P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

Polynomial Matrix Characterization Using Characteristic Values and Vectors.

by

P.J. Antsaklis

Publication No: 80/18

Department of E.E.

Imperial College

London SW7

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

Introduction

In control problems like pole allocation and regulation with stability the desired compensator is the solution of an equation involving polynomial matrices. To solve these equations one can use the coefficients of the polynomial entries of the matrices after reducing them to some canonical form. Many times this is difficult or even impossible. When one works with equations involving just polynomials one can work either with the coefficients or with the values obtained when certain values of the indeterminant s are "plugged in" ; the latter corresponds to polynomial representation by a number of points (using interpolation the coefficients can be determined). The motivation of this work is exactly this. To establish the necessary theoretical background so that equations involving polynomial matrices can be solved using "plug in" values.

Note that the zeros of the determinant of a polynomial matrix $P(s)$ alone do not fully characterize $P(s)$. Information about the structure is also necessary. Characteristic vectors or latent vectors or simply vectors $a_i^{k,j}$ are introduced to accommodate this in Theorems 1 and 4. The relation between $a_i^{k,j}$ and the Smith form of $P(s)$ is established in Corollaries 3 and 5. Theorem 6, Corollary 7 and Lemma 8 establish the relations between two polynomial matrices when they both satify relations of certain type involving $a_i^{k,j}$. Actually, when the two matrices satisfy exactly the same relations, then they are related by a unimodular

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

premultiplication, which is the generalization of the case when two polynomials of the same degree have the same roots; they are equal within a constant multiplication. Finally the relation between $a_i^{k,j}$ and the generalized eigenvectors of an equivalent to $\{P,Q,R,W\}$ system $\{A,B,C,E\}$ is established. The Appendix contains a detailed account of relations between several canonical forms of A (and $P(s)$) as well as a number of definitions.

It should be noted that the results presented here are of interest not only because of their relation to control applications or their relation to the solution of equations involving polynomial matrices. They are also of interest in their own right because they rigorously establish the relation between $P(s)$ and its "characteristic" values and vectors and by doing so, they generalize and combine results known from the eigenvalue-eigenvector theory of real matrices $(P(s) = sI - A)$ and the theory of polynomials $(P(s) = p(s))$.

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

Main Results

Let $P(s)$ be an $(m \times m)$ nonsingular matrix and let $P^{(k)}(s_i)$ denote the k^{th} derivative of $P(s)$ evaluated at $s = s_i$. If s_i is a zero of $|P(s)|$ repeated n_i times, define n_i to be the algebraic multiplicity of s_i ; define also the geometric multiplicity of s_i as the quantity $m - \text{rank } P(s_i)$ (see Appendix).

Theorem 1 There exist $(m \times 1)$ nonzero vectors $a_i^{1,j}, a_i^{2,j}, \ldots, a_i^{k_i^j,j}$ $j = 1,2,\ldots,\ell_i$ which satisfy

$$
\begin{aligned}
P(s_i)a_i^{1,j} &= 0 \\
P(s_i)a_i^{2,j} &= -P^{(1)}(s_i)a_i^{1,j} \qquad (1)\\
&\;\;\vdots \\
P(s_i)a_i^{k_i^j,j} &= -[P^{(1)}(s_i)a_i^{k_i^j-1,j} + \ldots + \frac{1}{(k_i^j-1)!}P^{(k_i^j-1)}(s_i)a_i^{1,j}]
\end{aligned}
$$

with $a_i^{1,1}, a_i^{1,2}, \ldots, a_i^{1,\ell_i}$ linearly independent if and only if s_i is a zero of $|P(s)|$ with algebraic multiplicity $n_i \geq \sum_{j=1}^{\ell_i} k_i^j$ and geometric multiplicity $m - \text{rank } P(s_i) \geq \ell_i$.

Lemma 2 Theorem 1 is satisfied for $P(s)$ and $a_i^{k,j}$ if and only if it is satified for $U(s)P(s)$ and $a_i^{k,j}$ where $U(s)$ is any unimodular matrix.

Proof First note that P and UP have exactly the same zeros of determinant with the same algebraic and geometric multiplicities.

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

Assume that P and $a_i^{k,j}$ satisfy (1). Then $U(s_i)P(s_i)a_i^{1,j} = UP(s_i)a_i^{1,j} = 0$, $-(UP)^{(1)}(s_i)a_i^{1,j} = -[U^{(1)}(s_i)P(s_i) + U(s_i)P^{(1)}(s_i)]a_i^{1,j} = -U(s_i)P^{(1)}(s_i)a_i^{1,j} = U(s_i)P(s_i)a_i^{2,j} = UP(s_i)a_i^{2,j}$ etc.
That is UP and $a_i^{k,j}$ also satisfy (1). The sufficiency proof is similar since $U^{-1}(s)$ is also a unimodular matrix QED.

It is known [3] that given $P(s)$ there exists a unimodular matrix $U(s)$ such that UP is column proper i.e. $C_c[UP(s)]$, the matrix with entries the coefficients of the highest power of s in each column of UP is of full rank. It is therefore clear, in view of Lemma 2, that without loss of generality we can assume in the proof of Theorem 1 that $P(s)$ is a column proper matrix.

Proof of Theorem 1 Let $d_1, d_2, .., d_m$ denote the column degrees of (the comumn proper matrix) $P(s)$. Write

$$P(s) = B_m^{-1}[\mathrm{diag}\,(s^{d_i}) - A_m S(s)]$$

where $S(s) \triangleq \mathrm{diag}\,([1, s, .., s^{d_i-1}]^T)$ and A_m, B_m † appropriate real matrices and observe that

$$B_c P(s) = (s - A_c)S(s) \tag{2}$$

where A_c and B_c are given by (A6); the column degrees d_i are the controllability indices of (A_c, B_c) while the above defined A_m and B_m make up the "nontrivial" rows of A_c, B_c. Repeated differentiations of (2) give

$$B_c P^{(k)}(s) = (s - A_c)S^{(k)}(s) + k\, S^{(k-1)}(s) \qquad k=1,2,.., \tag{3}$$

---

† $B_m^{-1} = C_c[P(s)]$; without loss of generality it is assumed that $C_c[P(s)]$ is in upper triangular form with 1s on the diagonal.

