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Ioannis K. Konstantopoulos and Panos J. Antsaklis
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556
email: antsakli@saturn.ece.nd.edu

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ROBUST STABILITY OF LINEAR CONTINUOUS AND DISCRETE-TIME SYSTEMS UNDER PARAMETRIC UNCERTAINTY

Ioannis K. Konstantopoulos and Panos J. Antsaklis
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556, U.S.A.
email: antsakli@saturn.ece.nd.edu

Abstract

Conditions for robust stability in linear continuous systems are derived, when all matrices of the state-space model are perturbed by independent uncertain parameters and static output feedback is applied. Two different approaches are presented. Then, improved conditions for robust stability in linear discrete-time systems with both unstructured and structured perturbations in the system matrix A are derived. Finally, a sufficient condition for robust stability when again all matrices of the state-space model are perturbed by independent uncertain parameters and static output feedback is applied, is derived. The analysis for all the problems studied above is based on the direct method of Lyapunov. Several examples are used to illustrate the results.

1 Introduction

The problem of robust stability of linear state-space models has been an active area of research for quite some time; see [1], [3], [17] for extensive discussion and references. For the cases of both structured and unstructured parametric uncertainty involving state-space models, results exist for both continuous ([2], [5], [6], [10], [12], [13], [16], [21], [22], [23]) and discrete-time systems ([5], [6], [7], [11], [14], [19], [20]). In all the above papers, the uncertain parameters describe the perturbation in either the open-loop system matrix A or the closed-loop system matrix A_c when state $(A + BK)$ or output feedback $(A + BKC)$

is applied. The uncertainty matrix ΔA for either A or A_c is assumed to be of the form $\Delta A = \sum_{i=1}^{m} \kappa_i A_i$, where $\kappa_i, i = 1,..,m$ denote the uncertain parameters and $A_i, i = 1,..,m$ are known constant matrices. Note that the uncertain parameters enter the uncertainty matrix linearly.

When all the matrices of a state-space model, that is the system matrix A, the input matrix B, and the output matrix C are perturbed and output feedback is applied, then existing literature methods can not be applied directly. This is because the system matrix of the closed-loop system contains now product-terms of the uncertain parameters.

In section 2, we study the linear continuous systems with the state-space description of (1), (2) below, where the state-space matrices are perturbed by independent uncertain parameters, as indicated in (3), (4). Note that the special case of $\kappa_i = \lambda_i = \mu_i, i = 1,..,m$ has been studied in [8] for the discrete-time case and in [18] for both discrete-time and continuous systems. In these papers, the rather restrictive assumption has been made that the system matrices are perturbed by the same uncertain parameters, as indicated above. Here, this assumption is relaxed and two different approaches are presented, distinct from [18], to address the more general problem. It should be noted that although only the static output feedback case is studied, the results apply to the dynamic output feedback case as well. This is because a dynamic output feedback controller of order r applied to a system of order n is equivalent to a static output feedback controller applied to an augmented system of order $n + r$; see for example [10], [18].

In section 3, we concentrate on linear discrete-time systems and present theorems, stemming from the direct method of Lyapunov, that provide sufficient conditions for the robust stability of uncertain systems. First, we study the case of unstructured perturbations ΔA in the system matrix A and then the case of structured perturbations, where $\Delta A = \sum_{i=1}^{m} \kappa_i A_i$. In both cases, we get bounds that improve the ones found via the methodology suggested in [11]. Then, we study the discrete-time systems with the state-space description of (85), (86) below, where again the state-space matrices are perturbed by independent uncertain parameters, as indicated in (87), (88). Note that this is the discrete-time counterpart to the continuous-time case of section 2; therefore all the comments of the previous paragraph, concerning the continuous case, apply here too. In section 4, illustrative examples for all the cases mentioned above are presented. Finally, in section 5, concluding remarks are briefly discussed.

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2 Continuous Systems

2.1 Problem Formulation

We consider linear continuous systems with the state space description

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) & (1) \\ y(t) &= Cx(t) & (2) \end{aligned}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^r$ is the input vector and $y \in \Re^q$ is the output vector. The state-space matrices are described by the following relations

$$A = A_0 + \sum_{i=1}^{m} \kappa_i A_i, \qquad B = B_0 + \sum_{i=1}^{m} \lambda_i B_i \tag{3}$$

$$C = C_0 + \sum_{i=1}^{m} \mu_i C_i \tag{4}$$

where κ_i, λ_i, μ_i denote the real, uncertain parameters which describe the perturbations for the state-space matrices A, B, C respectively. For simplicity, it has been assumed that each of the state matrices involves m distinct uncertain parameters. Note that the analysis that follows can be extended to the case, where the state matrices above have different number of perturbation parameters.

We consider the output feedback law

$$u(t) = Ky(t) \tag{5}$$

where K is a stabilizing ouput feedback matrix for the nominal system (A_0, B_0, C_0). Then the closed loop system is described by the following equations

$$\begin{aligned} \dot{x}(t) &= [\, A + BKC \,]\, x(t) \\ &= [\, A_0 + B_0KC_0 + \sum_{i=1}^{m} (\kappa_i A_i + \lambda_i B_i K C_0 + \mu_i B_0 K C_i) \\ &\quad + \sum_{i,j}^{m} \lambda_i \mu_j B_i K C_j \,]\, x(t) \end{aligned}$$

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$$= [\ \bar{A}_0 + \sum_{i=1}^{m} (\kappa_i A_i + \lambda_i E_i^{\lambda} + \mu_i E_i^{\mu}) + \sum_{i,j} \lambda_i \mu_j E_{ij}\ ]\ x(t) \tag{6}$$

where

$$\begin{aligned} \bar{A}_0 &= A_0 + B_0 K C_0 && (7)\\ E_i^{\lambda} &= B_i K C_0 && (8)\\ E_i^{\mu} &= B_0 K C_i && (9)\\ E_{ij} &= B_i K C_j && (10) \end{aligned}$$

The problem can now be formulated as follows:

"If K is a stabilizing output feedback matrix for the nominal continuous system described by (A_0, B_0, C_0), that is $\bar{A}_0$ stable, find the conditions that have to be satisfied by the parameters κ_i, λ_i, $\mu_i, i = 1, .., m$, so that the closed loop system of (6) remains asymptotically stable."

