I. K. Konstantopoulos, P. J. Antsaklis, "New Bounds for Robust Stability of Continuous and Discrete-Time Systems under Parametric Uncertainty," K ybernetika , Vol 31, No. 6, pp. 621-634, December 1995.

KYBERNETIKA — VOLUME 31 (1995), NUMBER 6, PAGES 623 – 636

NEW BOUNDS FOR ROBUST STABILITY OF CONTINUOUS AND DISCRETE–TIME SYSTEMS UNDER PARAMETRIC UNCERTAINTY

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New conditions for robust stability in linear continuous and discrete-time systems are derived, when all matrices of the state-space model are perturbed by uncertain parameters and static output feedback is applied. Also, new conditions for robust stability in linear discrete-time systems with both unstructured and structured perturbations in the system matrix A are derived. The analysis is based on the direct method of Lyapunov and several examples are used to illustrate the results.

1. INTRODUCTION

The problem of robust stability of linear state-space models has been an active area of research for quite some time; see [1], [4], [18] for extensive discussion and references. For the cases of both structured and unstructured parametric uncertainty involving state-space models, results exist for both continuous ([2], [6], [9], [14], [15], [17], [22], [23], [24]) and discrete-time systems ([6], [7], [10], [16], [20], [21]). In all the above papers, the uncertain parameters describe the perturbation in either the open-loop system matrix A or the closed-loop system matrix A_c, when state $(A + BK)$ or output feedback $(A + BKC)$ is applied. The uncertainty matrix ΔA for either A or A_c is assumed to be of the form $\Delta A = \sum_{i=1}^{m} \kappa_i A_i$, where κ_i, $i = 1, \ldots, m$ denote the uncertain parameters and A_i, $i = 1, \ldots, m$ are known constant matrices. Note that the uncertain parameters enter the uncertainty matrix linearly.

When all matrices of a state-space model, that is the system matrix A, the input matrix B, and the output matrix C are perturbed and output feedback is applied, then the above literature methods can not be applied directly, because the system matrix of the closed-loop system now contains product-terms of the uncertain parameters. However, this case of structured uncertainties in all state-space matrices, namely (A, B, C), has been investigated in [8] for discrete-time systems, under certain restrictions imposed on the uncertain parameters. Although sufficient conditions for stability are provided, no explicit analytic way is presented to derive the stability bounds for the general case, when no restrictions are imposed on the uncertain parameters. In [19], the same problem has been studied for both continuous and discrete-time systems.

Here, we present a new approach which is based on the selection of a positive definite matrix and a positive number. In Section 2, we study linear continuous systems with the state-space description of (1) below, where all state-space matrices are perturbed by uncertain parameters, as indicated in (2). The proposed approach gives results at least as good as the ones derived by the method of [19]. In Section 3, we present theorems, stemming from the direct method of Lyapunov, that provide sufficient conditions for the robust stability of linear discrete-time systems. First, we study the case of unstructured perturbations in the system matrix A and then the case of structured perturbations, (35) below. In both cases, the present approach improves previous results obtained via Lyapunov techniques in [10]. Finally, we study the discrete-time systems of (40) below, where again all state-space matrices are perturbed by uncertain parameters, as indicated in (2) and obtain results comparable to the ones provided by the method of [19]. In Section 4, illustrative examples for all the cases mentioned above are presented and in Section 5, concluding remarks are briefly discussed.

It should be mentioned that although the present paper presents analysis results, the theorems established for the discrete-time cases have already been used successfully for synthesis studies in [13]. Finally note that although only the static output feedback case is studied in both Sections 2 and 3, the results apply to the dynamic output feedback case as well. This is because a dynamic output feedback controller of order r applied to a system of order n is equivalent to a static output feedback controller applied to an augmented system of order $n+r$; see for example [9], [19].

It should also be noted that synthesis results for the cases of structured perturbations in all system matrices based on H_∞ techniques have also appeared in the literature, [3], [5], [25]. Note however that in [3], [5] no specific information about the uncertainty bounds that describe the uncertainty matrices is provided, and in [25] no explicit way is presented to compute the uncertainty bounds, which are decided experimentally via the ellipsoidal method. Here, as mentioned before, we present an analysis technique for state-space models without exogenous disturbances, and for a given controller we present simple ways to compute the stability bounds for the uncertain parameters that describe the uncertainty matrices.