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

Assume that vectors $a_i^{k,j}$ which satisfy (1) have been found. Premultiply the relations in (1) by B_c and use (2) and (3) to substitute $P(s)$ and its derivatives.

Then

$$\begin{aligned}(s_i - A_c)v_i^{1,j} &= 0\\ (s_i - A_c)v_i^{2,j} &= -v_i^{1,j}\\ &\vdots\\ (s_i - A_c)v_i^{k_i^j,j} &= -v_i^{k_i^j-1,j}\end{aligned} \tag{4}$$

where
$$\begin{aligned}v_i^{1,j} &= S(s_i)a_i^{1,j}\\ v_i^{2,j} &= [S(s_i)a_i^{2,j} + S^{(1)}(s_i)a_i^{1,j}]\\ &\vdots\\ v_i^{k_i^j,j} &= [S(s_i)a_i^{k_i^j,j} + \dots + \frac{1}{(k_i^j - 1)!} S^{(k_i^j-1)}(s_i)a_i^{1,j}]\end{aligned} \tag{5}$$

(4) implies that $(s_i - A_c)^{k_i^j} v_i^{k_i^j,j} = 0$ and $(s_i - A_c)^{k_i^j-1} v_i^{k_i^j,j} = v_i^{1,j} = S(s_i)a_i^{1,j} \neq 0$; furthermore $v_i^{1,1}, \dots, v_i^{1,\ell_i}$ are linearly independent since $a_i^{1,1}, \dots, a_i^{1,\ell_i}$ are linearly independent. In view of (A2) $v_i^{1,j}, \dots, v_i^{k_i^j,j}$ $j=1,2,..,\ell_i$ are ℓ_i chains of generalized eigenvectors of A_c corresponding to an eigenvalue s_i , each chain of length k_i^j $j=1,2,..,\ell_i$. Therefore $|s - A_c|$ (or $|P(s)|$ in view of (2) and the Appendix) has at least $\sum_{j=1}^{\ell_i} k_i^j$ zeros at s_i , which implies that the algebraic multiplicity $n_i \geq \sum_{j=1}^{\ell_i} k_i^j$; furthermore the geometric multiplicity of s_i is at least ℓ_i which implies that $\ell_i \leq n\text{-rank}(s_i - A_c)$ or, in view of the Appendix that $\ell_i \leq m\text{-rank}\, P(s_i)$.

Conversely, assume that s_i is a zero of $|P(s)|$ with algebraic and geometric multiplicities n_i and m-rank $P(s_i)$ respectively. The matrix A_c defined in (2) has an eigenvalue s_i with the same algebraic and geometric multiplicities. Out of the m-rank $P(s_i)$ chains of generalized eigenvectors of A_c which correspond to s_i, one can always choose ℓ_i ($\leq$ m-rank $P(s_i)$) distinct chains each of some length k_i^j (less than or equal to the actual lengths) with $\sum_{i=1}^{\ell_i} k_i^j \leq n_i$; call them $v_i^{1,j},\ldots,v_i^{k_i^j,j}$ $j=1,2,..,\ell_i$. Note that these eigenvectors satisfy (4); furthermore, because of the special structure of A_c and (2) it is straight forward to show that they are actually given by (5) where $a_i^{1,j},\ldots,a_i^{k_i^j,j}$ satisfy (1). Q.E.D.

<u>Corollary 3</u> There exist $(m \times 1)$ nonzero vectors $a_i^{1,j},\ldots,a_i^{k_i^j,j}$ $j=1,2,..,\ell_i$ which satisfy (1) with $a_i^{1,1},a_i^{1,2},\ldots,a_i^{1,\ell_i}$ linearly independent if and only if the Smith form of $P(s)$, $E_P(s)$, contains the factors $(s-s_i)^{k_i^j}$ $j=1,2,..,\ell_i$ in ℓ_i separate locations on the diagonal.

<u>Proof</u> In view of the proof of Theorem 1, (1) are satisfied iff A_c has a Jordan form of certain structure, or in view of the Appendix, iff $E_P(s)$ has the factors $(s-s_i)^{k_i^j}$ $j=1,2,..,\ell_i$ on the diagonal (see (A5),(A9) and (A11)). Q.E.D.

Theorem 1 implies that given $P(s)$ the maximum number of nonzero vectors $a_i^{k,j}$ which satisfy (1) is n_i , the algebraic multiplicity of s_i . If this maximum number of $a_i^{k,j}$ has been found,

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

then it is clear from the proof of Theorem 1 that ℓ_i will be equal to the geometric multiplicity m-rank $P(s_i)$ of s_i ; if it were less, then A_c would have its generalized eigenvectors corresponding to s_i distributed among less than m-rank $P(s_i)$ chains which is impossible. In this particular case, the numbers k_i^j $j=1,2,..,\ell_i$ are the lengths of the chains of the eigenvectors and appear as the exponents of the $(s-s_i)$ factors in ℓ_i locations in the Smith form of $P(s)$ (Corollary 3).

<u>Theorem 4</u> Let n be the degree of $|P(s)|$. There exist $n(m \times 1)$ nonzero vectors $a_i^{1,j}, a_i^{2,j}, \ldots, a_i^{k_i^j,j}$ $j=1,2,..,\ell_i$ $i=1,2,\ldots,\sigma$ $\left(\sum_{i=1}^{\sigma}\sum_{j=1}^{\ell_i} k_i^j = n\right)$ which satisfy (1) with $a_i^{1,1}, a_i^{1,2}, .., a_i^{1,\ell_i}$ linearly independent if and only if the zeros of $|P(s)|$ have σ distinct values s_i $i=1,2,..,\sigma$ each with algebraic multiplicity $n_i = \sum_{j=1}^{\ell_i} k_i^j$ and geometric multiplicity m-rank $P(s_i) = \ell_i$.