It has already been stated, in the introduction, that the static output feedback analysis includes the dynamic output feedback case. Note also that the results presented here apply to the linear state feedback case as well. To see this, let $\mu_i = 0, i = 1, .., m$ and $F = KC$ and consider the static state feedback law $u(t) = Fx(t)$. As indicated in [9] and [15], a necessary and sufficient condition for a linear time-invariant system to be stabilizable using a linear time-invariant dynamic state feedback law is that it is stabilizable using a static state feedback law. Hence with the definitions above, we see that our results, when applied to the static state feedback case are general enough to include the dynamic state feedback case.

At this point, we should mention that all the analysis techniques presented here are intended to deal with the problem of product terms of the uncertain parameters that enter the uncertainty matrix. When only the system matrix A is perturbed, or A together with either the input matrix B or the output matrix C, then no such product terms exist. In these cases, the present techniques can definitely be applied as well. Note however that this is a problem for which numerous approaches and useful results can be found in the literature, as indicated in the introduction above.

2.2 First Approach

Since K has been assumed to be a stabilizing gain matrix for the nominal system, then there exists a symmetric positive definite matrix P, which is the unique solution of the Lyapunov equation

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$$P\bar{A}_0 + \bar{A}_0^T P + 2I_n = 0 \tag{11}$$

We define

$$\begin{aligned} P_i &= PA_i + A_i^T P & (12)\\ P_i^\lambda &= PE_i^\lambda + (E_i^\lambda)^T P & (13)\\ P_i^\mu &= PE_i^\mu + (E_i^\mu)^T P & (14)\\ P_{ij} &= \frac{1}{2}\left[PE_{ij} + (E_{ij})^T P\right] & (15) \end{aligned}$$

and

$$\begin{aligned} K &= [\ \kappa_1\ \ \kappa_2\ \ \cdots\ \ \kappa_m\ ]^T & (16)\\ \Lambda &= [\ \lambda_1\ \ \lambda_2\ \ \cdots\ \ \lambda_m\ ]^T & (17)\\ M &= [\ \mu_1\ \ \mu_2\ \ \cdots\ \ \mu_m\ ]^T & (18)\\ \Theta &= [\theta_i] = [\ K^T\ \ \Lambda^T\ \ M^T\ ]^T & (19) \end{aligned}$$

where $P_i, P_i^\lambda, P_i^\mu, P_{ij} \in \Re^{n\times n}$, $K, \Lambda, M \in \Re^m$ and $\Theta \in \Re^{3m}$.

We also define

$$\begin{aligned} \tilde{P} &= [\ P_1^T\ \ P_2^T\ \ \cdots\ \ P_m^T\ ]^T & (20)\\ \tilde{P}_\lambda &= [\ (P_1^\lambda)^T\ \ (P_2^\lambda)^T\ \ \cdots\ \ (P_m^\lambda)^T\ ]^T & (21)\\ \tilde{P}_\mu &= [\ (P_1^\mu)^T\ \ (P_2^\mu)^T\ \ \cdots\ \ (P_m^\mu)^T\ ]^T & (22)\\ \Pi^\star &= [\ \tilde{P}^T\ \ \tilde{P}_\lambda^T\ \ \tilde{P}_\mu^T\ ]^T & (23) \end{aligned}$$

and

$$P_{\lambda\mu} = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ P_{m1} & P_{m2} & \cdots & P_{mm} \end{pmatrix} \tag{24}$$

$$\Pi = \begin{pmatrix} O_{mn} & O_{mn} & O_{mn} \\ O_{mn} & O_{mn} & P_{\lambda\mu} \\ O_{mn} & P_{\lambda\mu}^T & O_{mn} \end{pmatrix} \tag{25}$$

where $\tilde{P}, \tilde{P}_\lambda, \tilde{P}_\mu \in \Re^{mn\times n}$, $\Pi^\star \in \Re^{3mn\times n}$, $P_{\lambda\mu} \in \Re^{mn\times mn}$, $\Pi \in \Re^{3mn\times 3mn}$, and O_{mn} denotes the $\Re^{mn\times mn}$ square matrix that has zero elements.

Theorem 2.2.1 *When the output feedback law (5) is applied to the linear continuous system (1), (2) with structured uncertainties of (3), (4), then the closed loop system (6) remains asymptotically stable, when the uncertainty parameters satisfy the relation*

$$(\sum_{i=1}^{3m} \theta_i^2)\ \lambda_{max}(\frac{\alpha}{2}\ Z + \Pi) < 2 - \lambda_{max}[\frac{1}{2\alpha}(\Pi^\star)^T Z^{-1} \Pi^\star] \tag{26}$$

where $\theta_i, \Pi^\star$, and Π are defined in (19), (23), and (25) respectively, Z can be any positive definite matrix $\in \Re^{3mn \times 3mn}$, and $\lambda_{max}(A)$ denotes the maximum eigenvalue of the matrix A .

Proof: We consider the following Lyapunov function

$$V(x) = x^T P x \tag{27}$$

where P is the unique positive definite matrix defined in (11). The derivative of this function is

$$\begin{aligned}
\dot{V}(x) &= x^T[\ P\bar{A}_0 + \bar{A}_0^T P + \sum_{i=1}^{m} \kappa_i(\ PA_i + A_i^T P\ ) \\
&\quad + \sum_{i=1}^{m} \lambda_i(\ PE_i^\lambda + (E_i^\lambda)^T P\ ) + \sum_{i=1}^{m} \mu_i(\ PE_i^\mu + (E_i^\mu)^T P\ ) \\
&\quad + \sum_{i,j} \lambda_i \mu_j(\ PE_{ij} + E_{ij}^T P\ )\ ]\ x \\
&= x^T\ [\ -2I_n + \sum_{i=1}^{m}(\kappa_i P_i + \lambda_i P_i^\lambda + \mu_i P_i^\mu) +\ 2\sum_{i,j} \lambda_i \mu_j P_{ij}\ ]\ x \qquad (28) \\
&= x^T\ [\ -2I_n + (\Theta \otimes I_n)^T\ \Pi^\star +\ (\Theta \otimes I_n)^T\ \Pi\ (\Theta \otimes I_n)\ ]\ x \qquad (29)
\end{aligned}$$

where the Lyapunov equation (11) and definitions (12)-(25) have been used and $\otimes$ denotes the Kronecker product. For any two suitably dimensioned matrices X, Ψ, and any positive scalar α, the following matrix inequalities hold

