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THE TOTAL SYNTHESIS PROBLEM OF LINEAR MULTIVARIABLE CONTROL

PART I: NOMINAL DESIGN

Michael K. Sain
Department of Electrical Engineering
University of Notre Dame
Notre Dame, Indiana 46556

Bostwick F. Wyman
Department of Mathematics
Ohio State University
Columbus, Ohio 43210

R.R. Gejji
Department of Electrical Engineering
Western Michigan University
Kalamazoo, Michigan 49008

P.J. Antsaklis
Department of Electrical Engineering
University of Notre Dame
Notre Dame, Indiana 46556

Joseph L. Peczkowski
Energy Controls Division
Bendix Corporation
South Bend, Indiana 46620

ABSTRACT

The Total Synthesis Problem (TSP) of linear multivariable control consists of the Nominal Design Problem (NDP) and the Good Feedback Synthesis Problem (GFSP). NDP can be understood as an abstract kernel problem on localized modules, and design freedom amounts to the choice of a single good transfer function. Closed loop response constraints brought about by plants having zero modules which are not good can be handled conceptually in terms of the intersection of the plant zero module with the pole module of the feedforward response. These constraints are handled automatically by NDP.

1. INTRODUCTION

Synthesis of feedback control systems has had different meanings for different investigators. For example, writing on the subject of geometric methods [1] in linear multivariable control, Wonham [2] has described it as "the process by which one establishes the qualitative structural possibilities...such...as noninteraction...,loop stability, and regulation". By comparison, almost thirty-five years ago, Guillemin [3] proposed that synthesis of feedback control systems should involve a determination of the closed loop transfer function from specifications, followed by construction of appropriate compensation networks. In 1955 Truxal [4, pp. 303-305] addressed some of the standard objections to the Guillemin method, such as left-half-plane cancellation of plant poles, imperfect cancellation, and controller complexity.

Studies of these ideas have appeared in the modern linear multivariable control literature under the general title of model matching problems. For example, Morse [5] has studied model matching by means of geometric methods. From the input/output point of view, model matching has received considerable attention [6,7,8,9]. In this case, focus was placed [6] upon the equation

$$Z_1(s)\, Z(s) = Z_2(s),$$

with $Z_i(s)$, $i = 1,2$, being given rational matrices and with $Z(s)$ to be determined. A principal issue was the fact that $Z(s)$ might be required to have certain properties, such as being stable or proper, while $Z_1(s)$ or $Z_2(s)$ might not be so restricted. This led to the idea of studying the model matching equation over rings [10,11].

With the work of Bengtsson [12], the idea of a feedback realization could be naturally linked with the solution of the open loop or feedforward model matching equation. Generalized by Pernebo [13], these ideas have presented a number of new possibilities in the area of model matching.

In 1979, Peczkowski, Sain, and Leake introduced a generalization [14] of the model matching problem. The character of this generalization differs by regarding $Z(s)$ to be given, with $Z_1(s)$ and $Z_2(s)$ to be determined. Studies have been made [14,15,16] to determine the features of this concept in application to aircraft gas turbine engine control.

This paper presents the algebraic foundations of this generalized viewpoint, which we call total synthesis. The results are closest in spirit to the work of Pernebo [13], but differ in three important ways. First, only $Z(s)$ is assumed to be given, whereas Pernebo's treatment regards $Z_1(s)$ and $Z_2(s)$ to be given. Second, the treatment takes place in the original rings, where Pernebo's treatment makes a ring transformation. Third, the presentation is coordinate-free, whereas Pernebo's work involves only matrices.

After Section 2 on Notation and Preliminaries, Section 3 defines the Nominal Design Problem, the Feedback Synthesis Problem, and the Total Synthesis Problem. Section 4 addresses the Nominal Design Problem as a module theoretic kernel problem and resolves the issues with the use of localization. Section 5 presents a coordinate-free treatment of the effects of non-minimum-phase plants.

Because of paper length limitations, we have had to delete the proofs of two key results, one in Section 4 and one in Section 5. For the same reason, we have had to omit the section on causality in nominal design. These omissions will be added in a later report. The causality treatment will also take place in the original ring, without transformation.

2. NOTATION AND PRELIMINARIES

Let k be an arbitrary field. Then $k[s]$ is the principal ideal domain of polynomials in s with coefficients in k. As a commutative ring with no zero divisors, $k[s]$ admits the quotient field $k(s)$. Intuitively, $k(s)$ is just the set of rational transfer functions having coefficients in k. Our system functions are to be set up on $k[s]$-modules and on $k(s)$-vector spaces, which we now define.