<u>Proof</u> If (1) is satisfied for an s_i , according to the necessity proof of Theorem 1, there exist at least $\sum_{j=1}^{\ell_i} k_i^j$ linearly independent generalized eigenvectors of A_c corresponding to s_i in at least ℓ_i distinct chains. Since (1) is satisfied for $i=1,2,..,\sigma$, there exist $\sum_{i=1}^{\sigma}\sum_{j=1}^{\ell_i} k_i^j = n$ linearly independent [1] generalized eigenvectors. This implies that A_c has exactly $\sum_{j=1}^{\ell_i} k_i^j$ eigenvectors corresponding to s_i distributed in exactly ℓ_i chains (if A_c had more generalized eigenvectors corresponding to s_i , then, since the total is n , $\sum_{j=1}^{\ell_i} k_i^j$ for some other i must have been larger than the corresponding n_i which is impossible by Theorem 1);

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

therefore the algebraic and geometric multiplicities of s_i are exactly $\sum_{j=1}^{\ell_i} k_i^j$ and ℓ_i respectively. Sufficiency can be easily shown in manner analoguous to the sufficiency proof of Theorem 1.

Q.E.D.

Corollary 5 Let n be the degree of $|P(s)|$. There exist $n(m \times 1)$ nonzero vectors $a_i^{1,j},\ldots,a_i^{k_i^j,j}$ $j=1,2,..,\ell_i$ $i=1,2,..,\sigma$ $\left(\sum_{i=1}^{\sigma}\sum_{j=1}^{\ell_i} k_i^j = n\right)$ which satisfy (1) with $a_i^{1,1},\ldots,a_i^{1,\ell_i}$ linearly independent if and only if the Smith form of $P(s)$, $E_P(s)$, consists of factors $(s-s_i)^{k_i^j}$ $j=1,2,..,\ell_i$ in ℓ_i locations on the diagonal $(i=1,2,..,\sigma)$.

Proof Clear in view of the proof of Theorem 4 and the Appendix (see (A5), (A9) and (A11)). Q.E.D.

Theorem 4 implies that given $P(s)$ the maximum number of nonzero vectors $a_i^{k,j}$ which satisfy (1) for all possible s_i is n, the degree of $|P(s)|$. If this maximum number of $a_i^{k,j}$ have been found then the algebraic and geometric multiplicities are determined as well as the distribution of the factors $(s-s_i)$ in $E_P(s)$. In particular, in view of the divisibility property of the invariant factors in the Smith form of $P(s)$, $E_P(s)$ is completely determined in this case as the following example shows.

EX

Let $P(s) = \begin{bmatrix} s^2 & -1 \\ 0 & s \end{bmatrix}$ $P^{(1)}(s) = \begin{bmatrix} 2s & 0 \\ 0 & 1 \end{bmatrix}$, $P^{(2)}(s) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

$P^{(3)}(s) = 0$. (1) implies $(s_1 = 0)$: $P(0)a_1^{1,j} = 0$, $a_1^{1,1} = \begin{bmatrix}\alpha\\0\end{bmatrix}$ $(\alpha \neq 0)$

$P(0)a_1^{2,1} = -P^{(1)}(0)a_1^{1,1}$, $a_1^{2,1} = \begin{bmatrix}\beta\\0\end{bmatrix}(\beta \neq 0)$;

$P(0)a_1^{3,1} = -[P^{(1)}(0)a_1^{2,1} + \frac{1}{2!}P^{(2)}(0)a_1^{1,1}]$, $a_1^{3,1} = \begin{bmatrix}\gamma\\\alpha\end{bmatrix}$.

We stop here since $a_1^{4,1}$ etc. are zero i.e. $k_1^1 = 3$. Note that $\ell_1 = 1 =$ m-rank $P(0)$; this was also seen in the solution of $P(0)a_1^{1,j} = 0$ where dim{Null space $P(0)$} $= 1$ ($=$ m-rank $P(0)$) which implies that there is only one vector $a_1^{1,j}$ i.e. $\ell_1 = 1$. Observe that $\sum_{j=1}^{\ell_i} k_i^j = 3 = n_1$ the algebraic multiplicity of $s_1 = 0$ and that the total number of $a_i^{k,j}$ is $3 = n$ the degree of $|P(s)|$. The Smith form of $P(s)$ is $E_P(s) = \begin{bmatrix}1 & 0\\0 & s^3\end{bmatrix}$ since $(s - s_1) = s$ appears in $\ell_1 = 1$ locations with exponent $k_1^1 = 3$. Finally note that the generalized eigenvectors of the corresponding $A_c = \begin{bmatrix}0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$ (see (2)) are $v_1^{1,1} = [\alpha,0,0]^T$, $v_1^{2,1} = [\beta,\alpha, 0]^T$ and $v_1^{3,1} = [\gamma,\beta,\alpha]^T$ (see (5)) .

In view of the above, given two polynomial matrices, (1) can be used to find the relation between their Smith forms. Note that the relations between polynomial matrices which satisfy (1) are examined in detail in the following.

Assume that $P(s)$ is a given $(m \times m)$ polynomial matrix and let n be the degree of $|P(s)|$. $|P(s)|$ has σ distinct zeros s_i with algebraic and geometric multiplicities n_i $(\sum_{i=1}^{\sigma} n_i = n)$ and

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

ℓ_i ($=$ m-rank $P(s_i)$) respectively. In view of Theorem 4, there exist n $(m \times 1)$ nonzero vectors $a_i^{1,j}, \ldots, a_i^{k_i^j,j}$ $j=1,2,\ldots,\ell_i$ $i=1,2,\ldots,\sigma$ which satisfy (1) with $a_i^{1,1}, \ldots, a_i^{1,\ell_i}$ linearly independent $\left(\sum_{j=1}^{\ell_i} k_i^j = n_i\right)$.

<u>Theorem 6</u> $P(s)$ is a right divisor (rd) of a $(r \times m)$ polynomial matrix $M(s)$ if and only if $M(s)$ satisfies (1) with the same s_i and $a_i^{k,j}$.

