M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems,"
Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009,
Univ of Notre Dame, December 1993.

On the Stability of Logical Transitions in Hybrid Control Systems

Technical Report of the ISIS Group
at the University of Notre Dame
ISIS-93-009
December, 1993

M.D. Lemmon and P.J. Antsaklis
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556

Interdisciplinary Studies of Intelligent Systems

M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems," Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009, Univ of Notre Dame, December 1993.

On the Stability of Logical Transitions in Hybrid Control Systems

Michael Lemmon *and Panos J. Antsaklis
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556 USA

December 6, 1993

Abstract

Hybrid control systems consist of a discrete event system controller supervising the behaviour of a continuous-state plant. Since the plant and controller evolve over different types of state spaces, an interface is needed to interconnect the systems. The combined plant and interface form an equivalent DES representation of the plant's symbolic behaviour. In this paper stability of the DES plant is defined and discussed. For continuous state plants which are affine in their control policies, the interface yielding a stable DES plant can be characterized as a feasible point for a system of linear inequalities in an appropriate parameter space. As such inequality systems are efficiently solved using the method of centers algorithms, this paper provides an efficient method for the design of stable hybrid control system interfaces.

keywords: Hybrid Systems, Stability, Model Verification

*The partial financial support of the National Science Foundation (IRI91-09298 and MSS92-16559) is gratefully acknowledged.

M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems," Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009, Univ of Notre Dame, December 1993.

1 Introduction

Hybrid control systems arise when a discrete event system (DES) called the *DES controller* is used to supervise the behaviour of a continuous state system (CSS) called the *CSS plant*. The DES controller evolves in logical time over a state space consisting of a finite alphabet of symbols. The CSS plant evolves over a state space which is generally some subset of the Euclidean n-space. This paper considers CSS plants which evolve over continuous time. Because both systems evolve over distinctly different types of sets, an *interface* is needed to facilitate communication between the plant and controller. The interface must transform state trajectories into sequences of logical symbols and vice versa.

Prior work in hybrid control systems has proposed a variety of interface architectures. [1] used interfaces which were specific to the applications being considered. More recently, a variety of related interface architectures have been proposed [9] [7] [10] [8] in which the plant's state evolves over either a partition or open cover of the plant's state space. In these frameworks, the interface generates a sequence of logical symbols describing the plant's symbolic behaviour with respect to the underlying state space partition or covering. The combination of plant and interface can therefore be viewed as another discrete event system which is called the equivalent *DES plant*.

One approach to hybrid control system design synthesizes a DES controller which achieves a formal specification on the DES plant's logical behaviour. Early work in this area will be found in [7] and [11]. This approach is only effective if the DES plant represents a *valid* interpretation of the plant's logical (symbolic) behaviour. A "valid" DES plant is a DES whose symbolic behaviour truthfully predicts the symbolic behaviour of interest of the CSS plant. This paper shows that DES plant validity is closely related to the stability of CSS systems. Therefore statements about the "validity" of the DES plant are statements about the "stability" of the DES plant.

To date there has been relatively little formal and systematic work concerned with the stability of hybrid systems. [9] investigated the cyclic behaviour of a CSS plant evolving over a state space partition. Work reported in [7] introduced a notion of structural stability which is very similar to the definition of stability being used in this paper. Related work in [4] [5] identified a set of sufficient conditions on the system interface which insure the determinism (supervisability) of the DES plant.

This paper builds upon earlier work in [5]. For a large class of hybrid system models (section 2) a DES plant is defined. The transition or T-stability of this DES plant is formally defined (section 3) and several

M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems," Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009, Univ of Notre Dame, December 1993.

consequences of this definition are discussed. In particular, a set of sufficient conditions for T-stability will be derived in section 3. For plants which are affine in their control policies, these sufficient conditions form a system of linear inequalities characterizing the set of stable hybrid control system interfaces. These inequality systems are efficiently solved using "method of center" algorithms [6] such as the ellipsoid method [2] or path following algorithms [3]. The solution of these inequality systems are interfaces which "stabilize" the DES plant's interpretation of the CSS plant's behaviour. This paper therefore presents a method for designing stable hybrid control system interfaces. Section 4 summarizes the principal contributions of this paper.

2 Hybrid Control Systems

In this paper, a specific hybrid control system model is used. The model is a generalization of a framework suggested in [10]. The hybrid control system consists of three interconnected subsystems; the plant, the interface, and the controller. Figure 1 illustrates the assumed interconnections. The system interface consists of a pair of subsystems called the *actuator* and *generator*. The actuator is responsible for transforming control symbol sequences $\tilde{r}[n]$ into control vector trajectories, $\bar{r}(t)$. The generator is responsible for transforming the CSS plant's state trajectory, $\bar{x}(t)$ into a sequence of plant symbols, $\tilde{x}[n]$. Figure 1 illustrates the role of these two subsystems in the interface.

Definition 1 *A* hybrid control system, $\mathcal{H}$, *is the ordered 4-tuple,* $\mathcal{H} = (\mathcal{P}_c, C_d, \mathcal{A}, \mathcal{G})$, *consisting of 4 subsystems. These subsystems are the* CSS plant, $\mathcal{P}_c$, *the* DES controller, C_d, *the* interface generator, $\mathcal{G}$, *and the* interface actuator, $\mathcal{A}$.

