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Decoupling

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50.1 Introduction

Multi-input/multi-output systems are usually difficult for human operators to control directly, since changing any one input generally affects many, if not all, outputs of the system. As an example, consider the vertical landing of a vertical take off and landing jet or of a lunar landing rocket. Moving to a desired landing point to the side of the current position requires tilting the thrust vector to the side; but this reduces the vertical thrust component, which was balancing the weight of the craft. The aircraft therefore starts to descend, which is not desired. To move to the side at a constant height thus requires smooth, simultaneous use of both attitude control and throttle. It would be simpler for the pilot if a single control existed to do this maneuver; hence the interest in control methods that make the original system behave in a way that is easier to control manually. One example of such technique is when a compensator is sought that makes the compensated system diagonally dominant. If this can be achieved, it is then possible to regard the system as, to first order, a set of independent single-input/single-output systems, which is far easier to control than the original plant. Another approach is that of decoupling, where the system transfer matrix is made to be exactly diagonal. Each output variable is therefore affected by only one input signal, and each input/output pair can then be controlled by an easier-to-design single-input/single-output controller or manually by a human operator.

This chapter studies the problem of making the transfer function matrix of the system diagonal using feedback control and, in particular, state feedback, state feedback with dynamic precompensation, and constant output feedback control laws. This problem is referred to as the *dynamical decoupling* problem, as it leads to a compensated system where the input actions are decoupled; it is also called a noninteracting control problem for similar reasons. Stability is an important issue and it also is examined here. Conditions for decoupling with stability and algorithms to determine such control laws are described. The problems of block diagonal or triangular decoupling are also addressed. They are of interest when full diagonal decoupling using a particular form of feedback control, typically state feedback, is not possible. Note that the approach taken in this chapter follows the development in [14]. Static decoupling is also briefly discussed; references for approximate diagonal decoupling are provided in "Further Reading."

50.1.1 Diagonal Decoupling

Diagonal decoupling of a system with equal numbers of inputs and outputs is the most straightforward type of problem in the field of noninteracting control. The goal is to apply some form of control law to the system so as to make the i-th output of the closed-loop system independent of all but the i-th closed-loop input signal. Each output can then be controlled by a dedicated simpler single-input/single-output controller, or by a human operator. The main questions to be answered when investigating diagonal decoupling of a given system are

- Can it be decoupled at all?
- If so, what form of controller is required to achieve this?

Three classes of controllers that are customarily considered are

1. Constant output feedback $u = Hy + Gr$, where the output y of the system is simply multiplied by a constant gain matrix H and this is fed back as the control signal u, with r the new external input to the system and G a constant gain matrix
2. Linear state feedback $u = Fx + Gr$, where the control signal consists of a constant matrix F multiplying the internal state variable vector x of the system
3. State feedback plus precompensation, where a feedforward dynamic control system is added to the state feedback controller.

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Note that the compensator in class 3 corresponds to dynamic output feedback, where the input and output signal vectors r and y are multiplied by dynamic transfer function gain matrices rather than constant ones.

The problem of diagonally decoupling a square system was the first decoupling question to be studied, and it can be answered in a fairly straightforward fashion. First of all, diagonal decoupling by state feedback plus precompensation, or by dynamic output feedback, amounts to finding a transfer matrix that, when the open-loop transfer matrix is multiplied by it, produces a diagonal closed-loop transfer matrix. This problem is therefore closely related to the problem of finding an inverse for the open-loop plant. As a result of this, any square plant that has a full rank transfer matrix can be diagonally decoupled by this type of control. This result was proved by Rekasius [10]. A system that does not satisfy this condition does not have linearly independent outputs, so it follows that it is impossible to decouple by any form of controller. It is of great practical interest to establish whether a given plant can actually be decoupled by a simpler type of controller than this. Falb and Wolovich [3] established the necessary and sufficient condition under which diagonal decoupling by state feedback alone is possible. This condition, which can be easily tested from either a state-space or a transfer matrix model of the plant, can be expressed as follows.