Suppose that V_i is a k-vector space, for $i = 1, 2$. Then $V_1 \otimes V_2$ is the tensor product of V_1 with V_2, and may be denoted by $V_1 \otimes_k V_2$ to emphasize the fact that V_1 and V_2 are regarded as k-vector spaces. Observe that $k[s]$ admits the structure of a k-vector space; and choose V_1 to be $k[s]$. Let V be a k-vector space of finite dimension; and choose V_2 to be V. Define the k-vector space

$$k[s] \otimes_k V$$

of polynomials in s with coefficients in V. It turns out that this k-vector space admits the structure of a $k[s]$-module. To see this, write

$$\sum_{i=1}^{n} \{p_i(s) \otimes_k v_i\}$$

to represent a vector. Then scalar multiplication by a polynomial $p(s)$ in $k[s]$ is understood in the manner

$$p(s) \sum_{i=1}^{n} \{p_i(s) \otimes_k v_i\} = \sum_{i=1}^{n} \{[p(s)p_i(s)] \otimes_k v_i\} .$$

We shall write $V[s]$ for this $k[s]$-module. In an entirely similar way, we may develop the k-vector space

$$k(s) \otimes_k V$$

and equip it to be a $k(s)$-vector space, which carries the symbol $V(s)$. Clearly, $V[s] \subset V(s)$ in a natural way; and there is an insertion

$$i : V[s] \to V(s) ,$$

which is a morphism of $k[s]$-modules.

Now let W be another k-vector space of finite dimension, with $W[s]$ and $W(s)$ the corresponding $k[s]$-module and $k(s)$-vector space, respectively. Let

$$p : W(s) \to W(s)/W[s]$$

be the projection morphism from $W(s)$, regarded as a $k[s]$-module, onto the quotient module $W(s)/W[s]$.

By a <u>transfer function</u>, we shall mean a morphism

$$L(s) : V(s) \to W(s)$$

of $k(s)$-vector spaces. Notice that $L(s)$ is not a matrix, as we are yet in a coordinate-free mode. For specific calculations, bases in V and W may be chosen; in turn, these choices induce bases in $V(s)$ and $W(s)$; and then a matrix for $L(s)$ may be defined. This matrix will be denoted by $[L(s)]$.

Because $V(s)$ and $W(s)$ may be regarded as $k[s]$-modules, $L(s)$ can be regarded as a morphism of $k[s]$-modules. Thus the composition

$$L^{\#}(s) : V[s] \to W(s)/W[s] ,$$

given by

$$L^{\#}(s) = p \circ L(s) \circ i,$$

is a morphism of $k[s]$-modules, which we will call the Kalman input/output map associated with $L(s)$. By the <u>pole module</u> of $L(s)$, we shall mean the torsion $k[s]$-module

$$V[s]/\ker L^{\#}(s),$$

denoted by $X(L)$. Onto the pole module there is a controllability epimorphism $\hat{B}$ from $V[s]$, while into $W(s)/W[s]$ there is an observability monomorphism $\hat{C}$ from $X(L)$, as shown in the realization diagram, Figure 1. Denote by

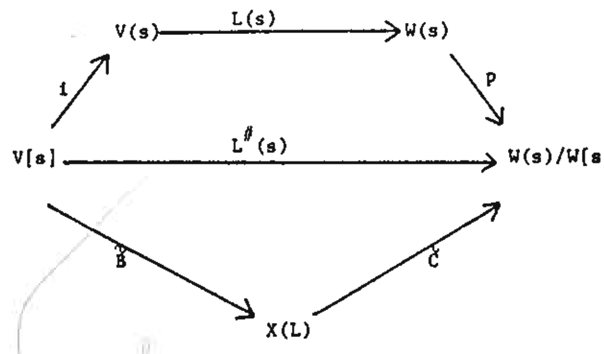

Figure 1. Realization Diagram.

$m(L)$, the minimal polynomial of $X(L)$. Let

$$S_g \subset k[s]$$

be closed under multiplication in $k[s]$, exclude the zero polynomial, and include the polynomial 1. We shall say that $p(s) \in k[s]$ is a <u>good polynomial</u> if $p(s) \in S_g$. Moreover, we shall say that $L(s)$ is a <u>good morphism</u> of $k(s)$-vector spaces if $m(L) \in S_g$. Alternatively, we can say that $L(s)$ is a <u>good transfer function</u>.