$$\begin{aligned}
0 &\leq (\ \alpha\ XZ^{\frac{1}{2}} - \Psi Z^{-\frac{1}{2}}\ )\ (\ \alpha\ XZ^{\frac{1}{2}} - \Psi Z^{-\frac{1}{2}}\ )^T \\
X\Psi^T + \Psi X^T &\leq \alpha XZX^T + \frac{1}{\alpha}\Psi Z^{-1}\Psi^T \qquad (30)
\end{aligned}$$

where Z can be any positive definite matrix of appropriate dimensions. Since $(\Theta \otimes I_n)^T \Pi^\star$ is a symmetric matrix, the previous inequality gives

$$\begin{aligned} 2(\Theta \otimes I_n)^T\,\Pi^\star &= (\Theta \otimes I_n)^T\,\Pi^\star + (\Pi^\star)^T\,(\Theta \otimes I_n) \\ &\leq \alpha\,(\Theta \otimes I_n)^T\,Z\,(\Theta \otimes I_n) + \frac{1}{\alpha}\,(\Pi^\star)^T\,Z^{-1}\,\Pi^\star \end{aligned} \tag{31}$$

Hence, (29) can be rewritten as follows

$$\begin{aligned} \dot{V}(x) &\leq x^T\,[\,-2I_n + \frac{\alpha}{2}\,(\Theta \otimes I_n)^T\,Z\,(\Theta \otimes I_n) \\ &\quad + \frac{1}{2\alpha}\,(\Pi^\star)^T\,Z^{-1}\,\Pi^\star + \,(\Theta \otimes I_n)^T\,\Pi\,(\Theta \otimes I_n)\,]\,x \\ &= x^T\,[\,-2I_n + \frac{1}{2\alpha}\,(\Pi^\star)^T\,Z^{-1}\,\Pi^\star + \,(\Theta \otimes I_n)^T(\frac{\alpha}{2}\,Z + \Pi)(\Theta \otimes I_n)\,]\,x \\ &\leq x^T\,[\,-2I_n + \frac{1}{2\alpha}\,(\Pi^\star)^T\,Z^{-1}\,\Pi^\star + \,\lambda_{max}(\frac{\alpha}{2}\,Z + \Pi)\,(\Theta^T\Theta)\,I_n\,]\,x \\ &= x^T\,\{\,[\,\lambda_{max}(\frac{\alpha}{2}\,Z + \Pi)\,(\sum_{i=1}^{3m}\theta_i^2) - 2\,]\,I_n + \,\frac{1}{2\alpha}\,(\Pi^\star)^T\,Z^{-1}\,\Pi^\star\,\}\,x \\ &= x^T\Phi x \end{aligned} \tag{32}$$

From this relation, we see that in order to maintain $\dot{V}(x) < 0$, it suffices to have $\lambda_{max}(\Phi) < 0$. We know that for any matrix $A \in \Re^{n \times n}$, the following property holds

$$\lambda_i(\beta I_n + A) = \beta + \lambda_i(A), \quad i = 1,..,m \tag{33}$$

where β can be any real number and $\lambda_i(A)$ denotes the ith eigenvalue of the matrix A.

Hence, $\dot{V}(x) < 0$ if the following inequality holds

$$\lambda_{max}[\frac{1}{2\alpha}\,(\Pi^\star)^T\,Z^{-1}\,\Pi^\star]\;+\;\lambda_{max}(\frac{\alpha}{2}\,Z + \Pi)\,(\sum_{i=1}^{3m}\theta_i^2) - 2 < 0 \tag{34}$$

Now (26) follows easily. **QED**

It has been stated that Z can be any positive definite matrix. For the cases, where $Z = I$ gives the best results, that is the largest bounds for the uncertainty parameters, we have the following Lemma. First, we define

$$\xi^\star = \lambda_{max}[\,(\Pi^\star)^T\,\Pi^\star\,] \tag{35}$$

$$\xi = \lambda_{max}(\Pi) \tag{36}$$

Lemma 2.2.2 *(a) If $\xi < 0$ and $|\xi| > \frac{\xi^\star}{8}$, then the closed loop system (6) remains asymptotically stable in the whole parameter space $\Re^{3m}$.*

(b) If $\xi < 0$ and α is selected so that $\alpha < min(\frac{\xi^\star}{4}, 2|\xi|)$, then the whole $\Re^{3m}$ outside the hypersphere with radius

$$R = \frac{2 - \frac{\xi^\star}{2\alpha}}{\frac{\alpha}{2} + \xi} \tag{37}$$

belongs to the solution space.

(c) If $\xi > 0$ and α is selected so that $\alpha > \frac{\xi^\star}{4}$, then the hypersphere with R in (37) above belongs to the solution space.

Proof: (a) If $\xi < 0$, then for $\frac{\alpha}{2} < |\xi|$, we have $\frac{\alpha}{2} + \xi < 0$. Similarly, for $\frac{\alpha}{2} > \frac{\xi^\star}{8}$, we have that $2 - \frac{1}{2\alpha}\xi^\star > 0$. Hence, by selecting $\frac{\xi^\star}{8} < \frac{\alpha}{2} < |\xi|$, we have that the solution space is the whole $\Re^{3m}$.

(b) By selecting $\alpha < min(\frac{\xi^\star}{4}, 2|\xi|)$, we have that $\frac{\alpha}{2} + \xi < 0$ and $2 - \frac{1}{2\alpha}\xi^\star < 0$, which implies that the solution lies outside the hypersphere with radius, as indicated in (37).

(c) If $\xi > 0$, and we select $\alpha > \frac{\xi^\star}{4}$, then both $\frac{\alpha}{2} + \xi > 0$ and $2 - \frac{1}{2\alpha}\xi^\star > 0$. Hence, the hypersphere with radius R of (37) belongs to the solution space. **QED**

It should be noted that Theorem 2.2.1 and Lemma 2.2.2 above provide only sufficient conditions that have to be satisfied by the uncertain parameters in order to maintain closed loop asymptotic stability.

2.3 Second Approach

The main results of this section are given in Theorem 2.3.2. Before we show these results, we need the following theorem, which has been proven in [4].