A square system can be diagonalized by state feedback alone if and only if the constant matrix B^* is nonsingular, where this matrix is defined below first from the state-space and then from the transfer matrix description of the system.

State-space representation. Let the given system be $\dot{x} = Ax + Bu$, $y = Cx + Du$ in the continuous-time case, or $x(k+1) = Ax(k) + Bu(k)$, $y(k) = Cx(k) + Du(k)$ in the discrete-time case; let A, B, C, D be $n \times n, n \times p, p \times n, p \times p$ real matrices, respectively; and assume for simplicity that the system is controllable and observable. Then the $p \times p$ matrix B^* is constructed as follows: If the i-th row of the direct feedthrough matrix D is nonzero, this becomes the i-th row of the constant matrix B^*. Otherwise, find the lowest integer, f_i, for which the i-th row of $CA^{f_i-1}B$ is nonzero. This then becomes the i-th row of the constant matrix B^*.

Transfer matrix representation. Let $T(s)$, with s the Laplace transform variable, be the $p \times p$ transfer function matrix of the continuous-time system; or $T(z)$, with z the Z-transform variable, be the transfer function matrix of the discrete-time system. Let $D(s)$ [or $D(z)$] be the diagonal matrix $D(s) = diag(s^{f_i})$ where the nonnegative integers $\{f_i\}$ are so that all rows of $\lim_{s\to\infty} D(s)T(s)$ are constant and nonzero. This limit matrix is B^*; that is,

$$\lim_{s\to\infty} D(s)T(s) = B^* \tag{50.1}$$

The integers $\{f_i\}$ are known as the decoupling indices of the system. They can be determined from either the state-space or the transfer function descriptions as described above; note that $f_i = 0$ corresponds to the i-th row of D being nonzero. In either case, of course, the resulting matrix B^* is the same. It should be noted that systems will generically satisfy the decoupling condition; that is, if all entries of the A, B, C (and D) matrices are chosen at random, the resulting B^* will have full rank. Diagonal decoupling by state feedback is therefore likely to be feasible for a wide variety of systems.

EXAMPLE 50.1:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 2 \end{pmatrix},$$
$$C = \begin{pmatrix} 3 & 6 & 1 \\ 2 & 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This gives $f_1 = 1$, $f_2 = 3$, and $B^* = \begin{pmatrix} -1 & 2 \\ -2 & 4 \end{pmatrix}$. This matrix is clearly singular; therefore, the system cannot be decoupled by state feedback.

EXAMPLE 50.2:

$$T(s) = \begin{pmatrix} \frac{1}{s} & \frac{2}{s+1} \\ \frac{4}{s+3} & \frac{8s}{s+4} \end{pmatrix}$$

This gives $B^* = \begin{pmatrix} 1 & 2 \\ 0 & 8 \end{pmatrix}$, with decoupling indices $f_1 = 1$, $f_2 = 0$. This system can therefore be diagonally decoupled by state feedback.

EXAMPLE 50.3:

$$T(s) = \begin{pmatrix} \frac{1}{s} & \frac{2}{s+1} \\ \frac{4}{s+3} & \frac{8}{s+4} \end{pmatrix}$$

The same as previously, but with the (2, 2) entry divided by s. We now obtain $B^* = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$, with decoupling indices $f_1 = 1$, $f_2 = 1$. B^* is now singular, so this system cannot be diagonally decoupled by state feedback alone.

50.1.2 Diagonal Decoupling with Internal Stability

A question of great practical interest is whether the closed-loop system that is obtained after decoupling can be made stable. It can be shown constructively (for instance, by use of the algorithm given below) that all of the poles that are evident from the diagonal closed-loop transfer matrix can be assigned any desired values. The question therefore becomes: Can the closed-loop system be made internally stable, where there are no "hidden" cancellations between unstable poles and zeros? Such unstable modes are particularly dangerous in practice, as they will not