3. PROBLEM STATEMENT

In this section, we define in a coordinate-free way the problem of designing, simultaneously, the complete set of controlled outputs desired from a plant and the corresponding complete set of inputs. The approach will be to characterize these two sets in terms of morphisms of $k(s)$-vector spaces, with each morphism acting upon a space of exogenous requests or commands. Because these morphisms must possess certain "good" qualities, as for example stability, which may not necessarily be shared by the plant, it is desirable at the outset to clarify carefully just what foundation can be taken as adequate for the task at hand. This requires a basis-independent definition of the issues.

Let R be a q-dimensional k-vector space of exogenous vectors; let U be an m-dimensional k-vector space of control vectors; and let Y be a p-dimensional k-vector space of plant output vectors. On these spaces, develop the k[s]-modules

$$R[s] = k[s] \otimes_k R$$

$$U[s] = k[s] \otimes_k U$$

$$Y[s] = k[s] \otimes_k Y$$

and the k(s)-vector spaces

$$R(s) = k(s) \otimes_k R$$

$$U(s) = k(s) \otimes_k U$$

$$Y(s) = k(s) \otimes_k Y.$$

Define the plant by the morphism

$$P(s) : U(s) \to Y(s)$$

of k(s)-vector spaces. As a transfer function, P(s) may be good or it may not. Define the desired plant response to exogenous vectors by the morphism

$$T(s) : R(s) \to Y(s)$$

of k(s)-vector spaces; and define the controls needed to generate such response by the morphism

$$M(s) : R(s) \to U(s)$$

of k(s)-vector spaces. Though P(s) is not required to be a good transfer function, it is required that T(s) and M(s) have that property. Further, since M(s) produces the control action which drives the plant P(s), it is of course not independent of the desired plant response T(s). This leads to the first of two basic sub-problems to be discussed.

The Nominal Design Problem (NDP) is to find pairs of good transfer functions (M(s),T(s)) such that the diagram of Figure 2 commutes.

$$\begin{tikzcd} & R(s) \arrow[dl, dashed, "M(s)"'] \arrow[dr, dashed, "T(s)"] & \\ U(s) \arrow[rr, "P(s)"'] & & Y(s) \end{tikzcd}$$

Figure 2. The Nominal Design Problem.

Notice that NDP is not an exact model matching problem [3, 4, 5] or a minimal design problem [6, 7, 8, 9], whose commutative diagrams would be that of Figure 3. In both these diagrams, of course, solid arrows are given morphisms, whereas dashed arrows are to be found.

Consider next the question of output feedback. By output feedback in the present context we shall mean a morphism

$$\begin{tikzcd} & R(s) \arrow[dl, dashed] \arrow[dr] & \\ U(s) \arrow[rr] & & Y(s) \end{tikzcd}$$

Figure 3. Model Matching.

$$C(s) : R(s) \oplus Y(s) \to U(s)$$

of k(s)-vector spaces. Here $R(s) \oplus Y(s)$ is the k(s)-vector (biproduct) space constructed in the usual way from R(s) x Y(s). Notice that any feedback scheme which is required to be a morphism $R(s) \to U(s)$ when the loop is broken and a morphism $Y(s) \to U(s)$ when there is no exogenous signal must be of this form, according to the following result.

Proposition 1 [17 , p. 212]

If U(s) is a k(s)-vector space and

$$C_R : R(s) \to U(s)$$

$$C_Y : Y(s) \to U(s)$$

are morphisms of k(s)-vector spaces, then there is a unique morphism $C(s) : R(s) \oplus Y(s) \to U(s)$ of k(s)-vector spaces such that the diagram of Figure 4 commutes.

$$\begin{tikzcd} R(s) \arrow[r, "e_R"] \arrow[dr, "C_R"'] & R(s) \oplus Y(s) \arrow[d, dashed, "C(s)"] & Y(s) \arrow[l, "e_Y"'] \arrow[dl, "C_Y"] \\ & U(s) & \end{tikzcd}$$

Figure 4. "Linear" Feedback.

Proof: See [17]. The morphisms e_R and e_Y have actions given by

$$e_R(r(s)) = (r(s),0) \; ; \; e_Y(y(s)) = (0,y(s)).$$

Moreover, an output feedback morphism C(s), when given, can always be used to define morphisms C_R and C_Y, as seen in Figure 5, according to the method

$$C_R = C(s) \circ e_R$$

$$C_Y = C(s) \circ e_Y.$$

$$\begin{tikzcd}
R(s) \arrow[r, "e_R"] \arrow[dr, dashed, "C_R"'] & R(s) \oplus Y(s) \arrow[d, "C(s)"] & Y(s) \arrow[l, "e_Y"'] \arrow[dl, dashed, "C_Y"] \\
 & U(s) &
\end{tikzcd}$$

Figure 5.