<u>Proof</u> $P(s)$ is a rd of $M(s)$ iff $P(s)$ is a greatest common right divisor (gcrd) of $P(s)$ and $M(s)$ or iff there exists a unimodular matrix U such that $U\begin{bmatrix} P \\ M \end{bmatrix} = \begin{bmatrix} P \\ 0 \end{bmatrix}$ [3].

Assume that such U exists i.e. P is a rd of M. Since P satisfies (1), $U\begin{bmatrix} P \\ M \end{bmatrix}$ also satisfies (1) which implies that $\begin{bmatrix} P \\ M \end{bmatrix}$ satisfies (1) because a premultiplication by a unimodular matrix does not affect these relations (see Lemma 2). Therefore M satisfies (1) with the same s_i and $a_i^{k,j}$.

Assume now that M satisfies (1). Let G be a gcrd of M and P i.e. $\hat{U}\begin{bmatrix} P \\ M \end{bmatrix} = \begin{bmatrix} G \\ 0 \end{bmatrix}$ with $\hat{U}$ a unimodular matrix. This implies that G satisfies the same n relations as M; the degree of $|G|$ is therefore at least n in view of Theorems 1 and 4. Note however that since G is a rd of P, $P = \tilde{P}G$ which implies that $\tilde{P}$ is a unimodular matrix. Therefore $M = \tilde{M}G = (\tilde{M}\tilde{P}^{-1})P$ and P is a rd of M. Q.E.D.

If $M(s)$ is a square matrix, the above proof can be used to show the following.

Corollary 7 $M(s) = U(s)P(s)$ with $U(s)$ unimodular if and only if the degree of $|M(s)|$ is n and $M(s)$ satisfies (1) with the same s_i and $a_i^{k,j}$.

An extension of Corollary 7, which gives insight into the relation between the vectors $a_i^{k,j}$ and the structure of $P(s)$, is the follwing lemma.

Consider the matrix $P(s)$ of Theorem 6 together with the corresponding s_i and $a_i^{k,j}$ which satisfy (1). Then

Lemma 8 Given an $(m \times m)$ polynomial matrix $\bar{P}(s)$, there exist unimodular matrices $U_1(s)$, $U_2(s)$ such that

$$P(s) = U_1(s)\bar{P}(s)U_2(s)$$

if and only if the degree of $|\bar{P}(s)|$ is n and there exist n $(m \times 1)$ nonzero vectors $\bar{a}_i^{k,j}$ which satisfy together with $\bar{P}(s)$ and s_i the same relations (1) satisfied by $a_i^{k,j}$, $P(s)$ and s_i. Furthermore if such U_1, U_2 exist

$$\begin{aligned}
\bar{a}_i^{1,j} &= U_2(s_i)a_i^{1,j} \\
\bar{a}_i^{2,j} &= U_2(s_i)a_i^{2,j} + U_2^{(1)}(s_i)a_i^{1,j} \\
&\vdots \\
\bar{a}_i^{k_i^j,j} &= U_2(s_i)a_i^{k_i^j,j} + \dots + \frac{1}{(k_i^j - 1)!} U_2^{(k_i^j - 1)}(s_i)a_i^{1,j}
\end{aligned} \tag{6}$$

<u>Proof</u> Assume that there exist U_1, U_2 such that $P = U_1 \bar{P} U_2$ with P, s_i and $a_i^{k,j}$ satisfying (1). Substitute P by $\bar{P} U_2$ in the relations since U_1 cancels out (see Lemma 2). Straight calculations show that vectors $\bar{a}_i^{k,j}$ given by (6) together with $\bar{P}$ and s_i satisfy the same relations (1).

Assume now that the degree of $|\bar{P}|$ is n and $\bar{P}$, $\bar{a}_i^{k,j}$ and s_i satisfy the same n relations (1), P, $a_i^{k,j}$ and s_i satisfy. In view of Corollary 5 and the remark following the corollary, P and $\bar{P}$ have exactly the same Smith forms; therefore, there exist unimodular matrices $\tilde{U}_1, \tilde{U}_2$ [2] such that $P = \tilde{U}_1 \bar{P} \tilde{U}_2$. Q.E.D.

Note that Lemma 8 applies to nonsquare matrices as well if n is taken to be the degree of the greatest common divisor of all highest order minors; e.g. given R, Q, $RU_3 = [\tilde{R}, 0]$, $QU_4 = [\tilde{Q}, 0]$; then $\tilde{R} = U_1 \tilde{Q} U_2$ (Lemma 8 applies here) and

$$RU_3 = U_1[\tilde{Q}, 0] \begin{bmatrix} U_2 & 0 \\ 0 & I \end{bmatrix} = U_1 Q \, U_4 \begin{bmatrix} U_2 & 0 \\ 0 & I \end{bmatrix} \quad (U_i \text{ unimodular}).$$

<u>Remark</u> If P, $a_i^{k,j}$, s_i and $\bar{P}$, $\bar{a}_i^{k,j}$, s_i satisfy the same n relations (1) where, in addition, $a_i^{k,j}$ and $\bar{a}_i^{k,j}$ satisfy (6) for some unimodular matrix $U_2(s)$, then P and $\bar{P}$ not only have the same Smith form, as it was shown in the sufficiency proof of Lemma 8, but they are related by: $P = \tilde{U}_1 \bar{P} U_2$. This is shown as follows: Assume that $P = \tilde{U}_1 \bar{P} \tilde{U}_2$ where $\tilde{U}_2 \neq U_2$. Lemma 8 implies that $a_i^{k,j}$, $\bar{a}_i^{k,j}$ and $\tilde{U}_2$ will also satisfy relations similar to (6). Equating $\bar{a}_i^{k,j}$ in the relations involving U_2 and $\tilde{U}_2$, we derive

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

n relations of type (1) satisfied by $a_i^{k,j}$, s_i and the polynomial matrix $U_2 - \tilde{U}_2$. According to Theorem 4, $|U_2 - \tilde{U}_2|$ must be of at least degree n which is false. Therefore $U_2 = \tilde{U}_2$. Note that this result agrees with Corollary 7 since if $U_2 = I$ i.e. $\bar{a}_i^{k,j} = a_i^{k,j}$, P and $\bar{P}$ are related by $P = U\bar{P}$.