2. CONTINUOUS SYSTEMS

We consider the linear continuous system with the state space description

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{1}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^r$ is the input vector and $y \in \Re^q$ is the output vector. The state-space matrices are described by

$$A = A_0 + \sum_{i=1}^{m} \theta_i A_i, \quad B = B_0 + \sum_{i=1}^{m} \theta_i B_i, \quad C = C_0 + \sum_{i=1}^{m} \theta_i C_i, \tag{2}$$

where θ_i, $i = 1, \ldots, m$ denote the real, uncertain parameters which describe the perturbations in A, B, C respectively. We consider the output feedback law

$$u(t) = Ky(t), \tag{3}$$

where K is a stabilizing ouput feedback matrix for the nominal system (A_0, B_0, C_0). Then, the closed loop system is described by

$$\begin{aligned} \dot{x}(t) &= [\, A + BKC \,]\, x(t) \\ &= \left[\bar{A}_0 + \sum_{i=1}^{m} \theta_i (A_i + E_i^B + E_i^C) + \sum_{i,j} \theta_i \theta_j E_{ij} \right] x(t), \end{aligned} \tag{4}$$

where

$$\bar{A}_0 = A_0 + B_0 K C_0, \quad E_i^B = B_i K C_0, \quad E_i^C = B_0 K C_i, \quad E_{ij} = B_i K C_j. \tag{5}$$

The problem can now be formulated as follows:

"If K is a stabilizing output feedback matrix for the nominal continuous system described by (A_0, B_0, C_0), that is $\bar{A}_0$ stable, find the conditions that have to be satisfied by the uncertain parameters θ_i, $i = 1, \ldots, m$, so that the closed loop system of (4) remains asymptotically stable."

Note that the approach presented here is intended to deal with the problem of product terms of the uncertain parameters. A second approach based on the methodology of Section 3.3 can be found in [11], [12]. When only the system matrix A is perturbed, or A together with either B or C, then no such product terms exist. In these cases, the present techniques can definitely be applied as well. Note however that this is a problem for which numerous approaches and useful results can be found in the literature, as indicated in the introduction above.

We now proceed with the solution of the problem stated above. Since K is a stabilizing gain matrix for the nominal system, there exists a symmetric positive definite matrix P, which is the unique solution of the Lyapunov equation

$$P\bar{A}_0 + \bar{A}_0^T P + 2I_n = 0. \tag{6}$$

Note that we have chosen $Q = 2I_n$ as the positive definite matrix needed in the Lyapunov equation above. This choice was made because it facilitates our computations below. This is not the case, however, in Section 3, where we study discrete-time systems and consider any positive definite matrix in the discrete-time Lyapunov equation of (22). The optimal choice of Q is not an issue of interest here. Note, however, that there exist optimization procedures for systematically choosing the Lyapunov matrix Q, as in [14].

We define

$$P_i = PA_i + A_i^T P, \qquad P_i^B = PE_i^B + (E_i^B)^T P \tag{7}$$

$$P_i^C = PE_i^C + (E_i^C)^T P, \qquad P_{ij} = PE_{ij} + (E_{ij})^T P \tag{8}$$

$$\tilde{P} = [\, P_1 \;\, P_2 \cdots P_m \,]^T, \qquad \tilde{P}_B = [\, P_1^B \;\, P_2^B \cdots P_m^B \,]^T \tag{9}$$

$$\tilde{P}_C = [\, P_1^C \;\, P_2^C \cdots P_m^C \,]^T, \qquad \Pi^\star = \tilde{P} + \tilde{P}_B + \tilde{P}_C \tag{10}$$

$$\Pi = \begin{pmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ P_{m1} & P_{m2} & \cdots & P_{mm} \end{pmatrix} \tag{11}$$

$$\Theta = [\,\theta_1 \;\; \theta_2 \;\; \cdots \theta_m\,]^T, \tag{12}$$

where $P_i, P_i^B, P_i^C, P_{ij} \in \Re^{n\times n}$, $\tilde{P}, \tilde{P}_B, \tilde{P}_C, \Pi^\star \in \Re^{mn\times n}$, $\Pi \in \Re^{mn\times mn}$, and $\Theta \in \Re^m$.

