was sufficiently large. A similar design procedure for discrete-time systems is developed in [7]. Finally, the decomposition is applicable to design criteria other than eigenvalue locations and output feedback problems.

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On the Dimensions of the Supremal (A, B)-Invariant and Controllability Subspaces

P. J. ANTSAKLIS AND T. W. C. WILLIAMS

***Abstract*—The dimensions of the supremal output-nulling (A, B)-invariant and controllability subspaces of a system are explicitly determined. The role of the input and output decoupling zeros is also discussed.**

Introduction

Among the key concepts of the geometric approach [3] are the concepts of the supremal output-nulling (A, B)-invariant and controllability subspaces [2] V^* and R^*, respectively. Their dimensions have been explicitly determined in [1] to be functions of the order n of the system, the number of zeros q and d_T, and the degree of the determinant of a polynomial matrix derived from the transfer matrix $T(s)$. The assumptions made in [1] of the system being completely controllable and $T(s)$ having full rank are relaxed here and it is shown that the same formulas are valid in the general case (Theorem 1 and Lemma 4). This is done by establishing the relation between R^* and the supremal controllability subspace of: 1) a controllable subsystem (Lemma 2), and 2) a subsystem with a transfer matrix of full rank (Lemma 4). In addition, the role played by the input and output decoupling zeros is discussed (Corollary 3) and their importance to the output stabilization problem is indicated.

Main Results

Let S denote the system $\dot{x}=Ax+Bu$; $y=Cx+Eu$ where $A\in R^{n\times n}$, $B\in R^{n\times m}$, $C\in R^{p\times n}$, and $E\in R^{p\times m}$. Let $T(s)=C(sI-A)^{-1}B+E$ be its transfer matrix which is assumed to be of full rank, i.e., rank $T(s)=\min(p,m)$.

A subspace V of the state space is an output-nulling invariant subspace [2] if and only if

$$\begin{pmatrix} A \\ C \end{pmatrix} V \subset \begin{pmatrix} I \\ 0 \end{pmatrix} V + \operatorname{Im}\begin{pmatrix} B \\ E \end{pmatrix}$$

or, equivalently, if and only if for some F

$$(A+BF)V\subset V\subset \ker(C+EF) \tag{1}$$

where Im and ker are the image and the kernel of linear maps. Let V^* be the supremal output-nulling invariant subspace. The supremal output-nulling controllability subspace R^* is defined by [2][1]

$$R^*=\langle A+BF^*|\hat{\mathfrak{B}}\cap V^*\rangle \tag{2}$$

where V^* and F^* satisfy (1) and $\hat{\mathfrak{B}}$ is the range of the map B restricted to $\ker E$, i.e., $\hat{B}$ is spanned by the columns of BG_1 where the columns of G_1 span $\ker E$. If $E=0$, V^* and R^* are the supremal (A, B)-invariant and controllability subspaces in $\ker C$. In the following, V^* and R^* will be referred to as the *supremal (A, B)-invariant and controllability subspaces*, respectively.

If the given system S is completely controllable, then the dimensions of V^* and R^* can be explicitly determined [1] as functions of n, the order of the system; q, the number of zeros; and d_T, the degree of the determinant of a polynomial matrix $X(s)$; these quantities are defined below for completeness.

The q zeros of S (also called invariant zeros) are those z_i (multiplicity included) for which

$$\operatorname{rank}\begin{pmatrix} z_i-A & B \\ -C & E \end{pmatrix} < n+\operatorname{rank} T(s)=n+\min(p,m). \tag{3}$$

$X(s)$ is a $p\times p$ polynomial matrix such that $X(s)T(s)$ has a proper right inverse ($p\leqslant m$), i.e.,

$$\lim_{s\to\infty} X(s)T(s)=K_T:\ \operatorname{rank} K_T=p. \tag{4}$$

Then $d_T \triangleq$ degree (det $X(s)$). Note that in [1] $X(s)$ was taken to be the "interactor," a matrix with special structure satisfying (4) and uniquely defined via a known algorithm. It should be noticed that if S can be decoupled via linear state feedback, then d_T is the sum of the "decoupling indices" of S [1]. The main theorem of this paper can now be stated.

