatrix} \tag{A 7}$$

where $\sum_1^{l_j} k_{ij} = n_j$.

The structure of J is determined by the generalized eigenvectors v_{ij}^k of A; they are used to construct P. To each distinct eigenvalue s_j correspond l_j chains of generalized eigenvectors each of length k_{ij}, $i = 1, l_j$, for a total of n_j linearly independent generalized eigenvectors.

Note that the characteristic polynomial of A, $\alpha(s)$, is

$$\alpha(s) = \prod_{i=1}^{\sigma}(s - s_j)^{n_j} \quad (= |sI - A|)$$

while the minimal polynomial of A, $\alpha_m(s)$, is $\prod_{i=1}^{\sigma}(s - s_j)^{\bar{n}_j}$ where $\bar{n}_j :=$ $\max_i k_{ij}$, that is the dimension of the largest block in J associated with s_j.

The Smith form of a polynomial matrix was defined above. It is not difficult to show the following result about the Smith form of $sI - A$, $E_A(s)$ (Antsaklis 1980): without loss of generality, assume that $k_{1j} \leq k_{2j} \leq \dots \leq k_{l_j j}$ $(= \bar{n})$, see also (A.4). If $E_A(s) = \text{diag}[\varepsilon_1(s), \varepsilon_2(s), \dots, \varepsilon_r(s)]$, then

$$\varepsilon_n(s) = (s - s_j)^{k_{l_j j}}(\cdot),\ \varepsilon_{n-1}(s) = (s - s_j)^{k_{l_j - 1, j}}(\cdot), \dots,$$

$$\varepsilon_{n-(l_j-1)}(s) = (s - s_j)^{k_{1j}}(\cdot) \tag{A 8}$$

with $\varepsilon_{n-l_j}(s) = \dots = \varepsilon_1(s) = 1$. That is, the n_j factors $(s - s_j)$ are factors of the l_j invariant polynomials $\varepsilon_{n-(l_j-1)}(s), \dots, \varepsilon_n(s)$; the exponents k_{ij} of $(s - s_j)$ are the dimensions of the matrices J_{ij}, $i = 1, l_j$ of the Jordan canonical form, or equivalently they are the lengths of the chains of the generalized eigenvectors of A corresponding to s_j. The relations in (A 8) are, of course, repeated for each distinct value of s_j, $j = 1, \sigma$, until the Smith form $E_A(s)$ is completely determined.

Example A.2: Let

$$A = J = \begin{bmatrix} J_1 & \\ & J_2 \end{bmatrix} = \begin{bmatrix} J_{11} & & \\ & J_{21} & \\ & & J_{12} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

that is, $s_1 = -3$, $n_1 = 3$, $l_1 = 2$ with $k_{11} = 2$, $k_{21} = 1$; $s_2 = -1$, $n_2 = l_2 = k_{12} = 1$. In view of (A.8), the Smith form of $sI - A$ is

$$E_A(s) = \begin{bmatrix} \varepsilon_1(s) & & & \\ & \varepsilon_2(s) & & \\ & & \varepsilon_3(s) & \\ & & & \varepsilon_4(s) \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & s-3 & \\ & & & (s-3)^2(s-1) \end{bmatrix}$$

Here $\alpha(s) = |sI - A| = (s - 3)^3(s - 1)$ and $\alpha_m(s) = (s - 3)^2(s - 1)$. □

It can be shown (Rosenbrock 1970, Wolovich 1974, Kailath 1980) that if $sI - A$ and $Q(s)$ satisfy relation (A.5), then the matrices

$$\begin{bmatrix} sI - A & B \\ -I_n & 0 \end{bmatrix}, \quad \begin{bmatrix} I_{n-m} & 0 & 0 \\ 0 & Q(s) & I_m \\ 0 & -S(s) & 0 \end{bmatrix}$$

are unimodularly equivalent and they have the same Smith forms. This implies that if $E_Q(s)$ is the Smith form of $Q(s)$,

$$E_A(s) = \begin{bmatrix} I_{n-m} & 0 \\ 0 & E_Q(s) \end{bmatrix} \tag{A 9}$$

It is now easy to show that $Q(s)$ has σ distinct roots s_j of $|Q(s)|$ each repeated n_j times (= algebraic multiplicity as defined before in Theorem A.1); the geometric multiplicity of s_j defined by $m - \text{rank}\, Q(s_j)$ equals l_j since $l_j = m - \text{rank}\, E_Q(s)$. If

$E_Q(s) = \text{diag}(\bar{\varepsilon}_1(s), \dots, \bar{\varepsilon}_m(s))$, then (see also (A.4))

for $k_{1j} \leq k_{2j} \leq \dots \leq k_{l_j j}$ $(= \bar{n})$

$$\bar{\varepsilon}_m(s) = (s - s_j)^{k_{l_j j}}(\cdot),\ \bar{\varepsilon}_{m-1}(s) = (s - s_j)^{k_{l_j-1,j}}(\cdot), \dots,$$

$$\bar{\varepsilon}_{m-(l_j-1)}(s) = (s - s_j)^{k_{1j}}(\cdot) \tag{A 10}$$

with $\bar{\varepsilon}_{n-l_j}(s) = \dots = \bar{\varepsilon}_1(s) = 1$. Compare with the Smith form $E_A(s)$ in (A 8). It is clear that $E_Q(s)$ and $E_A(s)$ or $Q(s)$ and $(sI - A)$ have the same non-unity invariant polynomials as is, of course, clear in view of (A 9). Note that the characteristic polynomial of $Q(s)$ is in this case $\delta(s) = |Q(s)| = \prod_{j=1}^{\sigma}(s - s_j)^{n_j}$ $(= \alpha(s) = |sI - A|)$ while the minimal polynomial of $Q(s)$ is

$$\delta_m(s) = |Q(s)| = \prod_{j=1}^{\sigma}(s - s_j)^{\bar{n}_j} \ (= \alpha_m(s))$$

Example A.3: Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q(s) = \begin{bmatrix} s^2 & -1 \\ 0 & s \end{bmatrix}$$

Note that if

$$S(s) = \begin{bmatrix} 1 & 0 \\ s & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then, $(sI - A)S(s) = BQ(s)$ as in (A 5) with $(sI - A)$, B left coprime and $S(s)$, $Q(s)$ right coprime. Notice that A is already in Jordan canonical form. In fact, $A = J = J_1$ with $s_1 = 0$, $l_1 = 1$, $k_{11} = 3$ and $n_1 = 3$. The Smith form of $sI - A$ is then (A 8)

$$E_A(s) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 & \\ & & & s^3 \end{bmatrix}$$

In view of (A 10), the Smith form of $Q(s)$ is

$$E_Q(s) = \begin{bmatrix} 1 & 0 \\ 0 & s^3 \end{bmatrix}$$

Note that this $Q(s)$ was also studied in Example A.1. □

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