Theorem 2.1. When the output feedback law (3) is applied to the linear continuous system of (1) with structured uncertainties of (2), then the closed loop system (4) remains asymptotically stable, when the uncertain parameters satisfy the relation

$$\left(\sum_{i=1}^{m} \theta_i^2\right) \;\; \lambda_{\max}\left(\frac{\alpha}{2}\, Z + \Pi\right) < 2 - \lambda_{\max}\left(\frac{1}{2\alpha}\,(\Pi^\star)^T\, Z^{-1}\Pi^\star\right), \tag{13}$$

where $\Pi^\star$ and Π are defined in (10) and (11) respectively, Z can be any positive definite matrix $\in \Re^{mn\times mn}$, α can be any positive number, and $\lambda_{\max}(A)$ denotes the maximum eigenvalue of the matrix A.

Proof. We consider the Lyapunov function $V(x) = x^T P x$, where P is the unique positive definite matrix of (6). The derivative of this function is

$$\begin{aligned} \dot{V}(x) \;\; &= \;\; x^T \Bigg[P\bar{A}_0 + \bar{A}_0^T P + \sum_{i=1}^{m} \theta_i \, [\, PA_i + A_i^T P + PE_i^B + (E_i^B)^T P \\ &\qquad + PE_i^C + (E_i^C)^T P\,] + \sum_{i,j} \theta_i\theta_j(\, PE_{ij} + E_{ij}^T P\,) \Bigg] \; x \\ &= \;\; x^T \; [\, -2I_n + (\Theta\otimes I_n)^T\,\Pi^\star + \; (\Theta\otimes I_n)^T\,\Pi\,(\Theta\otimes I_n)\,] \; x, \end{aligned} \tag{14}$$

where the Lyapunov equation (6) and definitions (7)-(12) have been used and $\otimes$ denotes the Kronecker product. For any two suitably dimensioned matrices X, Ψ, and any positive scalar α, the following matrix inequalities hold

$$\begin{aligned} 0 \;\; &\le \;\; \left(\alpha\, XZ^{\frac{1}{2}} - \Psi Z^{-\frac{1}{2}}\right)\left(\alpha\, XZ^{\frac{1}{2}} - \Psi Z^{-\frac{1}{2}}\right)^T \\ X\Psi^T + \Psi X^T \;\; &\le \;\; \alpha XZX^T + \frac{1}{\alpha}\Psi Z^{-1}\Psi^T, \end{aligned} \tag{15}$$

where Z can be any positive definite matrix of appropriate dimensions.
Since $(\Theta\otimes I_n)^T\Pi^\star$ is a symmetric matrix, the previous inequality gives

$$\begin{aligned} 2(\Theta\otimes I_n)^T\,\Pi^\star \;\; &= \;\; (\Theta\otimes I_n)^T\,\Pi^\star + (\Pi^\star)^T\,(\Theta\otimes I_n) \\ &\le \;\; \alpha\,(\Theta\otimes I_n)^T\,Z\,(\Theta\otimes I_n) + \frac{1}{\alpha}\,(\Pi^\star)^T\,Z^{-1}\,\Pi^\star. \end{aligned} \tag{16}$$

Hence, (14) can be rewritten as follows

$$\begin{aligned}
\dot{V}(x) &\le x^T \left[-2I_n + \frac{1}{2\alpha}(\Pi^\star)^T Z^{-1}\Pi^\star + (\Theta\otimes I_n)^T\left(\frac{\alpha}{2}Z+\Pi\right)(\Theta\otimes I_n)\right]x \\
&\le x^T \left[-2I_n + \frac{1}{2\alpha}(\Pi^\star)^T Z^{-1}\Pi^\star + \lambda_{\max}\left(\frac{\alpha}{2}Z+\Pi\right)(\Theta^T\Theta)\, I_n\right]x \\
&= x^T\left\{\left[\lambda_{\max}\left(\frac{\alpha}{2}Z+\Pi\right)\left(\sum_{i=1}^m \theta_i^2\right)-2\right]I_n + \frac{1}{2\alpha}(\Pi^\star)^T Z^{-1}\Pi^\star\right\}x \\
&= x^T\Phi x. \qquad (17)
\end{aligned}$$