The plant is the part of the model which represents the entire continuous-time and continuous-state (CSS) portion of the hybrid control system. The plant evolves over a subset $\bar{X}$ of $\Re^n$. $\bar{X}$ is called the continuous state system (CSS) plant's *state space*. The trajectory, $\bar{x}(t)$, of the state through $\bar{X}$ is described by a set of ordinary differential equations.

$$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{r}(t)) \tag{1}$$

where $\bar{x} \in \Re^n$ is the CSS plant state and $\bar{r}$ is the CSS plant's control vector. It is assumed that $\bar{r} \in \bar{R} \subset \Re^m$. $\bar{R}$ is called the CSS plant's *control space*. The function $f : \bar{X} \times \bar{R} \rightarrow \bar{X}$ is assumed to be Lipschitz continuous.

In a hybrid system, the plant is controlled by the application of logical directives. In this regard, the control vector trajectory $\bar{r}(t)$ is often piecewise constant. The CSS plant can therefore be modeled as a collection of dynamical systems as is done in the following definition.

Definition 2 *The* CSS plant,$\mathcal{P}_c$, *of a hybrid system,* $\mathcal{H}$, *is the ordered* $m+1$*-tuple,*

$$\mathcal{P}_c = (\tilde{X}, \Phi_t^{\bar{r}_1}, \cdots, \Phi_t^{\bar{r}_m}) \tag{2}$$

$\Phi_t^{\bar{r}_i} : \tilde{X} \to \tilde{X}$ *(where* $i = 1, \ldots, m$*) is the ith family of transition operators (indexed by* t*) generated by the differential equation*

$$\dot{\bar{x}}(t) = f(\bar{x}(t), \bar{r}_i) \tag{3}$$

where $\bar{r}_i$ *(for* $i = 1, \ldots, m$*) is a constant vector in* $\Re^m$ *and* $f : \Re^n \times \Re^m \to \Re^n$ *is the infinitesimal generator of the flow operator* $\Phi_t^{\bar{r}_i}$.

The controller is a discrete event system. The controller will often be called the *supervisor*. It receives as inputs a sequence $\tilde{x}[n]$ of symbols generated by the interface. These *plant symbols* are drawn from a finite alphabet $\tilde{X}$. The supervisor outputs a sequence $\tilde{r}[n]$ of symbols drawn from a finite alphabet $\tilde{R}$ of *control symbols*. The controller's dynamics are assumed to be modeled by a deterministic finite automaton (DFA), thereby leading to the following definition.

Definition 3 *The* DES controller, C_d, *for hybrid system* $\mathcal{H}$, *is the ordered 5-tuple,* $C = (\tilde{S}, \tilde{X}, \tilde{R}, \Phi, Q)$, *where* $\tilde{S}$ *is a finite alphabet of* controller state *symbols,* $\tilde{X}$ *and* $\tilde{R}$ *are finite alphabets of* plant and control symbols, *respectively.* $\Phi : \tilde{S} \times \tilde{X} \to \tilde{S}$ *is the* controller transition operator. $Q : \tilde{S} \to \Re$ *is the* controller enabling function. *The DES controller's dynamics are generated by the following recursive equations,*

$$\begin{aligned} \tilde{s}[n] &= \Phi(\tilde{s}[n-1], \tilde{x}[n]) && (4) \\ \tilde{r}[n] &= \text{argmax}_{\tilde{R}} Q(\tilde{s}[n]) && (5) \end{aligned}$$

The controller and plant communicate through an interface. The interface consists of two subsystems called the *actuator* and *generator*.

Definition 4 *The* interface generator, $\mathcal{G}$, *for hybrid system* $\mathcal{H}$ *is the ordered 4-tuple,* $\mathcal{G} = (\mathcal{B}, \tilde{X}, \alpha, T_x)$. $\mathcal{B} = \{b_i\}_{i=1,\ldots,p}$ *is a family of p continuously differentiable functionals,* $b_i : \Re^n \to \Re$ *over the state space such that* $\nabla_{\bar{x}} b_i(\bar{x}) \neq 0$ *when* $b_i(\bar{x}) = 0$ *and such that these functionals separate the state space. This collection of functionals will be called the* event basis *for the hybrid system.* $\tilde{X}$ *is a finite set of plant symbols.* $\alpha : \Re^n \to \tilde{X}$ *is a mapping from the CSS plant's state space onto the alphabet of plant symbols.* $T_x : \Re \times \tilde{X} \times \Re \to \Re$ *is a* *recursive mapping used to generate a sequence of* plant instants , $\tau_x[n]$. *(measured with respect to the CSS plant's clock). These plant instants are the times when the interface generator issues a plant symbol. The interface generator's output is represented by the following equation.*

$$\tilde{x}[n] = \alpha(\bar{x}(\tau_x[n])) \tag{6}$$

For the remainder of this paper, the event basis, $\mathcal{B}$, will be assumed to generate an open covering of the CSS plant's state space. Let $\mathbf{b}_i^+$ denote the subset of the state space for which $b_i(\bar{x}) > 0$. Let the set $\mathbf{b}_i^-$ denote the subset of the state space for which $b_i(\bar{x}) < 0$. This collection of sets, $\mathbf{b}_i^\pm$, associated with $\mathcal{B}$ are usually assumed to form an open cover of the CSS plant's state space. Define the following symbols,

$$\tilde{b}_i^+ = \text{EnterRegion}(\mathbf{b}_i^+) \tag{7}$$

$$\tilde{b}_i^- = \text{EnterRegion}(\mathbf{b}_i^-) \tag{8}$$

$$\tilde{\lambda} = \text{NULL} \tag{9}$$

The set $\mathbf{b}_i^\pm$, will be said to be the *preimage* of plant symbol $\tilde{b}_i^\pm$.