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be revealed by an examination of the transfer matrix. However, the hidden unstable state behavior they represent will very likely cause problems, such as burnout of internal electronic components of the system. It was shown by Gilbert [5] that a given plant may indeed have hidden fixed modes when it is diagonally decoupled by state feedback, with or without precompensation. Wolovich and Falb [15] then showed that these modes are the same for both cases; furthermore, they are a subset of the transmission zeros of the plant. In fact, they are those transmission zeros z_i that do not make any of the rows of the transfer matrix $T(s)$ equal to zero when evaluating $T(z_i)$; they are called *diagonal coupling zeros.* Thus, any plant with square, full-rank transfer matrix for which all the diagonal coupling zeros are in the left half-plane can be diagonally decoupled with internal stability by state feedback plus precompensation; or by state feedback alone if B^* is nonsingular. Therefore, there will never be any problems with internal stability when decoupling a minimum-phase system, as all of its transmission zeros are in the left half-plane.

An algorithm to diagonally decouple a system, when B^* has full rank, using state feedback is now presented. This algorithm is based on a procedure to obtain a stable inverse of a system that is described below. This procedure is applied to the system $D(s)T(s) = \hat{T}(s)$, where $D(s) = diag(s^{f_i})$ as in Equation 50.1, that can be shown to have a state-space realization $\{A, B, \hat{C}, \hat{D}\}$. In fact $\hat{D} = B^*$, which is assumed to have full rank p; and this implies that a proper right inverse of the system $\hat{T}(s)$ exists. Here it is assumed that the system has the same number of inputs and outputs, and this simplifies the selection of F, G as in this case they are unique; see the algorithm for the inverse below for the nonsquare case. In particular, if the state feedback $u = Fx + Gr$ with

$$F = -(B^*)^{-1}\hat{C}, \quad G = (B^*)^{-1} \tag{50.2}$$

is applied to the system $\dot{x} = Ax + Bu$, $y = \hat{C}x + B^*u$, then it can be shown that $\hat{T}_{F,G}(s) = D(s)T_{F,G}(s) = I_p$. This implies that if the state feedback $u = Fx + Gr$ with F, G as in Equation 50.2 is applied to the given system $\dot{x} = Ax + Bu$, $y = Cx + Du$ with transfer matrix $T(s)$, then

$$T_{F,G}(s) = D^{-1}(s) \tag{50.3}$$

which is diagonal with entries s^{-f_i}. Note that here the state feedback matrix F assigns all the n closed-loop eigenvalues at the locations of the n zeros of $\hat{T}(s)$; that is, at the zeros of $T(s)$ and of $D(s)$. The closed-loop eigenvectors are also appropriately assigned so the eigenvalues cancel all the zeros to give $D(s)T_{F,G}(s) = I_p$. This explains the control mechanism at work here and also makes quite apparent the changes necessary to ensure internal stability. Simply instead of $D(s)$ use $\hat{D}(s) = diag\{p_i(s)\}$ with $p_i(s)$ stable polynomials of degree s^{f_i}; that is, $p_i(s) = s^{f_i}$ + lower-degree terms. Then it can be shown that $\lim_{s\to\infty} \hat{D}(s)T(s) = B^*$ and that $\{A, B, \bar{C}, B^*\}$ is a realization of $\hat{D}(s)T(s) = \bar{T}(s)$. State feedback with

$$F = -(B^*)^{-1}\bar{C}, \quad G = (B^*)^{-1} \tag{50.4}$$

gives

$$T_{F,G}(s) = \hat{D}^{-1}(s) = diag\{p_i^{-1}(s)\} \tag{50.5}$$

which is stable. Note that in this case the closed-loop eigenvalues are at the assumed stable zeros of $T(s)$ and at the selected stable zeros of the polynomials $p_i(s)$, $i = 1, .., p$.

EXAMPLE 50.4:

Let $T(s) = \begin{pmatrix} \frac{s+1}{s^2} & 0 \\ \frac{1}{s(s-1)} & \frac{-1}{s-1} \end{pmatrix}$.