To maintain $\dot{V}(x)<0$, it suffices to have $\lambda_{\max}(\Phi)<0$. For any matrix $A\in\Re^{n\times n}$ and any real number β, we have $\lambda_i(\beta I_n + A) = \beta + \lambda_i(A)$, $i=1,\ldots,m$, where $\lambda_i(A)$ denotes the ith eigenvalue of the matrix A. Hence, $\dot{V}(x)<0$ if

$$\lambda_{\max}\left(\frac{1}{2\alpha}(\Pi^\star)^T Z^{-1}\Pi^\star\right) + \lambda_{\max}\left(\frac{\alpha}{2}Z+\Pi\right)\left(\sum_{i=1}^m \theta_i^2\right) - 2 < 0. \qquad (18)$$

Now (13) follows easily. □

It has been stated that Z can be any positive definite matrix. Note that the optimal selection of Z is not discussed here and remains an issue of future research. In the same respect, the positive number α that maximizes the stability bounds above can be selected experimentally, that is by testing several positive values of α and choosing the one that maximizes the stability region. Note that the above remarks for Z and α also hold for the discrete-time cases that are studied in Section 3 that follows.

For the cases, where $Z=I$ gives the largest bounds for the uncertain parameters, the following lemma can easily be proven; details in [11]. First, we define

$$\xi^\star = \lambda_{\max}\left((\Pi^\star)^T\Pi^\star\right), \qquad \xi = \lambda_{\max}(\Pi). \qquad (19)$$

Lemma 2.2. (a) If $\xi<0$ and $|\xi|>\frac{\xi^\star}{8}$, then the closed loop system (4) remains asymptotically stable in the whole parameter space $\Re^m$.

(b) If $\xi<0$ and α is selected so that $\alpha<\min\left(\frac{\xi^\star}{4},\, 2|\xi|\right)$, then the whole $\Re^m$ outside the hypersphere with radius

$$R^2 = \frac{2-\frac{\xi^\star}{2\alpha}}{\frac{\alpha}{2}+\xi} \qquad (20)$$

belongs to the solution space.

(c) If $\xi>0$ and α is selected so that $\alpha>\frac{\xi^\star}{4}$, then the hypersphere with R in (20) above belongs to the solution space.

3. DISCRETE–TIME SYSTEMS

3.1. Unstructured perturbations in A

We consider the linear discete-time system with the state-space description

$$x(k+1) = A\,x(k), \tag{21}$$

where $x \in \Re^n$ is the state vector and A an asymptotically stable matrix. Then, for every symmetric positive definite matrix Q, we can find a symmetric positive definite matrix P, which is the unique solution of the Lyapunov equation

$$A^T PA - P + Q = 0. \tag{22}$$

When A is perturbed by the matrix ΔA, then for the perturbed system

$$y(k+1) = (\,A + \Delta A\,)\,y(k) \tag{23}$$

the following theorem holds. First define

$$\Omega_1 = A^T P Z^{-1} PA. \tag{24}$$

Theorem 3.1.1. Consider the linear discrete-time system (21), where A is an asymptotically stable matrix that satisfies (22). Suppose that $A \to A + \Delta A$, then the perturbed system of (23) remains asymptotically stable, if

$$(\Delta A)^T\,(\alpha Z + P)\,(\Delta A) \;+\; \frac{1}{\alpha}\,\Omega_1 \;<\; Q \tag{25}$$

or

$$\sigma_{\max}(\Delta A) \;<\; \sqrt{\frac{\sigma_{\min}(Q) - \sigma_{\max}(\frac{1}{\alpha}\,\Omega_1)}{\sigma_{\max}(\alpha Z + P)}}, \tag{26}$$

where P, Q are defined in (22), Ω_1 in (24), Z can be any positive definite matrix $\in \Re^{n\times n}$, and α is any positive number that satisfies

$$\alpha \;>\; \frac{\sigma_{\max}(\Omega_1)}{\sigma_{\min}(Q)}. \tag{27}$$

Proof. We rewrite (22) as follows

$$(A+\Delta A)^T P(A+\Delta A) - P + Q - (\Delta A)^T P(\Delta A) - A^T P(\Delta A) - (\Delta A)^T PA = 0. \tag{28}$$

Using the direct method of Lyapunov, we see that $(A + \Delta A)$ remains an asymptotically stable matrix, if

$$\tilde{Q} = Q - (\Delta A)^T P(\Delta A) - A^T P(\Delta A) - (\Delta A)^T PA \;>\; 0. \tag{29}$$