Assume that, for an $(m \times m)$ polynomial matrix P(s) yet to be chosen, we have decided upon the degree of $|P(s)|$ as well as its zeros i.e. n, s_i and the algebraic multiplicities n_i . Clearly there are many matrices which satisfy these requirements e.g. diagonal matrices. If we specify the geometric multiplicities ℓ_i , our matrix has more structure e.g. the factors $(s - s_i)$ are appropriately distributed in our diagonal matrix. If k_i^j are also chosen, then, the Smith form of P(s) is completely defined (see Appendix) i.e. P(s) is defined within pre and post unimodular multiplication. This is equivalent to imposing the restriction that P(s) has to specify n relations of type (1) (P(s) of Theorem 6) without though restricting $a_i^{k,j}$ (other than being nonzero and $a_i^{1,1}, \ldots, a_i^{1,\ell_i}$ being linearly independent). If $a_i^{k,j}$ are also specified then P(s) is determined within a unimodular premultiplication (Corollary 7) .

If an $(r \times m)$ polynomial matrix M(s) satisfies n relations of type (1) (M(s) of Theorem 6) then, there exists an $(m \times m)$ polynomial matrix P(s) with specified structure which is a rd of M(s) i.e. $M = \tilde{M} P$. This is because in view of the above, there exists a matrix P(s) with degree of $|P(s)|$ equal to n which satisfies exactly the same n relations with the same $a_i^{k,j}$ and s_i (this $P(s)$ is specified within a unimodular premultiplication); in view of Theorem 6 this $P(s)$ is a rd of $M(s)$. If in the n relations which are satisfied by $M(s)$, the vectors $a_i^{k,j}$ are not specified, then the Smith form of $M(s)$ has factors $(s-s_i)^{k_i^j}$ in ℓ_i locations on the diagonal. In other words a rd $P(s)$ is not specified in this case, but it is only known that there exists a rd of the form $E_P U_2$ where E_P is a completely specified Smith form (it consists of the $(s-s_i)^{k_i^j}$) and U_2 an arbitrary unimodular matrix.

In view of the above, if it is known that an $(m \times m)$ polynomial matrix $P(s)$ satisfies n relations of type (1) ($P(s)$ of Theorem 6) i.e. $n, s_i, n_i, k_i^j, \ell_i, a_i^{k,j}$ are given, then $P(s)$ is specified within a unimodular premultiplication. In the following, some methods are outlined which can be used to determine an appropriate $P(s)$ matrix.†

(a) Let $[a_0, a_1, \ldots, a_r] \begin{bmatrix} I_m \\ I_m s \\ \vdots \\ I_m s^r \end{bmatrix} = P(s)$ where a_k are $m \times m$

matrices to be determined. If the n given relations (1) are written in terms of a_i, a linear system of equations is obtained

---

† These methods can be used to arbitrarily assign the poles of a system via feedback compensation; a detailed account of these techniques will be given in a future publication. Note that special cases have already appeared in [5] and [6].

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

with a_i as the unknowns. In order to have more unknowns than equations $m(r+1) \geq n$ i.e. $r \geq \frac{n}{m} - 1$ e.g. For s_i all distinct, solve $[a_o,\ldots,a_r] \begin{bmatrix} I_m \\ I_m s_i \\ \vdots \\ I_m s_i^r \end{bmatrix} \cdot a_i^{1,1} = 0 \quad i=1,2,\ldots,n$.

(b) Let $\text{diag}(s^{d_i}) - A_m S(s) = P(s)$ where $S(s) = \text{diag}\,[(1,s,\ldots,s^{d_i-1})^T]$ and A_m an $m \times n$ matrix to be determined. d_i are the column degrees of the (column proper) $P(s)$ ($\Sigma d_i = n$). The n relations (1) are written in terms of A_m and a linear system of equations is obtained with A_m as unknown. Note however that d_i are not completely free; they must satisfy certain inequalities involving k_i^j [2] .

(c) One can also use the Smith form of $P(s)$ (specified by n, s_i, n_i, k_i^j and ℓ_i) and the relations (6) of Lemma 8.

Assume that $P(s)$ is associated with a polynomial matrix description of a linear, time-invariant system; that is $P(D)z(t) = Q(D)u(t)$, $y(t) = R(D)z(t) + W(D)u(t)$ where u, y and z are the input, output and partial state respectively. Clearly in this case the roots s_i of $|P(s)|$ are the poles of the system; it will be shown that the nonzero vectors $a_i^{k,j}$ are closely related to the eigenvectors of an equivalent to $\{P,Q,R,W\}$ state-space description $\{A,B,C,E\}$ of the given system.

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

Def [4] $\{P_1, Q_1, R_1, W_1\}$ and $\{P_2, Q_2, R_2, W_2\}$ are equivalent iff there exist M_1, M_2, X_2, Y_1 polynomial matrices such that

$$\begin{bmatrix} M_2 & 0 \\ X_2 & I \end{bmatrix} \cdot \begin{bmatrix} P_1 & Q_1 \\ -R_1 & W_1 \end{bmatrix} = \begin{bmatrix} P_2 & Q_2 \\ -R_2 & W_2 \end{bmatrix} \cdot \begin{bmatrix} M_1 & -Y_1 \\ 0 & I \end{bmatrix} \tag{7}$$

with (M_2, P_2) left prime and (P_1, M_1) right prime.

From (7), $M_2P_1 = P_2M_1$. If $P_1 = s - A$, $P_2 = P$ then

$$M_2(s)(s - A) = P(s)M_1(s) \tag{8a}$$

with (M_2, P) left prime and $(s - A, M_1)$ right prime.