Theorem 2.3.1 *Consider $\dot{x}(t) = Ax(t)$, where A is a stability matrix; let $P = P^T > 0$ and $Q = Q^T > 0$, so that $A^TP + PA + Q = 0$. Suppose that $A \rightarrow A + \Delta A$, then $\dot{y}(t) = (A + \Delta A)y(t)$ remains asymptotically stable if any of these two equivalent inequalities holds*

$$(\Delta A)\, Q^{-1}\, (\Delta A)^T \;<\; \frac{1}{4}P^{-1}QP \tag{38}$$

$$(\Delta A)^T PQ^{-1}P(\Delta A) \;<\; \frac{1}{4}Q \tag{39}$$

Now, we define

$$\begin{aligned}
\Xi &= PQ^{-1}P && (40)\\
\tilde{A} &= [\ A_1^T\ \ A_2^T\ \ \cdots\ \ A_m^T\ ]^T && (41)\\
\tilde{E}_\lambda &= [\ (E_1^\lambda)^T\ \ (E_2^\lambda)^T\ \ \cdots\ \ (E_m^\lambda)^T\ ]^T && (42)\\
\tilde{E}_\mu &= [\ (E_1^\mu)^T\ \ (E_2^\mu)^T\ \ \cdots\ \ (E_m^\mu)^T\ ]^T && (43)\\
\Sigma^\star &= [\ \tilde{A}^T\ \ \tilde{E}_\lambda^T\ \ \tilde{E}_\mu^T\ ]^T && (44)
\end{aligned}$$

and

$$E_{\lambda\mu} = \begin{pmatrix} \frac{E_{11}}{2} & \frac{E_{12}}{2} & \cdots & \frac{E_{1m}}{2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{E_{m1}}{2} & \frac{E_{m2}}{2} & \cdots & \frac{E_{mm}}{2} \end{pmatrix} \tag{45}$$

$$\tilde{E}_{\lambda\mu} = \begin{pmatrix} \frac{E_{11}}{2} & \frac{E_{21}}{2} & \cdots & \frac{E_{m1}}{2} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{E_{1m}}{2} & \frac{E_{2m}}{2} & \cdots & \frac{E_{mm}}{2} \end{pmatrix} \tag{46}$$

$$\Sigma = \begin{pmatrix} O_{mn} & O_{mn} & O_{mn} \\ O_{mn} & O_{mn} & E_{\lambda\mu} \\ O_{mn} & \tilde{E}_{\lambda\mu} & O_{mn} \end{pmatrix} \tag{47}$$

where $\tilde{A}, \tilde{E}_\lambda, \tilde{E}_\mu \in \Re^{mn \times n}$, $\Sigma^\star \in \Re^{3mn \times n}$, $E_{\lambda\mu}$ and $\tilde{E}_{\lambda\mu} \in \Re^{mn \times mn}$, and $\Sigma \in \Re^{3mn \times 3mn}$.

Theorem 2.3.2 *When the output feedback law (5) is applied to the linear continuous systems (1), (2) with structured uncertainties of (3), (4), then the closed loop system (6) remains asymptotically stable, when the uncertainty parameters satisfy the relation*

$$\begin{aligned}
(\sum_{i=1}^{3m} \theta_i^2) \ &<\ -\frac{[\ \sigma_{max}(\alpha Z + \Xi)\ \sigma_{max}^2(\Sigma^\star)\ ]}{2\ [\sigma_{max}(\ \frac{1}{\alpha}\ \Xi\ Z^{-1}\ \Xi + \Xi\ )\ \sigma_{max}^2(\Sigma)]} \\
&+\ \frac{\sqrt{[\sigma_{max}(\alpha Z + \Xi)\ \sigma_{max}^2(\Sigma^\star)]^2\ + \sigma_{max}(\frac{1}{\alpha}\ \Xi\ Z^{-1}\ \Xi + \Xi)\ \sigma_{max}^2(\Sigma)\ \sigma_{min}(Q)}}{2\ [\sigma_{max}(\ \frac{1}{\alpha}\ \Xi\ Z^{-1}\ \Xi + \Xi\ )\ \sigma_{max}^2(\Sigma)]} \qquad (48)
\end{aligned}$$

where Ξ, $\Sigma^\star$, Σ are as defined in (40), (44), (47) above, and Z can be any positive definite matrix $\in \Re^{n \times n}$.

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Proof: Using (16)-(19), (41)-(47), we can rewrite (6) as follows

$$\dot{x}(t) = (\bar{A}_0 \; + \; \Delta A) \, x(t) \tag{49}$$

where

$$\Delta A = (\Theta \otimes I_n)^T \, \Sigma^\star \; + \; (\Theta \otimes I_n)^T \, \Sigma \, (\Theta \otimes I_n) \tag{50}$$

From Theorem 2.3.1, we see that (39) is a sufficient condition for (49) to remain asymptotically stable.

We define

$$\begin{aligned}
\Gamma_1 &= (\Theta \otimes I_n)^T \, \Sigma^\star && (51)\\
\Gamma_2 &= (\Theta \otimes I_n)^T \, \Sigma \, (\Theta \otimes I_n) && (52)\\
\Lambda_2 &= (P Q^{-1} P) \, \Gamma_2 \; = \; \Xi \, \Gamma_2 && (53)
\end{aligned}$$

With these definitions, we have

$$\begin{aligned}
(\Delta A)^T \, \Xi \, (\Delta A) &= \Gamma_1^T \, \Xi \, \Gamma_1 \; + \; \Gamma_2^T \, \Xi \, \Gamma_2 + \; \Gamma_1^T \, \Xi \, \Gamma_2 \; + \; \Gamma_2^T \, \Xi \, \Gamma_1 \\
&= \Gamma_1^T \, \Xi \, \Gamma_1 \; + \; \Gamma_2^T \, \Xi \, \Gamma_2 + \; \Gamma_1^T \, \Lambda_2 \; + \; \Lambda_2^T \, \Gamma_1 && (54)\\
&\leq \Gamma_1^T \, \Xi \, \Gamma_1 \; + \; \Gamma_2^T \, \Xi \, \Gamma_2 + \; \alpha \, \Gamma_1^T \, Z \, \Gamma_1 \; + \; \frac{1}{\alpha} \, \Lambda_2^T \, Z^{-1} \, \Lambda_2 && (55)\\
&= \Gamma_1^T \, (\alpha Z + \Xi) \, \Gamma_1 + \; \Gamma_2^T \, [\, \frac{1}{\alpha} \, \Xi \, Z^{-1} \, \Xi + \Xi \,] \, \Gamma_2 && (56)
\end{aligned}$$

where obviously (30) was used in (54) for $X = \Gamma_1^T$ and $\Psi = \Lambda_2^T$. From (56), we see that the following relations provide a sufficient condition for (39) to hold

$$\begin{aligned}
\sigma_{max}[(\Delta A)^T \, \Xi \, (\Delta A)] &\leq \sigma_{max}[\Gamma_1^T \, (\alpha Z + \Xi) \, \Gamma_1] + \; \sigma_{max}[\Gamma_2^T \, (\, \frac{1}{\alpha} \, \Xi \, Z^{-1} \, \Xi + \Xi \,) \, \Gamma_2] \\
&\leq \sigma_{max}(\alpha Z + \Xi) \, \sigma_{max}^2(\Sigma^\star) \, (\Theta^T \Theta) \\
&\quad + \sigma_{max}(\, \frac{1}{\alpha} \, \Xi \, Z^{-1} \, \Xi + \Xi \,) \, \sigma_{max}^2(\Sigma) \, (\Theta^T \Theta)^2 \\
&< \sigma_{min}(\frac{1}{4} \, Q) && (57)
\end{aligned}$$