In view of (15) for $X = (\Delta A)^T$, $\Psi = (PA)^T$ and (24), we have

$$(\Delta A)^T PA + A^T P(\Delta A) \le \alpha(\Delta A)^T Z(\Delta A) + \frac{1}{\alpha}\,\Omega_1 \tag{30}$$

$$Q - (\Delta A)^T(\alpha Z + P)(\Delta A) - \frac{1}{\alpha}\,\Omega_1 \le \tilde{Q}. \tag{31}$$

A sufficient condition for $\tilde{Q}$ to be positive definite is that the LHS of (31) is positive definite for some α, from which (25) follows easily. Note that α can be chosen as any positive number that satisfies (25). Next a sufficient lower bound for α is derived. For any positive definite matrices A, B

$$A < B \Leftrightarrow \sigma_{\max}(A) < \sigma_{\min}(B). \tag{32}$$

Defining $\Xi_1 = (\Delta A)^T\,(\alpha Z + P)\,(\Delta A) + \frac{1}{\alpha}\,\Omega_1$, we have

$$\begin{aligned} \sigma_{\max}(\Xi_1) &\le \sigma_{\max}\left((\Delta A)^T\,(\alpha Z + P)\,(\Delta A)\right) + \sigma_{\max}\left(\frac{1}{\alpha}\,\Omega_1\right) \\ &\le \sigma^2_{\max}(\Delta A)\,\sigma_{\max}(\alpha Z + P) + \sigma_{\max}\left(\frac{1}{\alpha}\,\Omega_1\right). \end{aligned} \tag{33}$$

In view of (32), (33), a sufficient condition for (25) to hold is

$$\sigma^2_{\max}(\Delta A)\,\sigma_{\max}(\alpha Z + P) + \sigma_{\max}\left(\frac{1}{\alpha}\,\Omega_1\right) < \sigma_{\min}(Q) \tag{34}$$

from which (26) follows easily. Note that α has to satisfy (27), in order to maintain the RHS of (26) positive. □

3.2. Structured perturbations in A

We consider the case where the asymptotically stable matrix A is perturbed by

$$\Delta A = \sum_{i=1}^{m} \theta_i A_i = (\Theta \otimes I_n)^T\,\tilde{A}, \tag{35}$$

where κ_i, $i = 1, \ldots, m$ denote real, uncertain parameters and A_i, $i = 1, \ldots, m$ are constant, known matrices, and the following definition has been used

$$\tilde{A} = [\,A_1^T \;\; A_2^T \;\; \cdots \;\; A_m^T\,]^T. \tag{36}$$

Theorem 3.2.1. The linear discrete-time system (23) with structured perturbations of the form of (35) remains asymptotically stable, when the uncertain parameters satisfy

$$\sum_{i=1}^{m} \theta_i^2 < \frac{\sigma_{\min}(Q) - \sigma_{\max}(\frac{1}{\alpha}\,\Omega_1)}{\sigma^2_{\max}(\tilde{A})\,\sigma_{\max}(\alpha Z + P)}, \tag{37}$$

where Ω_1, $\tilde{A}$ are defined in (24), (36) respectively, Z can be any positive definite matrix $\in \Re^{n\times n}$ and α is any positive number that satisfies (27).

Proof. It can easily be shown that

$$\sigma^2_{\max}(\Theta \otimes I_n) = \Theta^T\Theta = \sum_{i=1}^{m} \theta_i^2. \tag{38}$$

In view of (35), we easily get

$$\sigma^2_{\max}(\Delta A) < \left(\sum_{i=1}^{m} \theta_i^2\right) \sigma^2_{\max}(\tilde{A}). \tag{39}$$

Therefore, (37) follows easily as a sufficient condition for (26), under the condition that α satisfies (27). □

3.3. Perturbations in all system matrices

We consider the linear discrete-time system with the state space description

$$x(k+1) = Ax(k) + Bu(k), \qquad y(k) = Cx(k), \tag{40}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^r$ is the input vector and $y \in \Re^q$ is the output vector and, as in Section 2, the state-space matrices above are described by (2). We consider the output feedback law

$$u(k) = Ky(k), \tag{41}$$

where K is a stabilizing ouput feedback matrix for the nominal discrete-time system (A_0, B_0, C_0). Then, similarly to (4), the closed loop system is

$$x(k+1) = \left[\bar{A}_0 + \sum_{i=1}^{m} \theta_i \left(A_i + E_i^B + E_i^C\right) + \sum_{i,j} \theta_i\theta_j E_{ij}\right] x(k). \tag{42}$$

The problem can now be formulated as follows:

"If K is a stabilizing output feedback matrix for the nominal discrete-time system (A_0, B_0, C_0), that is $\bar{A}_0$ stable, find the conditions that have to be satisfied by the uncertain parameters θ_i, $i = 1, \dots, m$, so that the closed loop system (42) remains asymptotically stable."