From these facts, it follows that

$$\begin{aligned} y(s) &= P(s)u(s) \\ &= P(s)\,(C_R r(s) + C_Y y(s)), \end{aligned}$$

or

$$(1_{Y(s)} - P(s) \circ C_Y) y(s) = (P(s) \circ C_R)\, r(s) .$$

Under the technical assumption that

$$(1_{Y(s)} - P(s) \circ C_Y)^{-1} : Y(s) \to Y(s)$$

exists, we have

$$\begin{aligned} y(s) &= (1_{Y(s)} - P(s) \circ C_Y)^{-1} \circ P(s) \circ C_R\, r(s) \\ &= P(s) \circ (1_{U(s)} - C_Y \circ P(s))^{-1} \circ C_R\, r(s) \end{aligned}$$

which permits the identification

$$T(s) = P(s) \circ (1_{U(s)} - C_Y \circ P(s))^{-1} \circ C_R.$$

This leads to the second of the two basic sub-problems.

The Feedback Synthesis Problem (FSP) is to find an output feedback morphism $C(s)$ such that

$$M(s) = (1_{U(s)} - C_Y \circ P(s))^{-1} \circ C_R.$$

By itself, FSP is trivial, because of the choice

$$C_Y = 0 \quad , \quad C_R = M(s).$$

Thus we shall focus on the Good Feedback Synthesis Problem (GFSP). By way of example, if $k = \mathbb{R}$ and if S_g is the set of strictly Hurwitz polynomials, then GFSP may be understood as the Feedback Synthesis Problem with Internal Stability (FSPIS). Of course, even when $k = \mathbb{R}$, GFSP and FSPIS are not always the same, because S_g does not have to be the set of strictly Hurwitz polynomials. It may, for illustration, be a subset of those.

Finally, we define the Total Synthesis Problem (TSP) as the combination of NDP and GFSP. In this paper, we shall focus upon NDP.

4. NDP: THE ABSTRACT KERNEL PROBLEM

In this section, the set of all solution pairs $(M(s),T(s))$ to NDP will be described as the kernel of an abstract morphism. This treatment streamlines and extends that given by Gejji [18]. Some care must be taken to examine the distinction between "polynomial" solutions, "good" solutions, and "arbitrary rational" solutions, because we do not wish to require $P(s)$ to be a good transfer function.

A few additional notations are required. In particular, we need to introduce the localization [19] $S_g^{-1}k[s]$ of the ring $k[s]$. This is done by establishing an equivalence relation $\equiv$ on $S_g \times k[s]$, by a means analogous to that used in developing the quotient field $k(s)$ [17]. $S_g^{-1}k[s]$ is a commutative ring, which satisfies

$$k[s] \subset S_g^{-1}k[s] \subset k(s).$$

Regarded as rings, each of these three sets admits the structure of a $k[s]$-module. Consider the $k[s]$-module $V[s]$. $V[s]$ can be localized as well, by the construction

$$S_g^{-1}V[s] = (S_g^{-1}k[s]) \otimes_{k[s]} V[s].$$

Then a morphism $J(s) : V[s] \to W[s]$ of $k[s]$-modules has a localization

$$S_g^{-1}J(s) : S_g^{-1}V[s] \to S_g^{-1}W[s]$$

by the action

$$\begin{aligned} S_g^{-1}J(s) \left(\sum_{i=1}^{n} (p_i(s),q_i(s)) \otimes_{k[s]} v_i(s) \right) \\ = \sum_{i=1}^{n} (p_i(s),q_i(s)) \otimes_{k[s]} J(s)v_i(s). \end{aligned}$$

Some properties of localization, which are very useful in TSP, should be mentioned. Suppose that

$$J_1(s) : V_1[s] \to V_2[s]$$

$$J_2(s) : V_2[s] \to V_3[s]$$

are morphisms of $k[s]$-modules. Then the sequence

$$V_1[s] \xrightarrow{J_1(s)} V_2[s] \xrightarrow{J_2(s)} V_3[s]$$

is exact when

$$\ker J_2(s) = \operatorname{im} J_1(s).$$

In such a case, the localized sequence

$$S_g^{-1}V_1[s] \xrightarrow{S_g^{-1}J_1(s)} S_g^{-1}V_2[s] \xrightarrow{S_g^{-1}J_2(s)} S_g^{-1}V_3[s]$$