The plant symbol alphabet, $\tilde{X}$, used in this paper will be defined with respect to the event basis. Let $\tilde{X} = \{\tilde{b}_i^+\}_{i=1,\dots,p} \cup \{\tilde{b}_i^-\}_{i=1,\dots,p} \cup \{\tilde{\lambda}\}$, where $\tilde{\lambda}$ represents a "null" symbol. The generator mapping α is also defined with respect to the assumed event basis. It is assumed that this mapping will be used to generate plant symbols only when the plant state crosses the boundary of some $\mathbf{b}_i^\pm$ set. The generator mapping α, therefore takes the following form

$$\alpha(\bar{x}) = \begin{cases} \tilde{b}_i^+ & \text{if } b_i(\bar{x}) = 0 \text{ and } \nabla_{\bar{x}} b_i(\bar{x})\dot{\bar{x}}(t) > 0 \\ \tilde{b}_i^- & \text{if } b_i(\bar{x}) = 0 \text{ and } \nabla_{\bar{x}} b_i(\bar{x})\dot{\bar{x}}(t) < 0 \\ \tilde{\lambda} & \textit{otherwise} \end{cases} \tag{10}$$

The plant instant generator T_x is a recursive function generating a sequence of ***plant instant*** times, $\tau_x[n]$. In this paper, the plant instants represent times when the boundary of sets $\mathbf{b}_i^\pm$ are crossed by the CSS plant's state vector. The recursive function, T_x, generating this sequence of plant instants, $\tau_x[n]$ is given below.

$$\tau_x[0] = 0 \tag{11}$$

$$\tau_x[n+1] = T_x(\tau_x[n], \bar{x}(t)) \tag{12}$$

$$= \lim_{\epsilon \to 0+} \inf \{t > \tau_x[n] : \exists i, b_i(\bar{x}(t)) b_i(\bar{x}(\tau_x[n-1] + \epsilon)) < 0\} \tag{13}$$

The ***actuator*** converts a sequence $\tilde{r}[n]$ of control symbols into a plant input, $\bar{r}(t)$. As noted earlier, it is assumed that the resulting CSS control trajectory is piecewise constant. This yields the following definition

M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems," Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009, Univ of Notre Dame, December 1993.

Definition 5 *The* interface actuator. $\mathcal{A}$. *for the hybrid system* $\mathcal{H}$, *is the ordered 4-tuple,* $\mathcal{A} = (\bar{R}, \tilde{R}, \gamma, T_r)$, *where* $\gamma : \tilde{R} \to \bar{R}$ *is a mapping from the control symbols to the constant control vectors input to the plant,* $\bar{R}$ *and* $\tilde{R}$ *are the control spaces and alphabets, respectively.* $T_r : \Re \to \Re$ *is a recursive function generating a sequence of control instants using the formula,* $\tau_r[n] = T_r(\tau_x[n])$. *The actuator output is given by the following equation*

$$\bar{r}(t) = \sum_{n=0}^{\infty} \gamma(\tilde{r}[n]) I(t, \tau_r[n], \tau_r[n+1]) \tag{14}$$

where $I(t)$ *is an indicator function which is unity over the interval* $(\tau_r[n], \tau_r[n+1]]$ *and is zero elsewhere. It is assumed that the sequence of control and plant instants obey the following relation.*

$$\tau_x[n] < \tau_r[n] < \tau_x[n+1] \tag{15}$$

In a hybrid control system, the system formed by combining the plant and interface inputs and outputs sequences of logical symbols. The combined plant/interface is therefore a DES which is often called the *DES plant*. The input to the DES plant is a sequence of *control events* $(\tilde{r}[n], \tau_r[n])$. Each "control event" is an ordered pair consisting of a control symbol and a control instant. The output of the DES plant is a sequence of *plant events* $(\tilde{x}[n], \tau_x[n])$. Each "plant event" is an ordered pair consisting of a plant symbol and a plant instant. The DES plant can now be formally defined as a labeled directed graph.

Definition 6 *A* DES plant *of the hybrid control system* $\mathcal{H}$ *is a labeled digraph,* $\mathcal{P}_d = (\tilde{Z}, A)$. *The set of vertices,* $\tilde{Z}$, *is a finite alphabet of DES plant* state symbols. *The set of arcs,* A *is a subset of* $\tilde{Z} \times \tilde{Z}$. *Each arc is labeled by a* control event, $(\tilde{r}, \tau_r) \in \tilde{R} \times \Re$ *and a* plant event $(\tilde{x}, \tau_x) \in \tilde{X} \times \Re$.