Define

$$\tilde{E}_B = [(E_1^B)^T \ (E_2^B)^T \cdots (E_m^B)^T]^T, \qquad \tilde{E}_C = [(E_1^C)^T \ (E_2^C)^T \cdots (E_m^C)^T]^T \tag{43}$$

$$\Sigma^* = \tilde{A} + \tilde{E}_B + \tilde{E}_C, \qquad \Omega_2 = \bar{A}_0^T P Z^{-1} P \bar{A}_0 \tag{44}$$

$$\Sigma = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ E_{m1} & E_{m2} & \cdots & E_{mm} \end{pmatrix}, \tag{45}$$

where $\tilde{E}_B$, $\tilde{E}_C$, $\Sigma^* \in \Re^{mn\times n}$, and $\Sigma \in \Re^{mn\times mn}$.

Theorem 3.3.1. When the output feedback law (41) is applied to the linear discrete-time system (40) with structured uncertainties of (2), then the closed loop system (42) remains asymptotically stable, when the uncertainty parameters satisfy the relation

$$\left(\sum_{i=1}^{m} \theta_i^2\right) < \frac{-\sigma_{\max}^2(\Sigma^*)}{2\,\sigma_{\max}^2(\Sigma)} + \tag{46}$$
$$+\frac{\sqrt{\sigma_{\max}^2(\alpha Z+P)\,\sigma_{\max}^4(\Sigma^*)+2\,\sigma_{\max}(\alpha Z+P)\,\sigma_{\max}^2(\Sigma)\,[\,\sigma_{\min}(Q)-\sigma_{\max}(\frac{1}{\alpha}\,\Omega_2)\,]}}{2\,\sigma_{\max}(\alpha Z+P)\,\sigma_{\max}^2(\Sigma)},$$

where Σ^* and Ω_2 are defined in (44), Σ in (45), Z can be any positive definite matrix $\in \Re^{n\times n}$, and α can be any positive number that satisfies

$$\alpha > \frac{\sigma_{\max}(\Omega_2)}{\sigma_{\min}(Q)}. \tag{47}$$

Proof. Using (43)–(45), we can rewrite (42) as follows

$$x(k+1) = (\,\bar{A}_0 + \Delta A)\,x(k), \tag{48}$$

where

$$\Delta A = (\Theta\otimes I_n)^T\,\Sigma^* + (\Theta\otimes I_n)^T\,\Sigma\,(\Theta\otimes I_n). \tag{49}$$

From Theorem 3.1.1, we have the following sufficient condition for (48) to remain asymptotically stable

$$(\Delta A)^T\,(\alpha Z+P)\,(\Delta A) + \frac{1}{\alpha}\,\Omega_2 < Q, \tag{50}$$

where P, Q, and $\bar{A}_0$ satisfy the Lyapunov equation (22), that is

$$\bar{A}_0^T P \bar{A}_0 - P + Q = 0. \tag{51}$$

We define

$$\Gamma_1 = (\Theta\otimes I_n)^T\,\Sigma^*, \qquad \Gamma_2 = (\Theta\otimes I_n)^T\,\Sigma\,(\Theta\otimes I_n) \tag{52}$$
$$\Phi_1 = (\alpha Z+P)^{\frac{1}{2}}\,\Gamma_1, \qquad \Phi_2 = (\alpha Z+P)^{\frac{1}{2}}\,\Gamma_2. \tag{53}$$