Here

$$\lim_{s\to\infty} D(s)T(s) = \lim_{s\to\infty} diag(s,s)T(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = B^*.$$

Since B^* has full rank, the system can be decoupled using state feedback $u = Fx + Gr$. The system has one transmission zero at -1 and there are no diagonal coupling zeros, so it can be decoupled with internal stability. Let $\hat{D}(s) = \begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix}$.

A minimal (controllable and observable) realization of $\bar{T}(s) = \hat{D}(s)T(s)$ is $\{A, B, \bar{C}, B^*\}$ where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\bar{C} = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & -3 \end{pmatrix}.$$

In view now of Equations 50.4 and 50.5, for

$$F = -(B^*)^{-1}\bar{C} = \begin{pmatrix} -1 & -2 & 0 \\ 3 & 1 & -3 \end{pmatrix}$$

and

$$G = (B^*)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$T_{F,G}(s) = \hat{D}(s)^{-1} = \begin{pmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{pmatrix}.$$

The closed-loop eigenvalues are in this case located at the transmission zero of the plant at -1 and at the selected locations -1 and -2, the poles of $\hat{D}(s)^{-1}$. Note that it is not necessary to cancel the transmission zero at -1 in order to decouple the system since it is not a coupling zero; it could appear as a zero in the decoupled system instead. To illustrate this, consider Example 50.5 where $T(s)$ is the same except that the zero is now unstable at +1.

EXAMPLE 50.5:

Let $T(s) = \begin{pmatrix} \frac{s-1}{s^2} & 0 \\ \frac{1}{s(s-1)} & \frac{-1}{s-1} \end{pmatrix}$ where again

$$\lim_{s\to\infty} D(s)T(s) = \lim_{s\to\infty} diag(s,s)T(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = B^*.$$

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Since B^* has full rank, the system can be decoupled using state feedback. Since there are no diagonal coupling zeros, the system can be decoupled with internal stability. Write $T(s) = \begin{pmatrix} s-1 & 0 \\ 0 & 1 \end{pmatrix} T_N(s)$ and apply the algorithm to diagonally decouple $T_N(s)$. Now $D_N(s) = \begin{pmatrix} s^2 & 0 \\ 0 & s \end{pmatrix}$ and take

$$\hat{D}_N(s) = \begin{pmatrix} (s+2)(s+3) & 0 \\ 0 & s+1 \end{pmatrix}.$$

A minimal (controllable and observable) realization of $\tilde{T}_N(s) = \hat{D}_N(s)T_N(s)$ is $\{A, B, \tilde{C}_N, B_N^*\}$ where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\tilde{C}_N = \begin{pmatrix} 6 & 5 & 0; \\ 2 & 1 & -2 \end{pmatrix}$$

and

$$B_N^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = B^*.$$

In view now of Equations 50.4 and 50.5, for $F = -(B^*)^{-1}\tilde{C}_N = \begin{pmatrix} -6 & -5 & 0 \\ 2 & 1 & -2 \end{pmatrix}$ and

$$G = (B_N^*)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$(T_N)_{F,G}(s) = \hat{D}_N(s)^{-1} = \begin{pmatrix} \frac{1}{(s+2)(s+3)} & 0 \\ 0 & \frac{1}{s+1} \end{pmatrix}.$$

If now this state feeback is applied to the minimal realization $\{A, B, C\}$ of $T(s)$—note that A, B are the same as above—then

$$T_{F,G}(s) = \begin{pmatrix} s-1 & 0 \\ 0 & 1 \end{pmatrix} \hat{D}_N^{-1} = \begin{pmatrix} \frac{s-1}{(s+2)(s+3)} & 0 \\ 0 & \frac{1}{s+1} \end{pmatrix}.$$

Note that the unstable noncoupling transmission zero at +1 appears on the diagonal of the compensated system; the closed-loop eigenvalues are at the arbitrarily chosen stable locations -1,-2 and -3.