In this paper, the set of DES plant states, $\tilde{Z}$ will be defined with respect to the event basis, $\mathcal{B}$. As the plant alphabet, $\tilde{X}$, labels the CSS plant state's entry into sets $\mathbf{b}_i^{\pm}$, the DES plant states will be defined with respect to the plant symbols. Let $\mathcal{Z} = (\tilde{X}, [,], \sim, \vee)$ be a logical system in which the plant symbols, $\tilde{X}$, form the propositional variables and the other symbols ([,],$\sim$,and $\vee$) are used to form propositional logical formulae from the symbols in $\tilde{X}$. A formula is any concatenation of symbols and a *wff* or well-formed formula is any formula which satisfies the the usual formation rules for a propositional logic. With these definitions, the DES plant's state space, $\tilde{Z}$ will be taken to be any collection of wffs for the propositional logical system $\mathcal{Z}$.

Since each proper symbol used in constructing the symbols of $\tilde{Z}$ is associated with a "preimage" or subset in the state space, any state in $\tilde{Z}$ has an associated pre-image in the CSS plant's state space. Let $\tilde{x}$ and

M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems," Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009, Univ of Notre Dame, December 1993.

$\tilde{y}$ be elements of $\tilde{X}$ with pre-images $\mathbf{x}$ and $\mathbf{y}$, respectively. The preimage of any DES plant state can be determined from the following rules.

- the preimage of $\sim \tilde{x}$ is the opening of $\tilde{X} - \mathbf{x}$.
- the preimage of $[\tilde{x} \vee \tilde{y}]$ is $\mathbf{x} \cup \mathbf{y}$.

Note that with the preceding definition of the DES plant's state symbol, $\tilde{x}$, the symbol has a semantic interpretation that the CSS plant's state is in the preimage of that symbol. Each node of the DES plant is associated with a subset of the CSS plant's state space. Each arc has the semantic interpretation that the application of the control symbol will reliably transfer the plant's state between subsets of the CSS plant's state space.

3 Stability of Transition

The DES plant can be seen as a "logical" interpretation of the CSS plant's symbolic behaviour. If that interpretation accurately and reliably predicts the CSS plant's behaviour, then it is a valid interpretation or *model* of the CSS plant. As our approach to hybrid control system design involves synthesizing DES controllers for an extracted DES plant, the "validity" of the DES is a crucial question which has to be answered.

This paper defines DES plant validity in terms of the invariance of the plant and control event sequences to small perturbations in the CSS plant's state. An arc of the DES plant represents a transition of the CSS plant's state between two subsets of the CSS plant's state space. The labeling of that arc represents the symbolic behaviour of that transition. A valid DES plant would preserve that labeling under small perturbations of the initial CSS plant state. In particular, this means that the plant symbols generated by the transition must be unchanged with respect to transition perturbations. In addition, this means that the plant instants, must vary continuously with the assumed perturbations. This notion of "validity" is closely related to the Lyapunov stability of the underlying CSS plant and hence we can view the question of DES plant validity as a question about the hybrid system's *stability*.

The following definition formally introduces the notion of hybrid system *stability* used in this paper. This notion refers to the stability of an arc (transition) in the DES plant. Consequently, this notion of hybrid system stability is referred to as transition or T-stability.

M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems," Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009, Univ of Notre Dame, December 1993.

Definition 7 *Let $\mathcal{P}_d = (V, A)$ be a DES plant for the hybrid system $(\mathcal{P}_c, \mathcal{C}_d, \mathcal{A}, \mathcal{G})$. Consider an arc $(\tilde{u}, \tilde{w}) \in A$ which is labeled with plant event, $(\tilde{x}, \tau_x)$, and control event, $(\tilde{r}, \tau_r)$. Let* $\mathbf{u}$ *and* $\mathbf{w}$ *be the pre-images of the DES plant states, $\tilde{u}$ and $\tilde{w}$, respectively. Let $\bar{r} = \gamma(\tilde{r})$ where γ is the hybrid system's actuator mapping.*

The arc $(\tilde{u}, \tilde{w})$ is stable *if and only if for all $\bar{x}_u \in \mathbf{u}$ there exists an open neighborhood of size ϵ, $N(\bar{x}_u, \epsilon)$, and a finite time $0 < T(\bar{x}_u) < \infty$ such that*

- *$\Phi_T^{\bar{r}}(\bar{x})$ is an open neighborhood in* $\mathbf{w}$
- *and the plant symbols, $\tilde{x}$, issued by the transition are identical*

for all $\bar{x} \in N(\bar{x}_u, \epsilon)$.

In the preceding definition, the time, T, is a function of the vector $\bar{x}_u$. If T is independent of $\bar{x}_u$, the arc will be said to be ***uniformly stable***. The plant DES will be said to be (uniformly) T-stable if and only if all of its arcs are (uniformly) stable.

Figure 2 illustrates an unstable arc of a DES plant. This figure shows an arc, $(\tilde{u}, \tilde{w})$, which transfers the plant state from a set $\mathbf{u}$ to a set $\mathbf{w}$ such that the resulting trajectory is tangential to the boundary of a set $\mathbf{v}$ which is the pre-image of DES plant state, $\tilde{v}$. For this transition to occur in a stable manner, it is necessary that the transfer of the plant state occurs in manner such that the tranfer time, τ_x, is bounded and that the symbols issued by the interface generator are unchanged. Stability therefore requires that the "bundle" of state trajectories originating from an open neighborhood, $N(\bar{x}_u, \epsilon)$, not intersect the set $\mathbf{v}$. The figure shows that if such a perturbation results in this intersection, that the arc of the DES plant will not be stable since it fails to leave the symbolic labeling of the arc invariant under perturbations of the initial CSS plant state.