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or finally

$$[\sigma_{max}(\frac{1}{\alpha} \, \Xi \, Z^{-1} \, \Xi + \Xi \,) \, \sigma^2_{max}(\Sigma)] \, (\sum_{i=1}^{m} \theta_i^2)^2 +$$
$$+ \, [\, \sigma_{max}(\alpha Z + \Xi) \, \sigma^2_{max}(\Sigma^\star) \,] \, (\sum_{i=1}^{m} \theta_i^2) - \sigma_{min}(\frac{1}{4} \, Q) \;\; < \;\; 0 \tag{58}$$

We see that the 2 roots of (58) have opposite sign, and therefore the solution is as indicated in (48). **QED**

3 Discrete-Time Systems

3.1 Unstructured Perturbations in A

We consider linear discete-time systems with the state-space description

$$x(k+1) = A \, x(k) \tag{59}$$

where $x \in \Re^n$ is the state vector and A an asymptotically stable matrix. Therefore, for every symmetric positive definite matrix Q, we can find a symmetric positive definite matrix P, which is the unique solution of the Lyapunov equation

$$A^T P A - P + Q = 0 \tag{60}$$

When the state matrix A is perturbed by the matrix ΔA, then for the perturbed system

$$y(k+1) = (\, A + \Delta A \,) \, x(k) \tag{61}$$

the following theorem holds. First define

$$\Omega_1 = A^T P Z^{-1} P A \tag{62}$$

Theorem 3.1.1 *Consider the linear discrete-time system (59), where A is an asymptotically stable matrix that satisfies (60). Suppose that $A \rightarrow A + \Delta A$, then the perturbed system of (61) remains asymptotically stable, if*

$$(\Delta A)^T \, (\alpha Z + P) \, (\Delta A) \; + \; \frac{1}{\alpha} \, \Omega_1 \; < \; Q \tag{63}$$

or

$$\sigma_{max}(\Delta A) \; < \; \sqrt{\frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha} \, \Omega_1)}{\sigma_{max}(\alpha Z + P)}} \tag{64}$$

where P, Q are defined in (60), Ω_1 in (62), Z can be any positive definite matrix, and α is any positive number that satisfies

$$\alpha \; > \; \frac{\sigma_{max}(\Omega_1)}{\sigma_{min}(Q)} \tag{65}$$

Proof: We rewrite (60) as follows

$$(A + \Delta A)^T P (A + \Delta A) - P + Q - (\Delta A)^T P (\Delta A) - A^T P (\Delta A) - (\Delta A)^T P A = 0 \tag{66}$$

Using the direct method of Lyapunov, we see that $(A + \Delta A)$ remains an asymptotically stable matrix, if

$$\tilde{Q} = Q - (\Delta A)^T P (\Delta A) - A^T P (\Delta A) - (\Delta A)^T P A \; > \; 0 \tag{67}$$

Similarly to (30), we know that the following inequalities hold for any positive definite matrix Z and positive number α

$$\begin{aligned} 0 \;\; &\leq \;\; (\, Z^{\frac{1}{2}} X - \frac{1}{\alpha} Z^{-\frac{1}{2}} \Psi \,)^T \, (\, Z^{\frac{1}{2}} X - \frac{1}{\alpha} Z^{-\frac{1}{2}} \Psi \,) \\ X^T \Psi + \Psi^T X \;\; &\leq \;\; \alpha X^T Z X + \frac{1}{\alpha} \Psi^T Z^{-1} \Psi \end{aligned} \tag{68}$$

Applying (68) for $X = \Delta A$ and $\Psi = PA$, we have

$$\begin{aligned} (\Delta A)^T P A + A^T P (\Delta A) \;\; &\leq \;\; \alpha (\Delta A)^T Z (\Delta A) + \frac{1}{\alpha} \, \Omega_1 \\ (\Delta A)^T P (\Delta A) + (\Delta A)^T P A + A^T P (\Delta A) \;\; &\leq \;\; (\Delta A)^T (\alpha Z + P)(\Delta A) + \frac{1}{\alpha} \, \Omega_1 \end{aligned} \tag{69}$$

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In view of (67), (69), we see that

$$\tilde{Q} \;\geq\; Q - (\Delta A)^T(\alpha Z + P)(\Delta A) - \frac{1}{\alpha}\,\Omega_1 \tag{70}$$

Therefore, a sufficient condition for $\tilde{Q}$ to be positive definite is that the RHS of (70) is positive definite, from which (63) follows easily. Note that α can be chosen as any positive number that satisfies (63). Next a sufficient lower bound for α is derived. We know that the following equivalence holds for any two positive definite matrices A, B

$$A \;<\; B \;\Leftrightarrow\; \sigma_{max}(A) \;<\; \sigma_{min}(B) \tag{71}$$

Since

$$\begin{aligned} \sigma_{max}(A+B) \;&\leq\; \sigma_{max}(A) + \sigma_{max}(B) && (72)\\ \sigma_{max}(AB) \;&\leq\; \sigma_{max}(A)\,\sigma_{max}(B) && (73) \end{aligned}$$

we have

$$\begin{aligned} \sigma_{max}[(\Delta A)^T\,(\alpha Z + P)\,(\Delta A) \;+\; \frac{1}{\alpha}\,\Omega_1] \;&\leq\; \sigma_{max}[(\Delta A)^T\,(\alpha Z + P)\,(\Delta A)] \\ &\quad + \sigma_{max}(\frac{1}{\alpha}\,\Omega_1) \\ &\leq\; \sigma^2_{max}(\Delta A)\,\sigma_{max}(\alpha Z + P) + \;\sigma_{max}(\frac{1}{\alpha}\,\Omega_1) \end{aligned} \tag{74}$$

In view of (71), (74), we see that a sufficient condition for (63) to hold is

$$\sigma^2_{max}(\Delta A)\,\sigma_{max}(\alpha Z + P) + \sigma_{max}(\frac{1}{\alpha}\,\Omega_1) \;<\; \sigma_{min}(Q) \tag{75}$$

from which (64) follows easily. Note that α has to satisfy (65), in order to maintain the RHS of (64) positive. **QED**

3.2 Structured Perturbations in A

We consider the case that the perturbation matrix A is described by

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$$\Delta A = \sum_{i=1}^{m} \kappa_i A_i \tag{76}$$

where $\kappa_i, i = 1, .., m$ denote real, uncertain parameters and $A_i, i = 1, .., m$ are constant, known matrices.