Theorem 1: $\dim V^*=q+\dim R^*$ where $\dim R^*$ is: 1) zero when $p>m$, and 2) $n-q-d_T$ when $p<m$.[2]

Proof: If the system S is completely controllable, then the theorem is true as has been shown in [1] (see Theorem 11 and remarks preceding the example). Before the general case can be shown to be true, several properties must be proved.

Assume that the system S is not completely controllable. Then its state-space representation can be reduced to

$$\left(\begin{bmatrix} A_c & A_{c\bar{c}} \\ 0 & A_{\bar{c}} \end{bmatrix}, \begin{bmatrix} B_c \\ 0 \end{bmatrix}, [C_c, C_{\bar{c}}], E\right) \tag{5}$$

where the system S_c: $\{A_c, B_c, C_c, E\}$ is completely controllable.

Let V_c^* and R_c^* denote the supremal (A, B)-invariant and controllability subspaces of S_c.

Lemma 2: $\dim R^*=\dim R_c^*$.

Proof: Equation (2) can be written as $R^*=\langle A+BF^*|\hat{\mathfrak{B}}\cap V^*\cap \langle A|\mathfrak{B}\rangle\rangle$. Note that $V^*\cap\langle A|\mathfrak{B}\rangle=\operatorname{span}\begin{pmatrix} v_c^* \\ 0\end{pmatrix}$ with span $v_c^*=V_c^*$ as can be easily seen by restricting the system to the controllable subspace and using (5). Now $R_c^*=\langle A_c+B_cF_c^*|\hat{\mathfrak{B}}_c\cap V_c^*\rangle$ which implies that $\mathfrak{R}^*=\operatorname{span}\begin{pmatrix} r_c^* \\ 0\end{pmatrix}$ with span $r_c^*=R_c^*$; therefore, $\dim R^*=\dim R_c^*$. Q.E.D.

The system S_c is completely controllable; therefore, $\dim R_c^*(=\dim R^*)$ can be expressed [1] as a function of the order n_c of $S_c(A_c\in\mathfrak{R}^{n_c\times n_c})$, q_c, the number of invariant zeros of S_c, and d_T which is derived from the transfer matrix $T(s)$ (S and S_c have the same transfer matrix). In order to complete the proof of the theorem, the relations between n, q and n_c, q_c must be established. The order and the zeros of S and S_c are related via the input decoupling zeros defined below.

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[1] $\langle A|\mathfrak{B}\rangle \triangleq \mathfrak{B}+A\mathfrak{B}+\cdots+A^{n-1}\mathfrak{B}$.

[2] Note that $n-q-d_T=0$ when $p=m$.

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The l *input-decoupling zeros* (i.d.z.) of S are those z_i (multiplicity included) for which $\text{rank}[z_i-A, B]<n$ [6]. They correspond to the l uncontrollable eigenvalues of S. It is therefore clear that

$$n=n_c+l. \tag{6}$$

Let l_z denote the number of i.d.z. which are invariant zeros of S. Consider the composite matrix $P(s)$ and notice that if $p\leq m$, all of the l i.d.z. z_i reduce the normal $(n+p)$ row rank of $P(s)$. Therefore, in view of (3), when $p\leq m$, all i.d.z. of S are invariant zeros as well, i.e., $l=l_z$. When $p>m$, $l\geq l_z$

The invariant zeros of S_c are all of the invariant zeros of S except those which correspond to uncontrollable modes, that is, except those which are i.d.z.

$$q=q_c+l_z. \tag{7}$$

The proof of the main theorem can now be completed.

Proof of Theorem 1 (Continued): It is clear, in view of the above, that $\dim R^*=\dim R_c^*$ is: 1) zero when $p\geq m$, and 2) $n_c-q_c-d_T=(n-l)-(q-l_z)-d_T=n-q-d_T$ when $p<m$. Note also that $\dim V^*=q+\dim R^*$ is true for a system not necessarily controllable [4]. Q.E.D.