Figure 3 shows another example of an unstable arc. In this case, the DES plant arc, $(\tilde{u}, \tilde{w})$, transitions the CSS plant state from set $\mathbf{u}$ to $\mathbf{w}$. The flow field of the controlled system, however, is assumed to be discontinuous at state $\bar{x}_u$. In this example, the figure shows that the state trajectories, $\Phi_t^{r}(\bar{x})$, for some $\bar{x} \in N(\bar{x}_u, \epsilon)$, will never lie in $\mathbf{w}$. Consequently, the time T by which all transitions in the open neighborhood must have occurred is unbounded. This violates our preceding definition in that the plant instants do not vary in a continuous manner with the initial CSS state. Consequently, the DES plant is unstable.

Figure 4 shows an example of an arc, $(\tilde{u}, \tilde{w})$, which is not uniformly stable. Uniform T-stability requires that the transition time T identified in the definition be the same for all CSS plant states in $\mathbf{u}$. Now consider an *infinite* sequence of arcs which switch the CSS plant state between $\mathbf{u}$ and $\mathbf{w}$. Without the requirement

M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems," Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009, Univ of Notre Dame, December 1993.

that T be uniformly bounded away from zero, it is quite possible that the sequence of transitions may result in the CSS plant's state converging to a state which is neither in $\mathbf{u}$ or $\mathbf{v}$. The notion of uniform T-stability was introduced just to avoid this type of behaviour. Note that while any finite sequence of transitions would be T-stable, the limiting symbolic behaviour as the number of transitions becomes infinite can be considered unstable.

The following proposition shows that all sets in the event basis must be open subsets of the state space. Similar results have been discussed in [7].

Proposition 1 *If a DES plant, $\mathcal{P}_d$ of a hybrid system $\mathcal{H}$ is T-stable, then all pre-images of the DES plant's state space, $\tilde{Z}$, are open subsets of the CSS plant's state space.*

proof: Assume that the DES plant is stable and that the pre-image, $\mathbf{u}$, of DES plant state $\tilde{u}$ is closed. If the CSS plant state $\bar{x}_u$ which is on the boundary of $\mathbf{u}$ then it is also in $\mathbf{u}$. By the definition of T-stability there must exist an open neighborhood of $\bar{x}_u$ which is transfered to an open set in another set $\mathbf{w}$. The existence of such an open-neighborhood, however, contradicts the assumption that $\bar{x}_u$ is on the boundary of $\mathbf{u}$. Therefore all pre-images of the DES plant state symbol alphabet, $\tilde{Z}$, must be open subsets of the CSS plant's state space. •

A DES plant is said to be *deterministic* if and only if for any two arcs $(\tilde{u}, \tilde{v})$ and $(\tilde{u}, \tilde{w})$ which are labeled with the control event, $(\tilde{r}, \tau_v)$ and $(\tilde{r}, \tau_w)$, respectively, the target symbols, $\tilde{v}$ and $\tilde{w}$ are the same. The following proposition establishes the relationship between T-stability and determinism in the DES plant.

Proposition 2 *Assume that the pre-image of any DES plant state symbol $\tilde{z}$ is connected. If the DES plant is T-stable, then it is also deterministic.*

Proof: Consider the pre-image, $\mathbf{u}$, of a DES plant state symbol $\tilde{u}$. Without loss of generality assume that there are only two arcs, $a_v = (\tilde{u}, \tilde{v})$ and $a_w = (\tilde{u}, \tilde{w})$, originating in $\tilde{u}$. The following arguments are easily extended to more than two arcs. Let $\mathbf{u}_v$ denote the set of all CSS plant states $\bar{x} \in \mathbf{u}$ such that a_v is a valid arc. Let $\mathbf{u}_w$ denote the set of all CSS plant states in $\mathbf{u}$ such that a_w is a valid arc. Since there are only two arcs originating in $\tilde{u}$, the set $\mathbf{u}$ must equal the union of $\mathbf{u}_v$ and $\mathbf{u}_w$. $\mathbf{u}$, however, is assumed to be connected so that $\mathbf{u}_v$ and $\mathbf{u}_w$ have a non-null intersection. Since the arc is T-stable $\tilde{v}$ and $\tilde{w}$ must be the same, thereby implying that the arc is deterministic. •

M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems," Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009, Univ of Notre Dame, December 1993.

The definition of T-stability applies to a single arc of the DES plant. Two arcs the DES plant $(\tilde{u}_1, \tilde{w}_1)$ and $(\tilde{u}_2, \tilde{w}_2)$ are *connected* if $\tilde{w}_1 = \tilde{u}_2$. To following proposition extends T-stability to finite sequences of connected arcs.