Algorithm to Obtain a Proper Stable Right Inverse Using State Feedback

Let $\dot{x} = Ax + Bu$, $y = Cx + Du$ with A, B, C, D $n \times n$, $n \times m$, $p \times n$, $p \times m$ real matrices, respectively, and assume that the system is controllable and observable. Let $T(s)$ be its transfer function matrix. It is known that there exists a proper right inverse $T_R(s)$, such that $T(s)T_R(s) = I_p$, if and only if $rank D = p$. If, in addition, all the zeros of $T(s)$ (that is, the transmission zeros of the system) are stable, then a stable right inverse of order n can be constructed with $k(< n)$ of its poles equal to the k stable zeros of $T(s)$ with the remaining $n-k$ poles arbitrarily assignable. This can be accomplished as follows:

Let $T_{eq} = F[sI - (A+BF)]^{-1}BG + G$ where F, G are $n \times m$, $m \times p$, respectively, and note that

$$\begin{aligned} T(s)T_{eq}(s) &= [C(sI-A)^{-1}B + D] \\ &\quad [F[sI-(A+BF)]^{-1}BG + G] \\ &= (C+DF)[sI-(A+BF)]^{-1}BG + DG \\ &= T_{F,G}(s) \end{aligned} \tag{50.6}$$

which is the transfer matrix one obtains when the state feedback control law $u = Fx + Gr$ is applied to the given system. Note that the second line of Equation 50.6 results from application of a well-known formula for the matrix inverse. If now F, G are such that

$$C + DF = 0, \quad DG = I_p \tag{50.7}$$

then $T_{F,G}(s) = I_p$ and T_{eq} is a proper right inverse $T_R(s)$. The additional freedom in the choice of F when $p < m$ is now used to derive a stable inverse; when $p = m$, F, G are uniquely determined from $F = -D^{-1}C$, $G = D^{-1}$.

If the nonsingular $m \times m$ matrix M is such that $DM = (I_p \;\; 0)$, then $C + DF = C + DMM^{-1}F = C + (I_p \;\; 0)\begin{pmatrix} \hat{F}_1 \\ \hat{F}_2 \end{pmatrix} = 0$ from which $F = M\begin{pmatrix} -C \\ \hat{F}_2 \end{pmatrix}$ with $\hat{F}_2$ arbitrary. Also, from $DG = DMM^{-1}G = I_p$, $G = M\begin{pmatrix} I_p \\ \hat{G}_2 \end{pmatrix}$ with $\hat{G}_2$ arbitrary. The eigenvalues of $A + BF = A + BM\begin{pmatrix} -C \\ \hat{F}_2 \end{pmatrix} = A + (\hat{B}_1 \;\; \hat{B}_2)\begin{pmatrix} -C \\ \hat{F}_2 \end{pmatrix} = A - \hat{B}_1C + \hat{B}_2\hat{F}_2$ are the poles of $T_R(s)$. It can be shown that the uncontrollable eigenvalues of $(A - \hat{B}_1C \;\; \hat{B}_2)$ are exactly the (k) zeros of the system; they cannot be changed via $\hat{F}_2$. The remaining $n-k$ controllable eigenvalues can be arbitrarily assigned using $\hat{F}_2$. In summary, the steps to derive a stable proper inverse are

Step 1. Find nonsingular $m \times m$ matrix M such that $DM = (I_p \;\; 0)$.

Step 2. Calculate $(\hat{B}_1 \;\; \hat{B}_2) = BM$, and $A - \hat{B}_1C$.

Step 3. Find $\hat{F}_2$ that assigns the controllable eigenvalues of $(A - \hat{B}_1C \;\; \hat{B}_2)$ to the desired locations. The remaining uncontrollable eigenvalues are the stable zeros of the system.

Step 4.

$$\left\{A + BM\begin{pmatrix} -C \\ \hat{F}_2 \end{pmatrix}, BM\begin{pmatrix} I_p \\ \hat{G}_2 \end{pmatrix}, M\begin{pmatrix} -C \\ \hat{F}_2 \end{pmatrix}, M\begin{pmatrix} I_p \\ \hat{G}_2 \end{pmatrix}\right\} \tag{50.8}$$

where $\hat{G}_2$, a $(m-p) \times p$ arbitrary real matrix, is a stable right inverse.