is exact also. When applied to the sequences

$$\ker J_1(s) \to V_1[s] \xrightarrow{J_1(s)} V_2[s],$$

$$V_2[s] \xrightarrow{J_2(s)} V_3[s] \to V_3[s]/\mathrm{im}J_2(s),$$

this implies that

$$\ker S_g^{-1}J(s) = S_g^{-1} \ker J(s),$$

$$\mathrm{im}\, S_g^{-1}J(s) = S_g^{-1}\, \mathrm{im}\ J(s).$$

Further, if the $k[s]$-module $V[s]$ is free over $k[s]$, then $S_g^{-1}V[s]$ is free over $S_g^{-1}k[s]$, a fact which follows from the relation of the tensor product to the direct sum.

Next, examine NDP. Denote by $\mathrm{Hom}_k(R,U)$ the k-vector space of morphisms $R \to U$ of k-vector spaces. For simplicity, write

$$H(R,U) = \mathrm{Hom}_k(R,U).$$

Similarly, write

$$H(R,U)[s] = k[s] \otimes_k H(R,U)$$

$$= \mathrm{Hom}_{k[s]}\ (R[s],U[s]),$$

$$H(R,U)(s) = k(s) \otimes_k H(R,U)$$

$$= \mathrm{Hom}_{k(s)}\ (R(s),U(s)),$$

which are a $k[s]$-module and a $k(s)$-vector space, respectively. Notice that we have identified some naturally isomorphic tensor product structures. Now define

$$H(R,U)(s)_g = S_g^{-1}H(R,U)[s],$$

and observe that

$$H(R,U)[s] \subset H(R,U)(s)_g \subset H(R,U)(s).$$

Given bases in R and U, a morphism $X(s)$ in $H(R,U)(s)_g$ can be expressed as an $m \times q$ matrix with elements in $S_g^{-1}k[s]$. In an exactly analogous way, we can set up $H(R,Y)$ and $H(U,Y)$. Then similar developments follow. Given the plant $P(s)$, define a morphism

$$F : H(R,U)(s) \oplus H(R,Y)(s) \to H(R,Y)(s)$$

of $k(s)$-vector spaces by the action

$$F(M(s),T(s)) = P(s) \circ M(s) - T(s).$$

Clearly, $(M(s),T(s))$ satisfies the diagram of Figure 2 if and only if $(M(s),T(s))$ is in $\ker F$. Furthermore, since $F(0,-T(s)) = T(s)$, F is epic, with rank pq. Therefore $\ker F$ is an mq-dimensional $k(s)$-vector space.

F can be restricted to submodules of its domain. In particular, write

$$K(s) = \ker F$$

$$K(s)_g = \ker F|H(R,U)(s)_g \oplus H(R,Y)(s)_g$$

$$K[s] = \ker F|H(R,U)[s] \oplus H(R,Y)[s]\ .$$

Then $K[s]$ is a free $k[s]$-module of rank mq, $K(s)_g$ is the localization $S_g^{-1}K[s]$ which is a free $S_g^{-1}k[s]$-module of rank mq, and

$$K[s] \subset K(s)_g \subset K(s).$$

A $k[s]$-basis for $K[s]$ gives a $S_g^{-1}k[s]$ basis for $K(s)_g$, so that computations may be done in $k[s]$.

In fact, an explicit description of $K[s]$ may be given. Given $P(s)$, there exist morphisms

$$D(s) : U[s] \to U[s]$$

$$N(s) : U[s] \to Y[s]$$

of $k[s]$-modules, which induce morphisms $U(s) \to U(s)$ and $U(s) \to Y(s)$ of $k(s)$-vector spaces, such that (a) $D(s)$ is invertible on $U(s) \to U(s)$, (b) $P(s) = N(s) \circ D^{-1}(s)$, and (c) there exist morphisms $A(s) : U[s] \to U[s]$ and $B(s) : Y[s] \to U[s]$ of $k[s]$-modules such that

$$A(s) \circ D(s) + B(s) \circ N(s) = 1_{U[s]}.$$

The pair $(N(s),D(s))$ is called a <u>right coprime factorization</u> [6, 20] for $P(s)$. It is not unique, but any one may serve for the discussion. Given $(N(s),D(s))$, define a morphism

$$\alpha : H(R,U)[s] \to H(R,U)[s] \oplus H(R,Y)[s]$$

of $k[s]$-modules by the action

$$\alpha(X(s)) = (D(s) \circ X(s), N(s) \circ X(s))\ .$$

<u>Theorem 1</u>

The morphism α is monic; and the image of α is precisely $K[s]$.