Proposition 3 *If the DES plant is T-stable, then any finite sequence of connected arcs will also be T-stable.*

proof: Let $a[n]$ denote a finite sequence of connected arcs ($n = 1, \ldots, N$) for the assumed DES plant. Let $\tilde{u}[n]$ and $\tilde{u}[n+1]$ be the origin and terminus, respectively, for the arc $a[n]$. We can therefore associate with $a[n]$ a sequence of sets, $\mathbf{u}[n]$, consisting of the pre-images of $\tilde{u}[n]$ for $n = 0, \ldots, N$. The sequence of arcs, $a[n]$, will be said to be T-stable if for all $\bar{x} \in \mathbf{u}[0]$, there exists a sequence, $T[n]$, of positive and bounded times and there exists an open neighborhood, $N(\bar{x}, \epsilon)$, in $\mathbf{u}[0]$ such that $\prod_{n=1}^{N} \Phi_{T[n]}^{\tilde{r}[n]}(\mathbf{u}[0])$ is an open subset in $\mathbf{u}[N]$.

Consider the hypothesis, H_N, that a sequence of N arcs is T-stable. Clearly this hypothesis is true for $N = 1$ since this is the definition of a stable arc.

Now assume that the hypothesis is true for H_N and consider the sequence of arcs of length $N+1$. By T-stability, the arc from $\mathbf{u}[N]$ to $\mathbf{u}[N+1]$ is T-stable, so for all $\bar{x} \in \mathbf{u}[N]$ there exists a finite time $T[N+1]$ and there exists an open neighborhood, $N(\bar{x}, \epsilon)$, of $\bar{x}$ contained in $\mathbf{u}[N]$ such that $\Phi_{T[N+1]}^{\tilde{r}[N+1]}(N(\bar{x}, \epsilon))$ is an open set in $\mathbf{u}[N+1]$.

Now consider an open neighborhood $N(\bar{x}_0, \epsilon)$ of $\mathbf{u}[0]$. By the hypothesis this maps to another open neighborhood in $\mathbf{u}[N]$ in finite time. Consider the largest subset of this open neighborhood in $\mathbf{u}[N]$ which is mapped into $\mathbf{u}[N+1]$. By T-stability, we know such a neigborhood always exists. We can therefore map $N[0]$ into an open subset of $\mathbf{u}[N+1]$ in finite time, which implies the truth of hypothesis H_{N+1}. Therefore by mathematical induction, we know that the hypothesis is true for all finite N. •

The preceding proposition proved the stability of a finite sequence of T-stable transitions. As this number of transitions goes to infinity, it becomes natural to ask whether or not the "stability" of the transition is preserved. The following definition helps us discuss this problem more precisely.

Definition 8 *Consider an infinite sequence of connected arcs $a[n] = (\tilde{u}[n], \tilde{w}[n])$ where each arc is labeled by the control event $(\tilde{r}[n], \tau_r[n])$ and by the plant event $(\tilde{x}[n], \tau_x[n])$. The sequence of arcs will be said to be* convergent *if and only if there exists some $N > 0$ such that for all $n > N$, the plant symbol sequence $\tilde{x}[n]$ is periodic and the sequence of plant instants, $\tau_x[n]$ is convergent to a periodic sequence.*

The preceding definition states that the infinite sequence of transitions will be convergent if the logical behaviour settles to a periodic behaviour and the transition times $\tau_x[n]$ converge to a periodic sequence. There are, however, some infinite behaviours which a T-stable DES plant cannot possibly exhibit. One behaviour is given in the following proposition. This behaviour is the type of behaviour illustrated in figure 4. In this case, even though each individual arc or transition is stable, the lack of uniform stability allows the existence of undesirable asymptotic behaviour.

Proposition 4 *A uniformly T-stable DES plant will never generate a convergent infinite sequence of arcs where* $\lim_{n\to\infty} \tau_x[n] = 0$.

Proof: Assume that such a infinite sequence occurs, then this would imply that T goes to zero. Therefore there would be no positive finite T for all $\bar{x}$ in the plant's state space which satisfies the uniformity T-stability condition. •

The following proposition provides a general characterization of sufficient T-stability conditions. The conditions relate the controlled CSS plant's attractors and repellors to T-stable transitions. In this regard, the following proposition provides a clear connection between hybrid system T-stability and conventional notions of Lyapunov stability in CSS systems.

Proposition 5 *An arc $a = (\tilde{u}, \tilde{w})$ of a DES plant, $\mathcal{P}_d = (\tilde{Z}, A)$ of a hybrid system, $\mathcal{H}$ will be T-stable if*

- *any preimage, $\mathbf{v}$, of a DES plant state symbol $\tilde{v}$ not equal to $\mathbf{w}$ is a repellor for the system $(\bar{X}, \Phi_t^{\bar{r}_i})$,*
- *and the largest invariant set of the smooth dynamical system $(\bar{X}, \Phi_t^{\bar{r}_i})$ is properly contained in the preimage, $\mathbf{w}$, of $\tilde{w}$.*

Proof: The arc $(\tilde{u}, \tilde{w})$ is T-stable if the bundle of state trajectories leaving $\mathbf{u}$ (1) do not intersect any other pre-image, $\mathbf{v}$, of a DES plant state symbol, $\tilde{v}$, not equal to $\tilde{w}$ and (2) enter the preimage, $\mathbf{w}$ of state symbol $\tilde{w}$ in finite time. The first is guaranteed if all pre-images, $\mathbf{v}$, of state symbols not equal to $\tilde{w}$ are repellors of the controlled CSS plant. By the LaSalle invariance principle, the second condition is guaranteed if $\mathbf{w}$ properly contains the largest invariant set of the controlled CSS plant. •

The preceding proposition is only a sufficient condition for T-stability. Clearly, while it is highly desirable that all $\mathbf{v} \neq \mathbf{w}$ be repellors, it is by no means necessary that $\mathbf{w}$ contain an attractor. In fact, T-stability can be guaranteed if a set reachable from $\mathbf{u}$ is contained in $\mathbf{w}$. Even so, the proposition is extremely valuable.