If $P_1 = P$, $P_2 = s - A$, then

$$\bar{M}_2(s)P(s) = (s - A)\bar{M}_1(s) \tag{8b}$$

with $(\bar{M}_2, s - A)$ left prime and $(P, \bar{M}_1)$ right prime. Note that a special case of (8b) is (2) where $\bar{M}_2 = B_c$, $s - A = s - A_c$, $\bar{M}_1 = S(s)$.

Assume that $P(s)$ satisfies (1) together with s_i and $a_i^{k,j}$. Then $\bar{M}_2P$ also satisfies (1) with the same s_i and $a_i^{k,j}$. This can be shown as follows: $P(s_i)a_i^{1,j} = 0$ implies that $\bar{M}_2P(s_i)a_i^{1,j} = 0$; $[\bar{M}_2P(s_i)]^{(1)}a_i^{1,j} = \bar{M}_2^{(1)}(s_i)P(s_i)a_i^{1,j} + \bar{M}_2(s_i)P^{(1)}(s_i)a_i^{1,j} = \bar{M}_2(s_i)P^{(1)}(s_i)a_i^{1,j} = -\bar{M}_2(s_i)P(s_i)a_i^{2,j} = -[\bar{M}_2P(s_i)]a_i^{2,j}$ etc. Substitute now in all the relations (1) $\bar{M}_2P$ by $(s - A)\bar{M}_1$ (see (8b)). Then

$$\begin{aligned} (s_i - A)\, v_i^{1,j} &= 0 \\ (s_i - A)\, v_i^{2,j} &= -v_i^{1,j} \\ &\vdots \\ (s_i - A)\, v_i^{k_i^j,j} &= -v_i^{k_i^j-1,j} \end{aligned} \tag{9b}$$

where

$$\begin{aligned} v_i^{1,j} &= \bar{M}_1(s_i)a_i^{1,j} \\ v_i^{2,j} &= [\bar{M}_1(s_i)a_i^{2,j} + \bar{M}_1^{(1)}(s_i)a_i^{1,j}] \\ &\vdots \\ v_i^{k_i^j,j} &= [\bar{M}_1(s_i)a_i^{k_i^j,j} + \ldots + \frac{1}{(k_i^j-1)!}\bar{M}_1^{(k_i^j-1)}(s_i)a_i^{1,j}] \end{aligned} \tag{10b}$$

which implies that $v_i^{k,j}$ are the generalized eigenvectors of A corresponding to the eigenvalue s_i (compare with (4) and (5)).

That is, the vectors $a_i^{k,j}$ which satisfy (1) with $P(s)$ and s_i, determine the generalized eigenvectors $v_i^{k,j}$ of A of the equivalent state-space description via (10b).

Assume that $v_i^{k,j}$ are the generalized eigenvectors of A corresponding to the eigenvalue s_i i.e. they satisfy relations (9b). Then relation (8a) implies that there exist $a_i^{k,j}$ such that $a_i^{k,j}$, s_i together with $P(s)$ satisfy (1) where $a_i^{k,j}$ are given by

$$\begin{aligned} a_i^{1,j} &= M_1(s_i)\, v_i^{1,j} \\ a_i^{2,j} &= [M_1(s_i)\, v_i^{2,j} + M_1^{(1)}(s_i)\, v_i^{1,j}] \\ &\vdots \\ a_i^{k_i^j,j} &= [M_1(s_i)\, v_i^{k_i^j,j} + \frac{1}{(k_i^j-1)!} M_1^{(k_i^j-1)}(s_i)v_i^{1,j}] \end{aligned} \tag{10a}$$

This can be shown as follows:

$(s_i - A)\, v_i^{1,j} = 0$ implies that $P(s_i)M_1(s_i)\, v_i^{1,j} = 0$; let $a_i^{1,j} = M_1(s_i)\, v_i^{1,j}$. Differentiate (8a) and postmultiply by $v_i^{1,j}$:

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

$$M_2^{(1)}(s_i)(s_i - A)\ v_i^{1,j} + M_2(s_i)\ v_i^{1,j} = M_2(s_i)\ v_i^{1,j} = - M_2(s_i)(s_i - A)\ v_i^{2,j}$$

$$= - P(s_i)M_1(s_i)\ v_i^{2,j} = P^{(1)}(s_i)M_1(s_i)\ v_i^{1,j} + P(s_i)M_1^{(1)}(s_i)\ v_i^{1,j}$$

from which $P(s_i)\ [M_1(s_i)v_i^{2,j} + M_1^{(1)}(s_i)v_i^{1,j}] = - P^{(1)}(s_i)[M_1(s_i)v_i^{1,j}]$;

let $a_i^{2,j} = M_1(s_i)v_i^{2,j} + M_1^{(1)}(s_i)v_i^{1,j}$ etc.

That is, the generalized eigenvectors $v_i^{k,j}$ of A which satisfy (9b) determine vectors $a_i^{k,j}$ via (10a) which satisfy (1) with s_i and $P(s)$ where $P(s)$ is the corresponding to A matrix (see (8a)) of an equivalent polynomial matrix description.

Remark The above analysis can be used in the feedback compensation of systems described by polynomial matrices, not only to assign the closed loop poles but also the closed loop eigenvectors.

Finally note that all the results in this paper reduce to well known results when special cases are considered e.g. $P(s) = p(s)$ a polynomial $(m = 1)$ and $P(s) = sI - A$ $(m = n)$.

Conclusion

In this report several basic theorems were given, which establish the relations between a polynomial matrix $P(s)$ and its "characteristic" vectors $a_i^{k,j}$ and "characteristic" values s_i . This account is by no means complete. Extensions together with applications to control problems will be given in a future publication.

References:

[1] C.T. Chen, Introduction to Linear System Theory, Holt, Rinehart and Winston Inc., New York 1970.

[2] H.H. Rosenbrock, State-Space and Multivariable Theory, Nelson, London, 1970.

[3] W.A. Wolovich, Linear Multivariable Systems, Springer-Verlag, New York, 1974.

[4] P.A. Fuhrmann, "On Strict System Equivalence and Similarity", Int. J. Control, vol.25, no.1., pp 5-10, 1977.