We need again the definitions

$$K = [\kappa_i] = [\ \kappa_1 \ \ \kappa_2 \ \ \cdots \ \ \kappa_m\ ]^T \tag{77}$$
$$\tilde{A} = [\ A_1^T \ \ A_2^T \ \ \cdots \ \ A_m^T\ ]^T \tag{78}$$

Theorem 3.2.1 *The linear discrete-time system (61) with structured perturbations of the form of (76) remains asymptotically stable, when the uncertainty parameters satisfy*

$$\sum_{i=1}^{m} \kappa_i^2 < \frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\,\Omega_1)}{\sigma_{max}^2(\tilde{A})\ \sigma_{max}(\alpha Z + P)} \tag{79}$$

where Ω_1, Z, k_i, $\tilde{A}$, are defined in (62), (68), (77), (78) respectively, and α is a positive number that satisfies (65).

Proof: In view of (76), (77), (78) we have

$$\Delta A = (K \otimes I_n)^T\ \tilde{A} \tag{80}$$

In view of (80), we can rewrite (63) as follows

$$[\ (K \otimes I_n)^T\ \tilde{A}\ ]^T\ (\alpha Z + P)\ [\ (K \otimes I_n)^T\ \tilde{A}\ ]\ +\ \frac{1}{\alpha}\,\Omega_1\ <\ Q \tag{81}$$

It can be easily shown that

$$\sigma_{max}^2(K \otimes I_n) = \lambda_{max}[(K \otimes I_n)^T\ (K \otimes I_n)]\ = K^T K \tag{82}$$

From (71), (72), (73), (82), we get the following sufficient condition for (81) to hold

$$\sigma^2_{max}(\tilde{A})\,\sigma_{max}(\alpha Z+P)\,(K^T K)\;+\;\sigma_{max}(\frac{1}{\alpha}\,\Omega_1)\;<\;\sigma_{min}(Q) \tag{83}$$

or equivalently

$$(\sum_{i=1}^{m}\kappa_i^2)\,\sigma^2_{max}(\tilde{A})\,\sigma_{max}(\alpha Z+P)\;<\;\sigma_{min}(Q)-\sigma_{max}(\frac{1}{\alpha}\,\Omega_1) \tag{84}$$

from which (79) follows easily, under the condition that α has to be selected in order to satisfy (65). **QED**

3.3 Perturbations in all System Matrices

We consider the linear discrete-time systems with the state space description

$$x(k+1) = Ax(k)+Bu(k) \tag{85}$$
$$y(k) = Cx(k) \tag{86}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^r$ is the input vector and $y \in \Re^q$ is the output vector. Similarly to section 2.1, the state-space matrices above are described by (3), (4), that is

$$A = A_0+\sum_{i=1}^{m}\kappa_i A_i, \qquad B = B_0+\sum_{i=1}^{m}\lambda_i B_i \tag{87}$$

$$C = C_0+\sum_{i=1}^{m}\mu_i C_i \tag{88}$$

where κ_i, λ_i, μ_i denote the real, uncertain parameters which describe the perturbations for the state-space matrices A, B, C respectively. Once more, it has been assumed that each of the state matrices involves m distinct uncertain parameters and the analysis that follows can be extended to the case, where the state matrices above have different number of perturbation parameters.

We consider the output feedback law

$$u(k) = Ky(k) \tag{89}$$

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where K is a stabilizing ouput feedback matrix for the nominal discrete-time system (A_0, B_0, C_0). Then, similarly to (6), the closed loop system is described by

$$\begin{aligned} x(k+1) &= [A + BKC]\, x(k) \\ &= [\, \bar{A}_0 + \sum_{i=1}^{m}(\kappa_i A_i + \lambda_i E_i^{\lambda} + \mu_i E_i^{\mu}) + \sum_{i,j} \lambda_i \mu_j E_{ij} \,]\, x(k) \end{aligned} \tag{90}$$

where $\bar{A}_0$, E_i^{λ}, E_i^{μ}, E_{ij} are as defined in (7)-(10). The problem can now be formulated as follows:

"If K is a stabilizing output feedback matrix for the nominal discrete-time system described by (A_0, B_0, C_0), that is $\bar{A}_0$ stable, find the conditions that have to be satisfied by the uncertainty parameters κ_i, λ_i, $\mu_i, i = 1,..,m$, so that the closed loop system (90) remains asymptotically stable."

Before we present the main results of this section, we need to recall the previous definitions of (16)-(19), and (41)-(47). We also define

$$\Omega_2 = \bar{A}_0^T P Z^{-1} P \bar{A}_0 \tag{91}$$

Theorem 3.3.1 *When the output feedback law (89) is applied to the linear discrete-time systems (85), (86) with structured uncertainties of (87), (88), then the closed loop system (90) remains asymptotically stable, when the uncertainty parameters satisfy the relation*

$$\begin{aligned} (\sum_{i=1}^{3m} \theta_i^2) \;&<\; \frac{-\sigma_{max}^2(\Sigma^\star)}{2\,\sigma_{max}^2(\Sigma)} \;+ \\ &+\; \frac{\sqrt{\sigma_{max}^2(\alpha Z + P)\,\sigma_{max}^4(\Sigma^\star) \;+\; 2\,\sigma_{max}(\alpha Z + P)\,\sigma_{max}^2(\Sigma)\,[\,\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\,\Omega_2)\,]}}{2\,\sigma_{max}(\alpha Z + P)\,\sigma_{max}^2(\Sigma)} \end{aligned} \tag{92}$$

where $\Sigma^\star$, Σ, Ω_2, are as defined in (44), (47), (91) above, Z can be any positive definite matrix $\in \Re^{n \times n}$, and α can be any positive number that satisfies

$$\alpha \;>\; \frac{\sigma_{max}(\Omega_2)}{\sigma_{min}(Q)} \tag{93}$$

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Proof: Using (16)-(19), (41)-(47), we can rewrite (90) as follows

$$x(k+1) = (\ \bar{A}_0\ +\ \Delta A)\, x(k) \tag{94}$$

where

$$\Delta A = (\Theta \otimes I_n)^T\ \Sigma^\star\ +\ (\Theta \otimes I_n)^T\ \Sigma\ (\Theta \otimes I_n) \tag{95}$$