M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems," Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009, Univ of Notre Dame, December 1993.

Dynamical systems are always subject to unpredicted external disturbances which may force the plant state off of the controlled trajectory. In the event of such disturbances it is highly desirable that the system still be controlled to the desired final state. Consequently, while not necessary, it is certainly highly desirable that the pre-images of the DES plant state be repellors or attractors of the controlled flow field. In this regard, the above proposition insures the "robust" T-stability of the DES plant.

In the event that the controls enter linearly, the preceding proposition can be rewritten as a set of linear inequality constraints. Let the CSS plant's transition operators be generated by the differential equation

$$\dot{\bar{x}} = f_0(\bar{x}) + \sum_{i=1}^{m} r_i f_i(\bar{x}) \tag{16}$$

where r_i is the ith component of control vector $\bar{r}$ and $f_i : \Re^n \rightarrow \Re^n$ are a family of Lipschitz continuous functions. An arc $(\tilde{u}, \tilde{w})$ of the DES plant will be T-stable if there exists a continuously differentiable positive definite functional $V : \Re^n \rightarrow \Re$ which is zero on a closed proper subset of $\mathbf{w}$ and which satisfies the following conditions.

- for all $\bar{x} \in \Re^n$,

$$\begin{pmatrix} L_0V & L_1V & \cdots & L_mV \end{pmatrix} \begin{pmatrix} 1 \\ r_1 \\ \cdot \\ r_m \end{pmatrix} < 0 \tag{17}$$

- for all $\bar{x}$ in $\mathbf{v}$

$$\begin{pmatrix} L_0V & L_1V & \cdots & L_mV \end{pmatrix} \begin{pmatrix} 1 \\ r_1 \\ \cdot \\ r_m \end{pmatrix} > 0 \tag{18}$$

where L_iV is the Lie derivative of V with respect to the ith vector field, f_i. The two inequalities are conditions that the Lie derivative must satisfy in order for the sets $\mathbf{v}$ and $\mathbf{w}$ to be repellors and attractors, respectively.

The significance of the preceding inequality system is that it is linear in the control vector $\bar{r}$. Since the control vector is determined by the hybrid system's actuator mapping, these conditions also provide a method for determining an actuator mapping, γ, which insures that the DES plant will have T-stable transitions between states. The solution to this problem involves finding the feasible points for the inequality system.

There exist good (i.e., computationally efficient) numerical techniques for finding the feasible points of these systems of linear inequality. One class of algorithms for determining such points uses the so-called "method of centers" [6] to update a hypothesized feasible point. The method of centers computes a sequence of convex bodies and their centers in such a way that the computed centers converge to the feasible point. Depending on the analytic form of the convex bodies and the centers, different types of algorithms are obtained. Examples of such algorithms are the so-called ellipsoid method [2] and interior point methods based on logarithmic barrier functions [3]. Both of these algorithms converge after a finite number of updates and both algorithms have polynomial complexity. Therefore, not only does the inequality system allow the design of a T-stable interface, it provides a computationally efficient method which may be practical for DES plant with a large number of logical states.

4 Summary

One method for hybrid control system design involves extracting a DES representation for the plant's behaviour and then designing a DES controller for that extracted DES plant. For this approach to be successful the extracted DES plant must be a *model* (valid interpretation) of the CSS plant's symbolic behaviour. This paper has formally defined the notion of a valid DES plant using the concept of T-stability. T-stability requires that the DES plant logical symbols and transition times be well-behaved with respect to small perturbations in the plant's initial state. As such, this notion of T-stability is closely related to notions of the DES plant's structural stability. Sufficient conditions for T-stability were framed as a set of inequality constraints. The solution to the constraint system can, for affine plants, can be accomplished using efficient linear programming methods such as the ellipsoid algorithm or more recent interior point procedures. The solution is the hybrid system interface which guarantees the T-stability or validity of the DES plant. These procedures are capable of deciding if an arc of the DES plant arc is T-stable in finite time and with polynomial complexity. This paper has therefore presented a computationally efficient approach to the design of hybrid control system interfaces.

References

[1] A. Benveniste and P. Le Guernic (1990), "Hybrid dynamical systems and the signal language", *IEEE Transactions on Automatic Control*, Vol. 35, no. 5, pp 535-546, May 1990

M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems,"
Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009,
Univ of Notre Dame, December 1993.

[2] R.G. Bland, D. Goldfarb, M.J. Todd (1981), "The Ellipsoid Method: a Survey", *Operations Research*, Vol 29, pp. 1039-1091, 1981.

[3] C.C. Gonzaga (1992), "Path-Following Methods for Linear Programming", *SIAM Review*, Vol 34(2), pp. 167-224, June 1992.