[5] P.J. Antsaklis, Some New Matrix Methods Applied to Multivariable System Analysis and Design, PhD Thesis, Division of Engr., Brown Univ., 1976 .

[6] P.J. Antsaklis and W.A. Wolovich, "Arbitrary Pole Placement using Linear Output Feedback Compensation", Int. J. Control, vol.25, no.6, pp 915-925, 1977.

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

APPENDIX

Canonical Forms (Jordan, Smith and Controllable companion forms).

Let $(n \times n)A$ have σ distinct eigenvalues s_i each repeated n_i times $(\sum_{i=1}^{\sigma} n_i = n)$; n_i is the algebraic multiplicity of s_i. The geometric multiplicity of s_i, ℓ_i, is defined as $\ell_i = n - \text{rank}(s_i - A)$ i.e. the reduction in rank in $s-A$ when $s=s_i$.

There exists a similarity transformation matrix Q such that

$$AQ = QJ$$

where J is the Jordan canonical form of A.

$$J = \begin{bmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & \ddots A_\sigma \end{bmatrix}, \quad A_i = \begin{bmatrix} A_{i1} & & 0 \\ & A_{i2} & \\ 0 & & \ddots A_{i\ell_i} \end{bmatrix} \tag{A1}$$

where A_i $i=1,\ldots\sigma$ is an $(n_i \times n_i)$ matrix with eigenvalues s_i; A_i is a block diagonal matrix with ℓ_i $(n_i \geq \ell_i)$ matrices A_{ij} $j=1,..,\ell_i$ on the diagonal, of the form

$$(k_i^j \times k_i^j) \quad A_{ij} = \begin{bmatrix} s_i & 1 & 0 \ldots 0 \\ 0 & s_i & 1 \ldots 0 \\ \vdots & & \ddots \quad \vdots \\ \vdots & & \ddots \quad 1 \\ 0 & 0 & \ldots \quad s_i \end{bmatrix} \quad \left(\sum_{j=1}^{\ell_i} k_i^j = n_i\right).$$

The structure of the Jordan canonical form of A is determined by the generalized eigenvalues of A (they are used to construct Q). To each eigenvalue s_i correspond ℓ_i chains of generalized eigenvectors each chain of length k_i^j i.e. $v_i^{1,j}, v_i^{2,j}, \ldots, v_i^{k_i^j,j}$ $j=1,\ldots,\ell_i$ a total of n_i linearly independent generalized eigenvectors. The eigenvectors of a particular chain can be determined from

$$\begin{aligned}(s_i - A)v_i^{1,j} &= 0\\ (s_i - A)v_i^{2,j} &= -v_i^{1,j}\\ &\vdots\\ (s_i - A)v_i^{k_i^j,j} &= -v_i^{k_i^j-1,j}\end{aligned} \tag{A2}$$

where $(s_i - A)^{k_i^j} v_i^{k_i^j,j} = 0$ and $(s_i - A)^{k_i^j-1} v_i^{k_i^j,j} \neq 0$. [1].

If $\bar{n}_i \triangleq \max_j k_i^j$ (the dimension of the largest block associated with s_i) then

Minimal Polynomial $\Delta_m(s) = \prod_{i=1}^{\sigma} (s-s_i)^{\bar{n}_i}$ while the

Characteristic Polynomial $\Delta(s) = \prod_{i=1}^{\sigma} (s-s_i)^{n_i}$ ($= |s-A|$).

Note that A is cyclic iff there exists a vector b such that $\text{rank}\,[b,Ab,\ldots,A^{n-1}b] = n$

$\Leftrightarrow$ $\Delta(s) = \Delta_m(s)$ $\Leftrightarrow$ $n_i = \bar{n}_i$ $i=1,\ldots,\sigma$ i.e. only one block is associated with each distinct eigenvalue

$\Leftrightarrow$ $\ell_i = 1$ $i=1,\ldots,\sigma$ i.e. only one chain of generalized eigenvectors is associated with each distinct egenvalue. (A3)

$\Leftrightarrow$ $\text{rank}(s_i - A) = n-1$.

There exist unimodular matrices $U_1(s), U_2(s)$ such that

$$U_1(s)(s-A)U_2(s) = E_A(s)$$

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

are the dimensions of the matrices A_{ij} $j=1,..,\ell_i$ of the Jordan canonical form or equivalently, they are the lengths of the chains of the generalized eigenvectors corresponding to s_i.

In view of the divisibility property of $\varepsilon_i(s)$ it is therefore clear that if s_i and the dimensions of the submatrices of J are known, the Smith form of $s-A$ is uniquely determined. Furthermore note that the characteristic polynomial of $s-A$, is $\Delta(s) = \varepsilon_1(s)\,\varepsilon_2(s)\,\ldots\,\varepsilon_n(s)$, the minimal polynomial is $\Delta_m(s) = \varepsilon_n(s)$ and A is cyclic if $(s-s_i)$ is a factor only of $\varepsilon_n(s)$.

Ex

Let
$$J = \left[\begin{array}{c|c} A_1 & \\ \hline & A_2 \end{array}\right] = \left[\begin{array}{cc|c} A_{11} & & \\ & A_{12} & \\ \hline & & A_{21} \end{array}\right] = \left[\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

that is
$$\left\{\begin{array}{l} s_1 = -3 \;,\quad n_1 = 3 \;,\quad \ell_1 = 2 \\ k_1^1 = 2 \;,\quad k_1^2 = 1 \;\text{ and }\; s_2 = -1 \;,\quad n_2=\ell_2=k_2^1=1 \end{array}\right\}$$

Then the Smith form is

$$E_A(s) = \begin{bmatrix} \varepsilon_1(s) & 0 & 0 & 0 \\ 0 & \varepsilon_2(s) & 0 & 0 \\ 0 & 0 & \varepsilon_3(s) & 0 \\ 0 & 0 & 0 & \varepsilon_4(s) \end{bmatrix}$$

where
$$\varepsilon_1(s) = \varepsilon_2(s) = 1$$
$$\varepsilon_3(s) = (s-3)$$
$$\varepsilon_4(s) = (s-3)^2(s-1)$$

Clearly $\Delta(s) = (s-3)^3(s-1)$ and $\Delta_m(s) = (s-3)^2(s-1)$.