$T_{eq}(s)$ above is the open-loop equivalent to the state feedback control law. In view of Equations 50.6 and 50.7 the above algorithm selects F, G in a state feedback control law $u = Fx + Gr$

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so that the closed-loop transfer matrix $T_{F,G}(s) = I_p$ and the closed-loop system is internally stable; that is, all the n eigenvalues of $A + BF$ are stable. Note that when $p = m$, then F, G are uniquely given by $F = -D^{-1}C$, $G = D^{-1}$; the eigenvalues of $A+BF$ are then the n zeros of the system. In this development of stable inverses via state feedback, the approach in [1] was taken; see also [12] and [7].

In order to implement decoupling by state feedback in practice, it is often necessary to estimate the internal state variables by means of an observer. Certain plants can be decoupled by *constant output feedback*, avoiding the need for an observer. The necessary and sufficient conditions under which this is possible were proved by Wolovich [18]: it is that B^* not only be nonsingular, but also that the modified inverse transfer matrix $B^*T^{-1}(s)$ have only constant off-diagonal elements. This appears to be a very stringent condition, so diagonal decoupling by means of constant output feedback is not likely to be possible for any but a relatively small class of plants. This is in clear contrast with the state feedback case, as mentioned previously. If diagonal decoupling by output feedback is possible, any gain matrix H that achieves it must give all off-diagonal entries of B^*H equal to those of $B^*T^{-1}(s)$. It can therefore be seen that any constant matrix of the form $(B^*)^{-1}Z$ can be added to H, where Z is an arbitrary diagonal matrix, and still give a gain matrix that satisfies the required condition. There is thus a small amount of controller design freedom available, which can be used, for instance, to assign closed-loop poles to some extent. However, it does not appear possible to quantify this pole-placement freedom in any straightforward manner.

50.1.3 Block Decoupling

If diagonal decoupling by linear state feedback is not possible, an alternative to applying precompensation may still exist. It may be possible to use state feedback, or perhaps even output feedback, to reduce the system to a set of smaller subsystems that are independent; that is, decoupled. Controlling each of these small systems can then be performed in isolation from all the others, thus reducing the original plant control problem to several simpler ones. This is the idea behind block decoupling, where the goal is to transform the plant transfer matrix to one that is block diagonal rather than strictly diagonal. For square plants, each of these k diagonal blocks will also be square: the i-th will be taken to have p_i inputs and p_i outputs, with $\sum p_i = p$.

One question associated with block decoupling can be answered immediately: namely, any plant with nonsingular transfer matrix can be block decoupled by linear state feedback plus precompensation. This follows from the fact that any such system can be diagonally decoupled by this form of compensation and so is trivially of any desired block diagonal form. The two types of compensation that have to be addressed here are therefore state feedback and constant output feedback.

If we are interested in block decoupling a given system by state feedback, this implies that it was not fully diagonalizable by state feedback. Hence, the matrix B^* must have been singular. As the inverse of this matrix played a significant role in the development of diagonal decoupling compensators, it seems likely that overcoming this singularity may lead toward designing block decoupling compensators for systems that cannot be diagonalized by state feedback. An equivalent way of stating that B^* is singular is to note that, although all rows of $\lim_{s\to\infty} D(s)T(s)$ are certainly finite and nonzero, some of these rows must have been linearly dependent on the preceding ones. Suppose the i-th row is one such. It is then possible to add multiples of rows $1, \ldots, i-1$ to row i in order to zero out the i-th row in B^*; that is, to make what had been the leading coefficient vector of this row of $D(s)T(s)$ zero. If the new leading term in this row is now of order s^{-k}, multiplying the row by s^k yields a new finite and nonzero limit as s goes to infinity. If this row vector is independent of the preceding ones, we have now increased the rank of the modified B^*-like matrix; if not, the same process can be repeated until successful. This basic procedure leads to the following definition, which has proved to be very useful for studying block decoupling problems.