With definitions (52) and (53), we have

$$\begin{aligned}(\Delta A)^T\,(\alpha Z+P)\,(\Delta A) &= \Gamma_1^T\,(\alpha Z+P)\,\Gamma_1 + \Gamma_2^T\,(\alpha Z+P)\,\Gamma_2\\ &\quad + \Phi_1^T\,\Phi_2 + \Phi_2^T\,\Phi_1 &(54)\\ &\le 2\,\Gamma_1^T\,(\alpha Z+P)\,\Gamma_1 + 2\,\Gamma_2^T\,(\alpha Z+P)\,\Gamma_2, &(55)\end{aligned}$$

where (15) was used in (54) for $\alpha = 1$ and $Z = I$. Defining $\Xi_2 = (\Delta A)^T\,(\alpha Z+P)\,(\Delta A)+\frac{1}{\alpha}\,\Omega_2$, the following sufficient condition for (50) holds

$$\sigma_{\max}[\Xi_2] \le 2\,\sigma_{\max}\left(\,\Gamma_1^T\,(\alpha Z+P)\,\Gamma_1\,\right)$$

$$
\begin{aligned}
& + 2\,\sigma_{\max}\left(\,\Gamma_2^T\,(\alpha Z + P)\,\Gamma_2\,\right) \;+\; \sigma_{\max}\left(\frac{1}{\alpha}\,\Omega_2\right) \\
\leq\;& 2\,\sigma_{\max}(\alpha Z + P)\,\sigma^2_{\max}(\Sigma^*)\,(\Theta^T\Theta) \\
& + 2\,\sigma_{\max}(\alpha Z + P)\,\sigma^2_{\max}(\Sigma)\,(\Theta^T\Theta)^2 \;+\; \sigma_{\max}\left(\frac{1}{\alpha}\,\Omega_2\right) \\
<\;& \sigma_{\min}(Q)
\end{aligned} \tag{56}
$$

or finally

$$
\left[2\,\sigma_{\max}(\alpha Z + P)\,\sigma^2_{\max}(\Sigma)\right]\left(\sum_{i=1}^{m}\theta_i^2\right)^2 \;+\; \left[2\,\sigma_{\max}(\alpha Z + P)\,\sigma^2_{\max}(\Sigma^*)\right]\left(\sum_{i=1}^{m}\theta_i^2\right) + \left[\,\sigma_{\max}(\frac{1}{\alpha}\,\Omega_2) - \sigma_{\min}(Q)\,\right] \;<\; 0. \tag{57}
$$

Selecting α to satisfy (47), we see that the 2 roots of (57) have opposite sign, and therefore the solution is as indicated in (46). □

4. ILLUSTRATIVE EXAMPLES

Example 1. Consider the following uncertain continuous system

$$
A \;=\; \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} + \theta_1 \begin{pmatrix} 5 & 0 \\ -8 & 3 \end{pmatrix} + \theta_2 \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \tag{58}
$$

$$
B \;=\; \begin{pmatrix} 0 \\ -0.7 \end{pmatrix} + \theta_1 \begin{pmatrix} 0 \\ 1.5 \end{pmatrix} \tag{59}
$$

$$
C \;=\; (0 \quad 1) + \theta_1 (-3 \quad 0) + \theta_2 (0.3 \quad 2)\,. \tag{60}
$$

Using (13) for a given output feedback gain $K = 1$, $\alpha = 178.14$ and

$$
Z = \begin{pmatrix} 3.9214 & 0 & 0.0075 & 0.0302 \\ 0 & 3.9655 & 0 & 0 \\ 0.0075 & 0 & 3.9269 & -0.0211 \\ 0.0302 & 0 & -0.0211 & 3.9838 \end{pmatrix} \tag{61}
$$

we obtain

$$
\theta_1^2 \;+\; \theta_2^2 \;<\; (0.0522)^2 \tag{62}
$$

which improves the bound derived via the method of [19], which is

$$
\theta_1^2 \;+\; \theta_2^2 \;<\; (0.0520)^2. \tag{63}
$$

Example 2. Consider the uncertain discrete-time system (23) from [10] for

$$
A = \begin{pmatrix} 0.20 & 0.30 \\ 0.10 & -0.15 \end{pmatrix}. \tag{64}
$$

Using (26) of Theorem 3.1.1 for $Q = I_2$, $\alpha = 0.2702$ and

$$Z = \begin{pmatrix} 2.0399 & -0.2037 \\ -0.2037 & 1.4586 \end{pmatrix} \tag{65}$$

we obtain

$$\sigma_{\max}(\Delta A) < 0.6787 \tag{66}$$

which compares favorably to the result of [10]

$$\sigma_{\max}(\Delta A) < 0.6373. \tag{67}$$

Example 3. Consider the same nominal system as before, but now with structured perturbations of the form of (35), with $m = 3$ and

$$A_1 = \begin{pmatrix} 10 & 0.1 \\ -1 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0.5 & 9 \\ 0 & -3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0.6 \\ 1 & 0.3 \end{pmatrix}. \tag{68}$$