From Theorem 3.1.1, we have the following sufficient condition for (94) to remain asymptotically stable

$$(\Delta A)^T\ (\alpha Z + P)\ (\Delta A)\ +\ \frac{1}{\alpha}\, \Omega_2\ <\ Q \tag{96}$$

where P, Q, and $\bar{A}_0$ satisfy the Lyapunov equation (60), that is

$$\bar{A}_0^T P \bar{A}_0 - P + Q = 0 \tag{97}$$

We define

$$\Phi_1 = (\alpha Z + P)^{\frac{1}{2}}\, \Gamma_1 \tag{98}$$

$$\Phi_2 = (\alpha Z + P)^{\frac{1}{2}}\, \Gamma_2 \tag{99}$$

With definitions (51), (52) and (98), (99), we have

$$\begin{aligned}
(\Delta A)^T\ (\alpha Z + P)\ (\Delta A) &= \Gamma_1^T\ (\alpha Z + P)\ \Gamma_1\ +\ \Gamma_2^T\ (\alpha Z + P)\ \Gamma_2 \\
&\quad + \Gamma_1^T\ (\alpha Z + P)\ \Gamma_2\ +\ \Gamma_2^T\ (\alpha Z + P)\ \Gamma_1 \\
&= \Gamma_1^T\ (\alpha Z + P)\ \Gamma_1\ +\ \Gamma_2^T\ (\alpha Z + P)\ \Gamma_2 \\
&\quad + \Phi_1^T\ \Phi_2\ +\ \Phi_2^T\ \Phi_1 && (100) \\
&\leq \Gamma_1^T\ (\alpha Z + P)\ \Gamma_1\ +\ \Gamma_2^T\ (\alpha Z + P)\ \Gamma_2 \\
&\quad + \Phi_1^T\ \Phi_1\ +\ \Phi_2^T\ \Phi_2 && (101) \\
&= 2\,\Gamma_1^T\ (\alpha Z + P)\ \Gamma_1\ +\ 2\,\Gamma_2^T\ (\alpha Z + P)\ \Gamma_2 && (102)
\end{aligned}$$

where obviously (68) was used in (100) for $\alpha = 1$ and $Z = I$. From (102), we see that the following relations provide a sufficient condition for (96) to hold

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$$\begin{aligned}
\sigma_{max}[(\Delta A)^T\,(\alpha Z+P)\,(\Delta A)\;+\;\frac{1}{\alpha}\,\Omega_2] \;&\leq\; 2\,\sigma_{max}[\,\Gamma_1^T\,(\alpha Z+P)\,\Gamma_1\,] \\
&\quad +2\,\sigma_{max}[\,\Gamma_2^T\,(\alpha Z+P)\,\Gamma_2\,]\;+\;\sigma_{max}(\frac{1}{\alpha}\,\Omega_2) \\
&\leq\; 2\,\sigma_{max}(\alpha Z+P)\,\sigma_{max}^2(\Sigma^\star)\,(\Theta^T\Theta) \\
&\quad +2\,\sigma_{max}(\alpha Z+P)\,\sigma_{max}^2(\Sigma)\,(\Theta^T\Theta)^2 \\
&\quad +\sigma_{max}(\frac{1}{\alpha}\,\Omega_2) \\
&<\; \sigma_{min}(Q) \qquad (103)
\end{aligned}$$

or finally

$$\begin{aligned}
&[2\,\sigma_{max}(\alpha Z+P)\,\sigma_{max}^2(\Sigma)]\,(\sum_{i=1}^{m}\theta_i^2)^2\;+\;[2\,\sigma_{max}(\alpha Z+P)\,\sigma_{max}^2(\Sigma^\star)]\,(\sum_{i=1}^{m}\theta_i^2) \\
&\qquad +\,[\,\sigma_{max}(\frac{1}{\alpha}\,\Omega_2)-\sigma_{min}(Q)\,]\;<\;0 \qquad (104)
\end{aligned}$$

Selecting α to satisfy (93), we see that the 2 roots of (104) have opposite sign, and therefore the solution is as indicated in (92). **QED**

4 Illustrative Examples

Example 1 Consider the following uncertain continuous system

$$A=\begin{pmatrix}-1 & -1\\ 0 & 2\end{pmatrix}+\;\kappa_1\begin{pmatrix}5 & -2\\ -8 & 3\end{pmatrix} \qquad (105)$$

$$B=\begin{pmatrix}1\\ -3\end{pmatrix}+\;\lambda_1\begin{pmatrix}1\\ 0\end{pmatrix} \qquad (106)$$

$$C=(\,0\quad 1\,)+\;\mu_1\,(\,-3\quad 1\,) \qquad (107)$$

Note that this system was studied in [18] under the restriction that $\kappa_1=\lambda_1=\mu_1=\kappa$. First we apply (26) of Theorem 2.2.1. For a given output feedback gain $K=3$, let $\alpha=63$ and

$$Z = \begin{pmatrix} 0.9762 & -0.0001 & -0.0269 & 0.0003 & -0.0272 & -0.0001 \\ -0.0001 & 0.8706 & -0.0002 & 0.0106 & -0.0001 & -0.0111 \\ -0.0269 & -0.0002 & 0.8736 & 0.0005 & 0.0638 & 0.0254 \\ 0.0003 & 0.0106 & 0.0005 & 0.8471 & 0.0002 & 0.0289 \\ -0.0272 & -0.0001 & 0.0638 & 0.0002 & 0.8757 & -0.0252 \\ -0.0001 & -0.0111 & 0.0254 & 0.0289 & -0.0252 & 0.9391 \end{pmatrix} \tag{108}$$