The morphism α defined above can be localized; and, since localization of exact sequences gives exact sequences, we have the next result.

<u>Corollary 1</u>

The morphism

$$S_g^{-1}\alpha : H(R,U)(s)_g \to H(R,U)(s)_g \oplus H(R,Y)(s)_g$$

defined by

$$S_g^{-1}\alpha\ (X(s)) = (D(s) \circ X(s), N(s) \circ X(s))$$

is monic and has image precisely equal to $K(s)_g$.

With this background, we can summarize the basic character of NDP.

<u>Theorem 2</u>

The pair $(M(s),T(s))$ is a solution to the Nominal Design Problem if and only if there exists a good transfer function

$$X_g(s) : R(s)_g \to U(s)_g$$

such that

$$M(s) = D(s) \circ X_g(s)$$

$$T(s) = N(s) \cdot X_g(s),$$

where $(N(s),D(s))$ is any right coprime factorization of $P(s)$.

The good transfer function $X_g(s)$ thus becomes the crucial design parameter in NDP. The design situation, then, may be sketched as in Figure 6 . In this figure, the viewpoint is at the "computational" level of Theorem 1 and the morphism α of k[s]-modules. Localization of the diagram in Figure 6 brings us to the

$$\begin{tikzcd}
 & R[s] \arrow[ddl, "M(s)"'] \arrow[d, "X(s)"] \arrow[ddr, "T(s)"] & \\
 & U[s] \arrow[dl, "D(s)"'] \arrow[dr, "N(s)"] & \\
U(s) \arrow[rr, "P(s)"'] & & Y(s)
\end{tikzcd}$$

Figure 6. NDP at the k[s]-Module Level.

diagram of Figure 7, which is at the level of Theorem 2. Here we have written $R(s)_g$ for $S_g^{-1}R[s]$,

$$\begin{tikzcd}
 & R(s)_g \arrow[ddl, "M_g(s)"'] \arrow[d, "X_g(s)"] \arrow[ddr, "T_g(s)"] & \\
 & U(s)_g \arrow[dl, "D_g(s)"'] \arrow[dr, "N_g(s)"] & \\
U(s) \arrow[rr, "P(s)"'] & & Y(s)
\end{tikzcd}$$

Figure 7. NDP at the $S_g^{-1}k[s]$-Module Level.

$M_g(s)$ for $S_g^{-1}M(s)$, and so forth. Notice that

$$S_g^{-1}V(s) = V(s),$$

so that $U(s)$ and $Y(s)$ are unchanged. The subscript g on M, X, T, D and N may be omitted, when in context.

5. THE ZEROS OF $T(s)$

If k were equal to $\mathbb{R}$, the real numbers, and if S_g were the set of nonzero, strictly Hurwitz polynomials, then NDP consists in finding pairs $(M(s), T(s))$ which satisfy the diagram of Figure 2 and which are stable in the sense $m(M) \in S_g$ and $m(T) \in S_g$. In this case, because $M(s)$ is a stable transfer function, we expect that no right-half-plane zero of $P(s)$ can be cancelled by a pole of $M(s)$ and thus that the right-half-plane zeros of $T(s)$ contain those of $P(s)$. Such a statement was established under rather stringent hypotheses in [21], and related work can be found in [14 , 22]. Those results, however, are for the matrix case, and thus are not coordinate-free. In this section, we present a precise and general algebraic result which gives earlier work as a special case. This result is coordinate-free and makes use of the theory of the zero module introduced in [20].

Intuitively, we show that "the zero module of $T(s)$ maps onto the zero module of $P(s)$ modulo the intersection of the zeros of $P(s)$ with the poles of $M(s)$".

The pole module of $M(s)$ has the description

$$X(M) = \frac{M(s)R[s] + U[s]}{U[s]} \subset \frac{U(s)}{U[s]} .$$

From [20], we have that the zero module of $T(s)$ can be described by

$$Z(P) = \frac{P^{-1}(Y[s]) + U[s]}{\ker P(s) + U[s]} .$$

Let

$$Z_0(P) = \frac{P^{-1}(Y[s]) + U[s]}{U[s]} \subset \frac{U(s)}{U[s]} ;$$

and write

$$\pi : Z_0(P) \to Z(P)$$

for the natural epimorphism defined by factoring out $\ker P(s)$. Notice that

$$\frac{\ker P(s) + U[s]}{U[s]} \subset Z_0(P).$$

Now $X(M)$ and $Z_0(P)$, as described above, are both submodules of $U(s)/U[s]$. Therefore, their intersection $X(M) \cap Z_0(P)$ can be formed. Moreover, under π, this intersection produces a submodule

$$i(M,P) = \pi(X(M) \cap Z_0(P)) \subset Z(P),$$

which has the intuitive interpretation of "the intersection of the poles of $M(s)$ with the zeros of $P(s)$".