[4] M.D. Lemmon, J.A. Stiver, P.J. Antsaklis, "Event Identification and Intelligent Hybrid Control", in *Hybrid Systems*, Lecture Notes in Computer Science, Vol 736, R.L. Grossman, Anil Nerode, Anders Ravn, and Hans Rischel (editors), pp. 267-296, Springer Verlag, Berlin, 1993.

[5] M.D. Lemmon, J.A. Stiver, and P.J. Antsaklis (1993), "Learning to coordinate control policies of hybrid systems", In *Proceedings of the American Control Conference*, pp. 31-35, San Francisco, CA June 1993

[6] A.S. Nemirovsky, D.B. Yudin (1983), *Problem Complexity and Method Efficiency in Optimization*, Wiley and Sons, 1983.

[7] A. Nerode and W. Kohn (1993), "Models for Hybrid Systems: Automata, Topologies, Stability", in *Hybrid Systems*, Lecture Notes in Computer Science, Vol 736, R.L. Grossman, Anil Nerode, Anders Ravn, and Hans Rischel (editors), pp. 317-356, Springer Verlag, Berlin, 1993.

[8] P. Peleties and R. DeCarlo (1993), "A modeling strategy for hybrid systems based on event structures", In *Discrete Event Dynamical Systems: Theory and Applications 3*, pp. 39-69, Kluwer Academic Publishers, Boston, 1993.

[9] P.J. Ramadge (1990), "On the periodicity of symbolic observations of piecewise smooth discrete-time systems", *IEEE Transactions on Automatic Control*, vol 35, no 7, pp. 807-812, July 1990.

[10] J.A. Stiver and P.J. Antsaklis (1991), "A novel discrete event system approach to modeling and analysis of hybrid control systems", In *Proceedings of the Twenty-Ninth Annual Allerton Conference on Communications, Control, and Computing*, University of Illinois at Urbana-Champaign, Oct. 1991.

[11] J.A. Stiver and P.J. Antsaklis (1993), "DES Supervisor Design for Hybrid Control Systems", *Proc. of the Thirty-First Annual Allerton Conference on Communication Control and Computing*, Univ. of Illinois at Urbana-Champaign, Sept 29-Oct. 1 1993, To appear

M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems,"
Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009,
Univ of Notre Dame, December 1993.

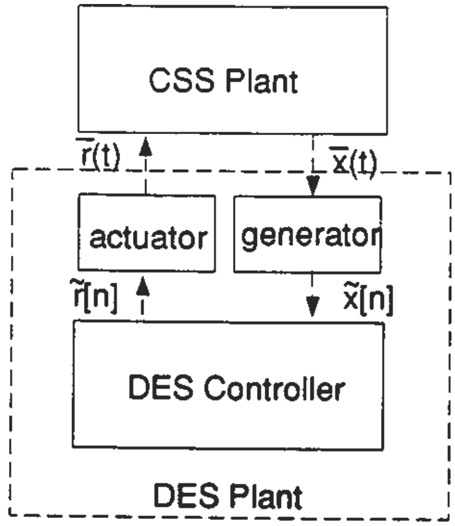

Figure 1: Hybrid Control System Block Diagram

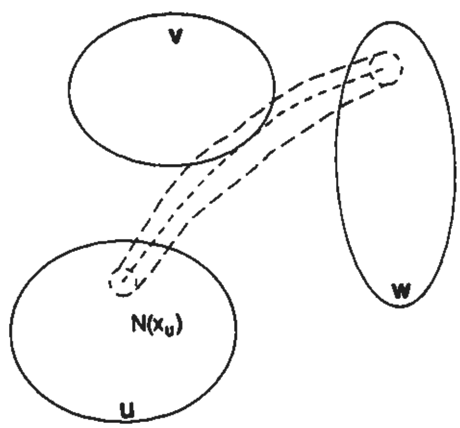

Figure 2: Example of a hybrid system whose logical transition is unstable

M. D. Lemmon and P. J. Antsaklis, "On The Stability of Logical Transitions in Hybrid Control Systems," Technical Report of the ISIS (Interdisciplinary Studies of Intelligent Systems) Group, No. ISIS-93-009, Univ of Notre Dame, December 1993.

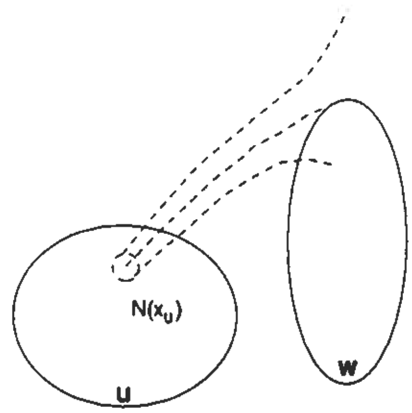

Figure 3: This figure shows an example of unstable arcs in which the transition time T various in a discontinous manner with perturbations of the initial CSS plant state.

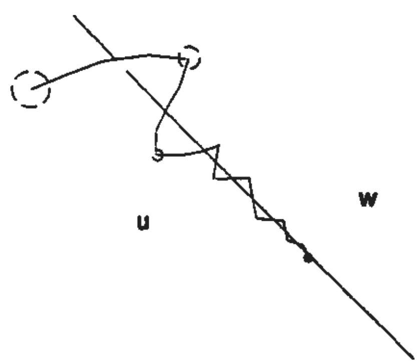

Figure 4: Example of a hybrid system which is not uniformly T-stable.