Then

$$\kappa_1^2 \; + \; \lambda_1^2 \; + \; \mu_1^2 \; < \; (0.1806)^2 \tag{109}$$

Then, we apply (48) of Theorem 2.3.2 for the same output feedback gain K. Now, the largest hypersphere radius is obtained, when we select $Q = 2I$, $\alpha = 0.003$ and

$$Z = \begin{pmatrix} 0.9518 & 0.0094 \\ 0.0094 & 0.9982 \end{pmatrix} \tag{110}$$

In that case, we obtain

$$\kappa_1^2 \; + \; \lambda_1^2 \; + \; \mu_1^2 \; < \; (0.0293)^2 \tag{111}$$

Now, we consider a different problem. Specifically, we want to find the output feedback gain K that maximizes the hypersphere radius, when the second approach is followed; note that $K = 3$ gave the best results for the first approach. Now, the largest bound is obtained, when we consider output feedback gain $K = 1$, $Q = 2I$, $\alpha = 0.0008$ and

$$Z = \begin{pmatrix} 5.3644 & -1.9444 \\ -1.9444 & 6.7889 \end{pmatrix} \tag{112}$$

Then

$$\kappa_1^2 \; + \; \lambda_1^2 \; + \; \mu_1^2 \; < \; (0.0696)^2 \tag{113}$$

Note the considerable improvement compared to (111).

Example 2 Consider the following uncertain discrete-time system from [11]

$$x(k) = (A + \Delta A)\, x(k) \tag{114}$$

where

$$A = \begin{pmatrix} 0.20 & 0.30 \\ 0.10 & -0.15 \end{pmatrix} \tag{115}$$

Using (64) of Theorem 3.1.1 for $Q = I$, $\alpha = 0.2702$ and

$$Z = \begin{pmatrix} 2.0399 & -0.2037 \\ -0.2037 & 1.4586 \end{pmatrix} \tag{116}$$

we obtain

$$\sigma_{max}(\Delta A) \ < \ 0.6787 \tag{117}$$

which compares favorably to the result of [11], which is

$$\sigma_{max}(\Delta A) \ < \ 0.6373 \tag{118}$$

__Example 3__ Consider the same nominal system as before, but now with structured perturbations of the form of (76), with $m = 3$ and

$$A_1 = \begin{pmatrix} 10 & 0.1 \\ -1 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.5 & 9 \\ 0 & -3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0.6 \\ 1 & 0.3 \end{pmatrix} \tag{119}$$

Using (79) of Theorem 3.2.1 for $Q = I$, $\alpha = 0.40$ and

$$Z = \begin{pmatrix} 1.3462 & -0.1184 \\ -0.1184 & 0.8786 \end{pmatrix} \tag{120}$$

we obtain

$$\kappa_1^2 \ + \ \kappa_2^2 \ + \ \kappa_3^2 \ < \ (0.0606)^2 \tag{121}$$

that is a sphere with radius $R = 0.0606$, whereas the method suggested in [11] gives

$$|\kappa_i| \ < \ 0.0348 \qquad i = 1, 2, 3 \tag{122}$$

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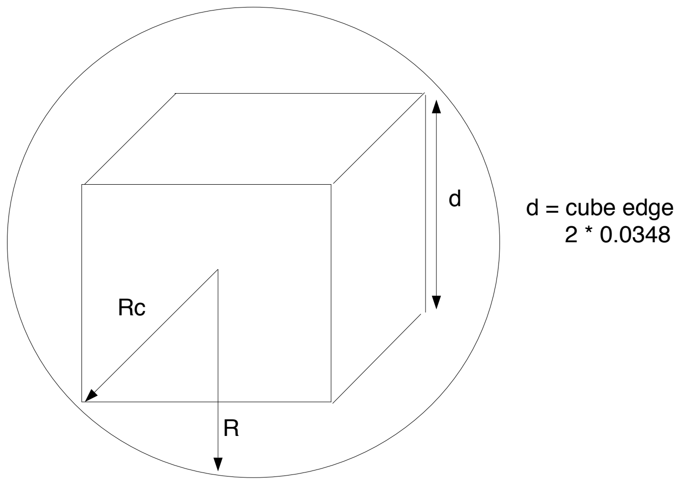

Figure 1: Example 3

Note that, as we can see in Fig. 1, the cube implied by (122) is completely included in the sphere of (121), which shows that our bound is less conservative than the one of [11].

Example 4 Consider the following uncertain discrete-time system

$$A = \begin{pmatrix} -1 & 1.20 \\ 0.10 & -0.15 \end{pmatrix} + \kappa_1 \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \tag{123}$$

$$B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{124}$$

$$C = \begin{pmatrix} 1.2 & -1.5 \end{pmatrix} + \mu_1 \begin{pmatrix} -1 & 1 \end{pmatrix} \tag{125}$$

For a given output feedback gain of $K = 0.80$, we use (92) of Theorem 3.3.1 for $Q = 2I$, $\alpha = 0.35$ and

$$Z = \begin{pmatrix} 0.8160 & 0.0345 \\ 0.0345 & 1.2865 \end{pmatrix} \tag{126}$$

to obtain

$$\kappa_1^2 + \lambda_1^2 + \mu_1^2 < (0.2654)^2 \tag{127}$$

Note that the values of α and Z in all the examples above have been decided experimentally to give the best bound, that is the largest radius for the hypersphere within which the uncertain parameters can vary. Note also that for all the cases that Q had to be selected in the discrete Lyapunov equations (examples 2-4 above), the choice of $Q = \alpha I$ gave the best results.

5 Conclusions

In this paper, conditions for robust stability in continuous systems with output feedback controllers, when independent uncertain parameters describe the perturbations of all the state-space matrices, have been derived. Two different approaches have been presented. Then, conditions for robust stability in linear discrete-time systems with both unstructured and structured perturbations in the system matrix A have been derived. These conditions provide bounds that improve the ones found via the methodology suggested in [11]. A sufficient condition for robust stability in discrete-time systems with output feedback controllers, when independent uncertain parameters describe the perturbations of all the state-space matrices, has also been derived. The approach for all the problems studied above is based on Lyapunov techniques and several examples have been used to illustrate the results.

Issues to be addressed include reduction of conservatism; the extension of the present results to the case where the bounds do not have to be necessarily symmetric with respect to the origin; and the study of the case where the parameters are nonlinear functions of an uncertainty. Another issue that would be of considerable interest is the development of a systematic way-procedure, possibly based on optimization techniques, to obtain the optimal positive matrix needed for our theorems above, where with optimal matrix we mean the matrix that could give the best bounds.

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