It is now possible to state a "pole-zero cancellation theorem".

Theorem 3

The morphism $M(s) : R(s) \to U(s)$ of k(s)-vector spaces induces a morphism

$$\mu : Z(T) \to Z(P)/i(M,P)$$

of k[s]-modules. The morphism μ is epic if the rank of $T(s)$ equals the rank of $P(s)$.

This theorem shows in a coordinate-free way to what extent the zeros of $P(s)$ "appear in" the zeros of $T(s)$.

Let us examine one detailed consequence of the theorem. Suppose N is any finitely generated torsion k[s]-module. Let m_N be the minimal polynomial of N. The support of N, denoted supp(N), is the set of all zeros of m_N. These zeros need not be elements of k itself. They could lie in an extension field. For example, if k is $\mathbb{R}$, the support is a set of complex numbers. Note that this definition can be extended to modules over arbitrary rings [19]. We shall say that an element in a support is good if it is the zero of some polynomial in S_g. The support itself will be good if each of its elements is good.

The fundamental fact here needed about supports is this: Suppose that

$$N_1 \subset N_2$$

and

$$N_3 = N_2/N_1$$

where N_i, $i = 1,2,3$, are each finitely generated torsion $\mathbb{R}[s]$-modules; then

$$\mathrm{supp}(N_1) \subset \mathrm{supp}(N_2)$$

and

$$\mathrm{supp}(N_3) \subset \mathrm{supp}(N_2).$$

In fact,

$$\mathrm{supp}(N_2) = \mathrm{supp}(N_1) \cup \mathrm{supp}(N_3).$$

Notice that supp(X(M)) is just the set of poles of M(s), with multiplicities ignored, and that supp (Z(T)) is just the set of multivariable zeros of T(s), with multiplicities ignored. In particular, if $k = \mathbb{R}$, M(s) is (classically) stable when and only when supp (X(M)) is a subset of the open left-half-plane.

Next, the epimorphism μ of Theorem 3 will be interpreted in terms of supports. The resulting statement, while intuitive, is a severe weakening of the result---because not only module structure, but even multiplicity, is ignored.

Consider the submodule i(M,P) defined above. Clearly, the properties of support imply

$$\mathrm{supp}\ (i(M,P)) \subset \mathrm{supp}\ (X(M)).$$

Furthermore

$$\mathrm{supp}\ (Z(P)) = \mathrm{supp}\ (i(M,P)) \cup \mathrm{supp}\ (Z(P)/i(M,P)) .$$

Now, from the construction of μ, $Z(T)/\ker \mu$ is isomorphic to $Z(P)/i(M,P)$; and so

$$\mathrm{supp}\ (Z(P)/i(M,P)) \subset \mathrm{supp}\ (Z(T)).$$

Assume that M(s) is a good transfer function. Then supp (X(M)) is good; and this implies that supp (i(M,P)) is good. It follows that every element in supp (Z(P)) which is not good must occur in supp (Z(P)/i(M,P)), and hence also in supp (Z(T)).

Corollary 2

Suppose that (M(s), T(s)) is a solution to NDP. Then any zero of P(s) which is not good is also a zero of T(s).

Remark

The structural constraints discussed in this section are handled automatically by the approach of Section 4. The use of the transfer function $X_g(s)$ as a design tool fully parameterizes all solutions to NDP and fully respects the structural constraints arising from the fact that P(s) may have some zeros which are not good.

6. CONCLUSIONS

The Total Synthesis Problem (TSP) has been described. Intuitively introduced by Peczkowski, Sain, and Leake in 1979 [14], TSP consists of the Nominal Design Problem (NDP) and the Good Feedback Synthesis Problem (GFSP). It has been shown that NDP can be understood as an abstract kernel problem on localized modules, and that design freedom amounts to the choice of a single morphism $X_g(s)$. Further, the idea of plants with a zero module which is not good has been discussed, and it has been shown in a coordinate free way just what type of constraint this imposes on T(s). This generalizes the comments [14,21]. The treatment is nearest in spirit to that of Pernebo [13], but differs essentially in that both control action and plant response are considered at the same time---a departure from the model matching problem, in that no transformation of rings is used, and in being coordinate-free. Space limitations here require that proofs be omitted; and there was no space at all for discussions of causality of the pair (M(s),T(s)). This will be treated in a subsequent work, and will again be addressed without a ring transformation.