Theorem 1 gives the dimension of V^* and R^* explicitly in terms of the order n, the number of zeros q, and d_T, a quantity which depends only on the transfer matrix or the controllable and observable part of the system. If the linear state feedback control law $u=F^*x+v$ [see (1) and (2)] is applied to S, $\dim V^*$ modes will be hidden from the output (they become unobservable) and this is the maximum possible number of modes with this property; $\dim R^*$ of these modes is arbitrarily assignable, while the rest coincide in value with the q invariant zeros of S [1]. One can go one step further and determine which ones of the uncontrollable and/or unobservable eigenvalues can be arbitrarily assigned and which ones remain fixed in the process. This is done by determining the relation between the supremal (A, B)-invariant and controllability subspaces of S and the corresponding subspaces of the systems obtained from S by isolating the controllable part, the observable part, or the controllable and observable part.

The controllable subsystem $S_c:\{A_c, B_c, C_c, E\}$ has been defined already. In a similar way, the observable part of S can be isolated and the subsystem $S_0:\{A_0, B_0, C_0, E\}$ is defined. The controllable and observable subsystem $S_{c0}:\{A_{c0}, B_{c0}, C_{c0}, E\}$ can be derived by, say, taking the observable part of S_c.

In addition to the order n_c, n_0, n_{c0}, and the number of invariant zeros q_c, q_0, q_{c0} of S_c, S_0, and S_{c0},[3] respectively, we shall need the input, output, and input–output decoupling zeros [6].

The l *input-decoupling zeros (i.d.z.)* have already been defined above.

The r *output-decoupling zeros (o.d.z.)* of S are those z_i (multiplicity included) for which $\text{rank}[z_i-A^T, C^T]^T<n$. They correspond to the r unobservable eigenvalues of S. The k values z_i which are common between the i.d. and o.d. zeros are the k *input–output decoupling zeros (i.o.d.z.)* and they correspond to those (k) eigenvalues of S which are both uncontrollable and unobservable.

Let r_z and k_z denote the number of o.d. and i.o.d. zeros which are also invariant zeros of S, respectively. Proceeding in a similar fashion as for l and l_z, it can be shown that for $p\geq m$, $r=r_z$ and $r\geq r_z$ for $p<m$. These relations together with $l=l_z$ for $p\leq m$ and $l\geq l_z$ for $p>m$ imply that $k=k_z$ always. Summarizing,

$$\begin{array}{lllll} \text{for} & p<m & l=l_z & r\geq r_z & k=k_z \\ & p=m & l=l_z & r=r_z & k=k_z \\ & p>m & l\geq l_z & r=r_z & k=k_z. \end{array} \tag{8}$$

It is not difficult to see that the following relations among the order of S, S_c, S_0, and S_{c0} are satisfied.

$$n=n_c+l=n_0+r=n_{c0}+l+r-k. \tag{9}$$

Furthermore, the number of invariant zeros are related by [see also (7)]

$$q=q_c+l_z=q_0+r_z=q_{c0}+l_z+r_z-k_z. \tag{10}$$

We are now ready to state and prove the following corollary of Theorem 1.

Corollary 3: $\dim V^*=\dim V_c^*+l_z=\dim V_0^*+r=\dim V_{c0}^*+(l_z+r-k)$ and $\dim R^*=\dim R_c^*=\dim R_0^*+(r-r_z)=\dim R_{c0}^*+(r-r_z)$.

Proof: If Theorem 1 is applied to S, S_c, S_0, S_{c0} and the relations (8), (9), and (10) are used, the corollary is easily derived.[4] In particular,

$$(p\geq m) \quad \dim V^*=q=q_{c0}+l_z+r_z-k_z=\dim V_{c0}^*+(l_z+r-k)$$

$$\dim R^*=0=\dim R_{c0}^*=\dim R_{c0}^*+(r-r_z)$$

$$(p<m) \quad \dim V^*=n-d_T=(n_{c0}-d_T)+l+r-k=\dim V_{c0}^*+(l_z+r-k)$$

$$\dim R^*=\dim V^*-q=\dim V^*-(q_{c0}+l_z+r_z-k_z)=\dim R_{c0}^*+(r-r_z).$$

The remaining relations are proved in a completely analogous fashion. Q.E.D.

If the given system S is completely controllable (but not necessarily observable), then $l=l_z=k=k_z=0$ and $\dim V^*=\dim V_{c0}^*+r$, $\dim R^*=\dim R_{c0}^*+(r-r_z)$ which are the relations derived in [1].