Using (37) of Theorem 3.2.1 for $Q = I_2$, $\alpha = 0.40$ and

$$Z = \begin{pmatrix} 1.3462 & -0.1184 \\ -0.1184 & 0.8786 \end{pmatrix} \tag{69}$$

we obtain

$$\theta_1^2 + \theta_2^2 + \theta_3^2 < (0.0606)^2 \tag{70}$$

that is a sphere with radius $R = 0.0606$, whereas the method of [10] gives

$$|\theta_i| < 0.0348 \qquad i = 1, 2, 3. \tag{71}$$

As we see in Fig. 1, the cube of (71) is completely included in the sphere of (70), which shows that our bound is less conservative than the one of [10].

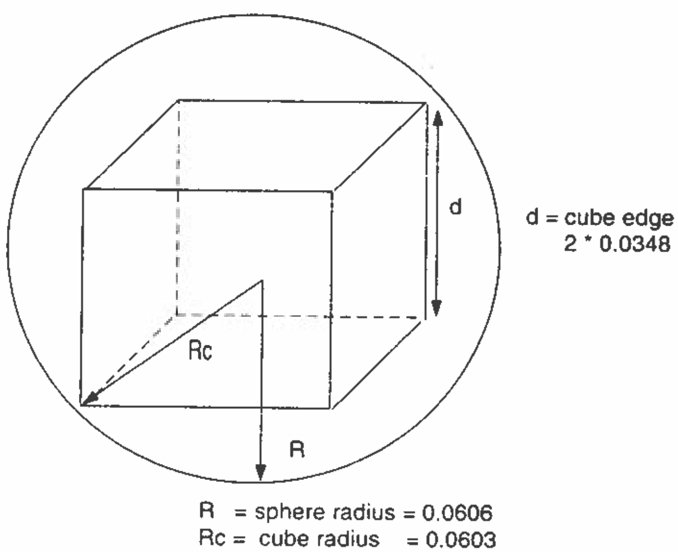

Fig. 1. Example 3.

Example 4. Consider the following uncertain discrete-time system with independent uncertain parameters describing the perturbation matrices

$$A = \begin{pmatrix} -1 & 1.20 \\ 0.10 & -0.15 \end{pmatrix} + \theta_1 \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \theta_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{72}$$

$$C = (\,1.2 \quad -1.5\,) + \theta_3\,(\,-1 \quad 1\,)\,. \tag{73}$$

For a given gain $K = 0.80$, we use (46) for $Q = 2I_2$, $\alpha = 0.35$ and

$$Z = \begin{pmatrix} 0.8160 & 0.0345 \\ 0.0345 & 1.2865 \end{pmatrix} \tag{74}$$

to obtain

$$\theta_1^2 + \theta_2^2 + \theta_3^2 < (0.2636)^2. \tag{75}$$

Note that α and Z in all the examples above have been decided experimentally to give the largest radius for the hypersphere within which the uncertain paremeters vary. Note also that for all the cases that Q had to be selected in the discrete Lyapunov equations (Examples 2–4), $Q = \alpha I$ gave the best results.

5. CONCLUSIONS

In this paper, a novel approach for robust stability of linear continuous and discrete-time systems under parametric uncertainty has been presented. The approach is based on Lyapunov techniques and several examples have been used to illustrate the results. The main point of this approach is the selection of a positive definite matrix Z and a positive number α that maximize the stability region, within which the uncertain parameters vary. Note that the theorems presented here have also been used in [13] for the design of output feedback controllers that maintain the robust stability and optimal performance of discrete-time systems.

Issues that remain to be addressed include reduction of conservatism, the extension of the present results to the case where the bounds do not have to be necessarily symmetric with respect to the origin, and the study of the case where the parameters are nonlinear functions of an uncertainty. Another issue that would be of considerable interest is the development of a systematic way-procedure, possibly based on optimization techniques, to obtain the optimal positive matrix Z needed for our theorems above, where with optimal matrix we mean the matrix that could give the best bounds.

(Received February 24, 1995.)

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