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8. REFERENCES

1. W.M. Wonham, Linear Multivariable Control: A Geometric Approach. New York: Springer-Verlag, 1979, Second Edition.
2. W.M. Wonham, "Geometric Methods in the Structural Synthesis of Linear Multivariable Controls", Proceedings 1977 Joint Automatic Control Conference, pp. 594-600.
3. M.R. Aaron, "Synthesis of Feedback Control Systems by Means of Pole and Zero Location of the Closed Loop Function", AIEE Transactions, Volume 70, Part II, pp. 1439-1445, 1951.
4. J.G. Truxal, Automatic Feedback Control System Synthesis. New York: McGraw-Hill, 1955.
5. A.S. Morse, "Structure and Design of Linear Model Following Systems", IEEE Transactions on Automatic Control, Volume 18, Number 4, pp. 346-353, August 1973.
6. W.A. Wolovich, P.J. Antsaklis, and H. Elliott, "On the Stability of Solutions to Minimal and Nonminimal Design Problems", IEEE Transactions on Automatic Control, Volume 22, Number 1, pp. 88-94, February 1977.
7. S.H. Wang and E.J. Davison, "A Minimization Algorithm for the Design of Linear Multivariable Systems", IEEE Transactions on Automatic Control, Volume 18, pp. 220-225, 1973.
8. G.D. Forney, "Minimal Bases of Rational Vector Spaces, with Applications to Multivariable Linear Systems", SIAM Journal on Control, Volume 13, pp. 493-520, 1975.
9. M.K. Sain, "A Free-Modular Algorithm for Minimal Design of Linear Multivariable Systems", Sixth IFAC World Contress, Part IB, Sec. 9.1, Aug. 1975.

1.L

10. B.D.O. Anderson and N.T. Hung, "Multivariable Design Problem Reduction to Scalar Design Problems", in Alternatives for Linear Multivariable Control, M.K. Sain, J.L. Peczkowski, and J.L. Melsa, Eds. Chicago: National Engineering Consortium, 1978, pp. 88-95.
11. A.S. Morse, "System Invariants under Feedback and Cascade Control", in Lecture Notes in Economics and Mathematical Systems, Volume 131. New York: Springer-Verlag, 1975.
12. G. Bengtsson, "Feedback Realizations in Linear Multivariable Systems", IEEE Transactions on Automatic Control, Volume 22, Number 4, pp. 576-585, August 1977.
13. L. Pernebo, "An Algebraic Theory for the Design of Controllers for Linear Multivariable Systems -Part I, II", IEEE Transactions on Automatic Control, Volume 26, Number 1, pp. 171-194, February 1981.
14. J.L. Peczkowski, M.K. Sain, and R.J. Leake, "Multivariable Synthesis with Inverses", Proceedings 1979 Joint Automatic Control Conference, pp. 375-380.
15. M.K. Sain and A. Ma, "Multivariable Synthesis with Reduced Comparison Sensitivity", Proceedings 1980 Joint Automatic Control Conference, Volume 1, Part WP8-B.
16. M.K. Sain, R.M. Schafer, and K.P. Dudek, "An Application of Total Synthesis to Robust Coupled Design", Proceedings 1980 Allerton Conference on Communication, Control, and Computing, pp. 386-395.
17. S. MacLane and G. Birkhoff, Algebra. New York: Macmillan, 1967.
18. R.R. Gejji, "On the Total Synthesis Problem of Linear Multivariable Control", Ph.D. Dissertation, Department of Electrical Engineering, University of Notre Dame, 1980.
19. M.F. Atiyah and I.G. MacDonald, Introduction to Commutative Algebra. Reading, Mass.: Addison-Wesley, 1969.
20. B.F. Wyman and M.K. Sain, "The Zero Module and Essential Inverse Systems", IEEE Transactions on Circuits and Systems, Volume 28, Number 2, pp. 112-126, February 1981.
21. V.H.L. Cheng and C.A. Desoer, "Limitations on the Closed-Loop Transfer Function due to Right-Half Plane Transmission Zeros of the Plant", IEEE Transactions on Automatic Control, Volume 25, Number 6, pp. 1218-1220, December 1980.
22. R.J. Leake, J.L. Peczkowski, and M.K. Sain, "Step Trackable Linear Multivariable Plants", International Journal of Control, Volume 30, Number 6, pp. 1013-1022, December 1979.