Corollary 3 suggests that when the control law $u=F^*x+v$ is used [see (1) and (2)], in which case the maximum possible number of eigenvalues becomes unobservable, only the unobservable eigenvalues which do not correspond to invariant zeros, i.e., only $r-r_z$ eigenvalues among the uncontrollable and/or unobservable eigenvalues can be arbitrarily changed. (Note that $r-r_z$ is nonzero only when $p<m$.) The remaining (l_z+r_z-k) uncontrollable and/or unobservable eigenvalues become unobservable without changing their value. Note that the $(l-l_z)$ uncontrollable eigenvalues which do not correspond to invariant zeros do not become unobservable (note that $l-l_z$ is nonzero only when $p>m$). This can be explained as follows. An uncontrollable eigenvalue will remain fixed in value and uncontrollable under linear state feedback; however, this type of compensation can affect the direction of the corresponding eigenvector and make the eigenvalue unobservable. It is easy now to see that when z_i is both an uncontrollable and unobservable eigenvalue (it corresponds to an i.o.d. zero), it must be an invariant zero of the system as well. Therefore, only the l_z uncontrollable eigenvalues can become unobservable.

It is thus clear that if $p>m$ and some of the uncontrollable eigenvalues which do not coincide with invariant zeros are undesirable, then linear state feedback compensation cannot be used to "hide" these modes from the output. This gives a nonsolvability condition for the output stabilization problem using linear state feedback since, as it is known, all the controllable eigenvalues can be arbitrarily altered, while the unstable uncontrollable eigenvalues (which remain fixed) can only become, at best, unobservable from the output.

Theorem 1 was derived under the assumption that the transfer matrix $T(s)$ is of full rank $\min(p, m)$. In the following, the dimensions of V^* and R^* of a system which does not satisfy this assumption will be found.

Let $T(s)=C(sI-A)^{-1}B+E$ and $\text{rank}\,T(s)=\rho<\min(p, m)$. $\tilde{T}(s)$ is defined as a $\rho\times m$ rational matrix consisting of ρ linearly independent rows of $T(s)$. It is thus clear that $\tilde{T}(s)$ has full row rank ρ and it is the transfer matrix of the system $\tilde{S}:\{A, B, \tilde{C}, \tilde{E}\}$ where $\tilde{C}$ and $\tilde{E}$ consist of the corresponding ρ rows of C and E, respectively.

Since $\tilde{T}(s)$ has full rank, Theorem 1 can be applied to $\tilde{S}$ and the dimensions of the supremal (A, B)-invariant and controllability subspaces $\tilde{V}^*$ and $\tilde{R}^*$ can be readily obtained. We will now determine the relations between V^*, R^* and $\tilde{V}^*, \tilde{R}^*$ of S and $\tilde{S}$, respectively.

Lemma 4: $R^*=\tilde{R}^*$.

Proof: Let

$$P(s)\triangleq\begin{pmatrix} s-A & B \\ -C & E\end{pmatrix} \quad \text{and} \quad \tilde{P}(s)\triangleq\begin{pmatrix} s-A & B \\ -\tilde{C} & \tilde{E}\end{pmatrix}.$$

[3]The invariant zeros of S_{c0} are the transmission zeros of S and they can also be found from $T(s)$.

[4]Note that all systems have the same transfer matrix $T(s)$; therefore, they have the same d_T.

$$\ker P(s)=\ker\begin{pmatrix} I & 0 \\ C\,(s-A\,)^{-1} & I \end{pmatrix}$$

$$P(s)=\ker\begin{pmatrix} s-A & B \\ 0 & T\,(s\,) \end{pmatrix}=\ker[s-A,B]\cap\ker[0,T(s)].$$

But $\ker T(s)=\ker\tilde{T}(s)$ since $(p-\rho)$ rows of $T(s)$ are linearly dependent on the remaining ρ rows which are exactly those of $\tilde{T}(s)$.

Consequently,

$$\ker P(s)=\ker\begin{pmatrix} s-A & B \\ 0 & \tilde{T}(s) \end{pmatrix}$$

$$=\ker\begin{pmatrix} I & 0 \\ -\tilde{C}(s-A)^{-1} & I \end{pmatrix}\begin{pmatrix} s-A & B \\ 0 & \tilde{T}(s) \end{pmatrix}=\ker\tilde{P}(s).$$

This implies that $R^*=\tilde{R}^*$ since, as it can be shown using [5], $\ker P(s)$ and $\ker\tilde{P}(s)$ define any and all output-nulling controllability subspaces of the systems S and $\tilde{S}$. Q.E.D.