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

Assume that (A,B), where $(n \times m)B$ has full rank $m(\leq n)$ is a controllable pair. There exists an equivalence transformation matrix Q such that

$$AQ = QA_c \quad , \qquad B = QB_c$$

with (A_c, B_c) in controllable companion form,

$$A_c = [A_{ij}] \qquad (d_i \times d_j)A_{ij} \begin{cases} \begin{bmatrix} & 0 & \\ x & x \ \ldots & x \end{bmatrix} & \text{for } i \neq j \\[2ex] \begin{bmatrix} 0 & I_{d_i-1} \\ \vdots & \\ 0 & \\ x & x \ \ldots \ x \end{bmatrix} & \text{for } i = j \end{cases} \tag{A6}$$

i and $j=1,2,..,m$

$$B_c = [\, B_i \,] \quad i=1,2,..,m \qquad (d_i \times m)B_i = \begin{array}{c} \begin{bmatrix} & & 0 & & \\ 0 & \ldots & 0 \; 1 & x \ldots & x \end{bmatrix} \\ \uparrow \ \text{ith column} \end{array}$$

$d_i \quad i=1,2,..,m$ are the controllability indices of (A,B). The m nontrivial $(\sum_{k=1}^{j} d_k)$th $j=1,2,..,m$ rows of A_c and B_c define the matrices $(m \times n)A_m$ and $(m \times m)B_m$ respectively where B_m is in upper triangular form with 1s on the diagonal [3] .

The structure of A_c implies that $\text{rank}(s_i - A) = \text{rank}(s_i - A_c) \geq n-m$ for any s_i, which in turn implies that the geometric multiplicity ℓ_i of s_i satisfies

$$\ell_i (\triangleq n - \text{rank}(s_i - A)) \leq m \ . \quad i=1,2,..,\sigma \ . \tag{A7}$$

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

There exist two polynomial matrices $(n \times m)S(s)$ and $(m \times m)P(s)$, closely related to the controllable pair (A_c, B_c), which satisfy the identity

$$B_c P(s) = (s - A_c)S(s) \tag{A8}$$

These matrices are defined by:

$$S(s) = \mathrm{diag}([1, s, \ldots, s^{d_i - 1}]^T), \quad P(s) \triangleq B_m^{-1}[\mathrm{diag}(s^{di}) - A_m S(s)] [3]$$

It can be known that the system matrices

$$\left[\begin{array}{c|c} s - A_c & B_c \\ \hline -I_n & 0 \end{array}\right] \quad \text{and}$$

$$\left[\begin{array}{cc|c} I_{n-m} & 0 & 0 \\ 0 & P(s) & I_m \\ \hline 0 & -S(s) & 0 \end{array}\right] \quad \text{are unimodularly equivalent [2]}^{\dagger};$$

this implies that the Smith forms $E_A(s)$ $E_P(s)$ of $s - A_c$ (or $s - A$) and $P(s)$ respectively satisfy the relation:

$$E_A(s) = \begin{bmatrix} I_{n-m} & 0 \\ 0 & E_P(s) \end{bmatrix} \tag{A9}$$

$\dagger$ therefore the system representations $\dot{x} = A_c x + B_c u$, $y = x$ and $Pz = u$, $y = Sz$ are equivalent.

P. J. Antsaklis, "Polynomial Matrix Characterization Using Characteristic Values and Vectors," Publication No. 80/18, Dept. of Electrical Engineering, Imperial College, July 1980.

It is now clear that $|s-A| = |P(s)|$ which implies that $|P(s)|$ has σ distinct roots s_i each repeated n_i times (n_i is the <u>algebraic multiplicity</u> of s_i). Furthermore the <u>geometric multiplicity</u> of s_i, ℓ_i, is given by $\ell_i = m\text{-rank } P(s_i)^{\dagger}$ since $\ell_i = n\text{-rank } E_A(s_i) = n - [(n-m) + \text{rank } E_P(s_i)] = m\text{-rank } E_P(s_i)$.

If

$$E_P(s) = \begin{bmatrix} \bar{\varepsilon}_1(s) & & \\ & \ddots & \\ & & \bar{\varepsilon}_m(s) \end{bmatrix} \tag{A10}$$

where $\bar{\varepsilon}_k$ divides $\bar{\varepsilon}_{k+1}$ $k=1,2,..,m-1$ then assuming that

$$k_i^1 \le k_i^2 \le \dots \le k_i^{\ell_i} (=\bar{n}_i) \quad \text{we have}$$

$$\begin{aligned} \bar{\varepsilon}_m(s) &= (s-s_i)^{k_i^{\ell_i}} \; (\; . \;) \\ \bar{\varepsilon}_{m-1}(s) & \;\; (s-s_i)^{k_i^{\ell_i-1}} \; (\; . \;) \\ &\;\;\vdots \\ \bar{\varepsilon}_{m-(\ell_i-1)}(s) &= (s-s_i)^{k_i^1} \; (\; . \;) \end{aligned} \tag{A11}$$

which are completely analoguous to (A5).

The <u>characteristic polynomial</u> of $P(s)$ is $\Delta(s) = |P(s)| = \prod_{i=1}^{\sigma} (s-s_i)^{n_i}$ while the <u>minimal polynomial</u> is

$$\Delta_m(s) = \prod_{i=1}^{\sigma} (s-s_i)^{\bar{n}_i}$$

---

† It has been shown that $\ell_i \le m$.

P(s) is _cyclic_ (or _simple_) iff there exists a vector g so that P(s),g are relatively left prime

$$\Leftrightarrow \quad \Delta(s) = \Delta_m(s) \qquad (n_i = \bar{n}_i \quad i=1,2,..,\sigma) \tag{A12}$$

$$\Leftrightarrow \quad \ell_i = \text{m-rank } P(s_i) = 1 \quad i=1,2,..,\sigma \; .$$