The interactor $X_T(s)$ of $T(s)$ is the unique polynomial matrix of the form $X_T(s) = H(s)\Delta(s)$, where $\Delta(s) = diag(s^{\hat{f}_i})$ and $H(s)$ is a lower triangular polynomial matrix with 1s on the diagonal and the nonzero off-diagonal elements divisible by s, for which

$$\lim_{s\to\infty} X_T(s)T(s) = K_T \tag{50.9}$$

is finite and full rank. The interactor can be found from the transfer matrix of the system [16]; from a state-space representation [4]; or from a polynomial matrix fraction description for it [13]. The basic procedure can be illustrated by applying it to two examples discussed previously.

EXAMPLE 50.6:

$$T(s) = \begin{pmatrix} \frac{1}{s} & \frac{2}{s+1} \\ \frac{4}{s+3} & \frac{8s}{s+4} \end{pmatrix}$$

. This gives $B^* = \begin{pmatrix} 1 & 2 \\ 0 & 8 \end{pmatrix}$, with decoupling indices $f_1 = 1$, $f_2 = 0$. B^* is nonsingular, so it satisfies the definition of the desired matrix K_T. Thus, $K_T = B^* = \begin{pmatrix} 1 & 2 \\ 0 & 8 \end{pmatrix}$ here, and $X_T(s) = diag(s^{f_1}, s^{f_2}) = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$.

EXAMPLE 50.7:

$$T(s) = \begin{pmatrix} \frac{1}{s} & \frac{2}{s+1} \\ \frac{4}{s+3} & \frac{8}{s+4} \end{pmatrix}$$

$B^* = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$, which is singular, with decoupling indices $f_1 = 1$, $f_2 = 1$. Subtracting 4 times row 1 of $diag(s^{f_i})T(s)$ from row 2 eliminates the linearly dependent leading coefficient

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vector. The resulting lower-degree polynomial row vector can then be multiplied by s, so as to again obtain a finite limit as s goes to infinity. We then have

$$\hat{T}_1(s) = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}$$

$$T(s) = \begin{pmatrix} 1 & \frac{2s}{s+1} \\ \frac{-12s}{s+3} & \frac{-24s^2}{(s+1)(s+4)} \end{pmatrix}.$$

Unfortunately, $\hat{T}_1(s)$ has limit as s goes to infinity of

$$\begin{pmatrix} 1 & 2 \\ -12 & -24 \end{pmatrix},$$

which is still singular. We therefore have to repeat the procedure, this time adding 12 times row 1 to row 2 to eliminate the leading coefficients and multiplying the resulting row by s to give it a finite limit. We then obtain $\begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix}\hat{T}_1(s) = \begin{pmatrix} 1 & \frac{2s}{s+1} \\ \frac{36s}{s+3} & \frac{96s^2}{(s+1)(s+4)} \end{pmatrix}$, which has limit as s goes to infinity of $\begin{pmatrix} 1 & 2 \\ 36 & 96 \end{pmatrix}$. This is clearly nonsingular, so $K_T = \begin{pmatrix} 1 & 2 \\ 36 & 96 \end{pmatrix}$ for this system. The interactor is then

$$\begin{aligned} X_T(s) &= \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 12 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} \\ &\quad \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \\ &= \begin{pmatrix} s & 0 \\ -4s^3 + 12s^2 & s^3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ -4s^2 + 12s & 1 \end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & s^3 \end{pmatrix} \end{aligned}$$

which is of the desired form $H(s)\Delta(s)$.

It can be seen that, if B^* is nonsingular, no additional row operations are needed to modify it to give the nonsingular K_T. Thus, in this case $B^* = K_T$ and $D(s) = X_T(s)$. But we already know that diagonal decoupling by state feedback is possible in this case; that is, diagonalization by state feedback is possible if and only if the interactor of the system is diagonal. This suggests the following general result.

A square system can be block decoupled by state feedback if and only if its interactor is of this same block diagonal structure.