Let $\tilde{q}$ be the number of invariant zeros of $\tilde{S}$ and notice that $\tilde{q}$ depends on the choice of the ρ independent rows of $T(s)$. Clearly, now,

$$\dim V^*=q+\dim R^*=q+\dim\tilde{R}^*=\dim\tilde{V}^*+(q-\tilde{q}),$$

that is, the system $\tilde{S}$ can be used to determine the dimensions of V^* and R^* of S.

Concluding Remarks

The dimensions of V^* and R^* of a system were explicitly determined, and they were shown to be functions of the order of the system, the number of invariant zeros, and the degree of the determinant of a matrix derived from the transfer matrix of the system. The role played by the invariant zeros as well as by the input and output decoupling zeros was discussed and their importance to the output stabilization problem was indicated.

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An Iterative Method for Generalized Complementarity Problems

SHU-CHERNG FANG

***Abstract*—Given a generalized complementarity problem (i.e., complementarity problem over a cone), Habetler and Price introduced an iterative method to solve it under the conditions that the cone is solid and the function is continuous and strongly copositive on the cone. In this paper, we provide an easier iterative method to solve this problem provided that the function is Lipschitz continuous and strongly monotone on the (maybe nonsolid) cone. A separate consideration is given to polyhedral cones.**

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I. Introduction

Given a closed convex cone K in the n-dimensional Euclidian space R^n, its dual cone K^* (i.e., the set of all vectors y in R^n such that the inner product $\langle x,y\rangle\geqslant 0$ for all x in K) and a function f from K to R^n. The generalized complementarity problem is to find all solution vectors x such that

$$x\in K,\quad f(x)\in K^*$$

and

$$\langle x,f(x)\rangle=0. \tag{1}$$

This problem was first introduced by Habetler and Price [6] and later refined by Karamardian [9], [10]. When $K=R^n_+$ (i.e., the positive orthant of R^n), the problem becomes the common complementarity problem: to find solution vectors x such that

$$x\geqslant 0,\quad f(x)\geqslant 0$$

and

$$\langle x,f(x)\rangle=0. \tag{2}$$

Moreover, when f has the form $f(x)=Mx+q$ where M is a given n by n matrix and q is an n-vector, the problem is called the linear complementarity problem. A considerable number of authors have contributed to the computational algorithms for solving linear and/or nonlinear complementarity problems. For example, see Cottle and Dantzig [3], Lemke [11], [12], Merrill [15], Eaves and Saigal [4], and Fisher and Gould [5].

Habetler and Price [7] introduced an iterative method to solve the generalized complementarity problem when the cone K is solid (i.e., K has interior points) and f is continuous and strongly copositive on K. Their work is theoretically correct, but is not so easy to apply. In this paper, we propose an applicable iterative method under the assumption that f is Lipschitz continuous and strongly monotone on K (maybe nonsolid). Then a separate consideration will be given to the case when K is a polyhedral cone.

II. Preliminaries

The well-known result that a given generalized complementarity problem is equivalent to the corresponding variational inequality problem can be stated in the following theorem.

Theorem 1: Given a closed convex cone K and a function f from K to R^n, x is a solution to the generalized complementarity conditions (1) if and only if x is a solution to the corresponding variational inequalities.

$$x\in K$$

and

$$\langle x'-x,f(x)\rangle\geqslant 0 \qquad \text{for all } x' \text{ in } K. \tag{3}$$

Proof: See Karamardian [9].

For a given function f from K to R^n, we say that f is strongly monotone on K if there is a scalar $\alpha>0$ such that

$$\langle x'-x,f(x')-f(x)\rangle\geqslant\alpha\|x'-x\|^2$$

for each x', x in K. The following existence and uniqueness theorem is given by Morè.

Theorem 2: For any generalized complementarity problem, if the function f is continuous and strongly monotone on the closed convex cone K, then there is one and only one solution to the generalized complementarity conditions (1).

Proof: See Morè [16].

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