A proof of this result is based on the fact that state feedback matrices F, G can always be found that make the closed-loop transfer matrix equal to the inverse of its interactor; see the algorithms discussed previously and [6], [2]. Thus, if this matrix is block diagonal, so is the closed-loop transfer matrix. The state feedback that achieves this form can be found in an analogous manner to the state feedback matrices determined above that diagonally decouple the system.

Note that the structure algorithm of Silverman [11] is quite closely related to the interactor. This method determines a polynomial matrix $X(s)$ such that $\lim_{s\to\infty} X(s)T(s)$ is finite and nonsingular; however, $X(s)$ is not of any particular structure, unlike the interactor. This makes $X_T(s)$ better suited to obtaining clear block decoupling results.

Another question that generalizes naturally from the diagonal case is that of stability. The only fixed modes when diagonalizing were the diagonal coupling zeros, which were all zeros of the original plant that were not also zeros of any of the rows of the plant transfer matrix. In the case of block decoupling, the only fixed poles are the block coupling zeros, which are all zeros of the plant that are not also zeros of one of the $(p_i \times m)$ row blocks of $T(s)$. As in the diagonal case, these zeros must be cancelled by closed-loop poles in the decoupled transfer matrix, so creating unobservable modes; all other poles can be assigned arbitrarily.

Finally, it may be possible to achieve block decoupling by the simpler constant output feedback compensation. It can be shown that the interactor also allows a simple test for this question. In fact, block decoupling by constant output feedback is possible if and only if the interactor of the system is block diagonal and the modified inverse system $K_T T^{-1}(s)$ has only constant entries outside the diagonal blocks. The output feedback gain matrix H that achieves block decoupling is such that $K_T H$ is equal to the constant term in $K_T T^{-1}(s)$ outside the diagonal blocks. This is very similar to the diagonal decoupling result. As there, a certain degree of flexibility exists in the design of H, due to the fact that the diagonal blocks of $K_T T^{-1}(s)$ are essentially arbitrary; this can be used to provide a small amount of pole assignment flexibility when decoupling.

50.1.4 Decoupling Nonsquare Systems

The previous development has been primarily for plants with equal numbers of inputs and outputs. Plants that are not square present additional complications when studying decoupling. For instance, if there are more outputs than inputs, it is clearly impossible to assign a separate input to control each output individually; diagonal decoupling in its standard form is therefore not feasible. Similarly, decoupling the system into several independent square subsystems is also impossible. On the other hand, plants with more inputs than outputs present the opposite difficulty: there are now more input variables than are required to control each output individually.

Fortunately, the classical decoupling problem can be generalized in a straightforward fashion to cover nonsquare plants as well as square ones. In view of the preceding remarks, it is clear that systems with more outputs than inputs ($p > m$) must be analyzed separately from those with more inputs than outputs ($p < m$). The former case leads to decoupling results that are barely more complicated than those for the square case; the additional design freedom available in the latter case means that conditions that were necessary and sufficient for $p = m$ become only sufficient for $p < m$.

Taking the case of more inputs than outputs ($p < m$), the following results can be shown to hold for diagonal decoupling. First, any such plant that is right-invertible (that is, for which the transfer matrix is of full rank, p) can be decoupled by state feedback plus precompensation; this follows from the close con-

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nections between this type of decoupling control law and finding a right inverse of the system. If we restrict ourselves to state feedback, two sufficient conditions for diagonal decoupling can be stated. First, the plant can be diagonalized by state feedback if its matrix B^* is of full row rank, p. This is extremely easy to test, but can be somewhat conservative. A tighter sufficient condition is as follows: The plant can be diagonalized by state feedback if a constant $(m \times p)$ matrix G can be found for which the B^* matrix of the square-modified transfer matrix $T(s)G$ is nonsingular.

It may be thought that these two sufficient conditions are identical. To see that they are not, consider the following simple example: $T(s) = \begin{pmatrix} \frac{1}{s} & \frac{1}{s^2} & \frac{1}{s} \\ \frac{s+2}{s^2} & \frac{1}{s^2} & \frac{1}{s} \end{pmatrix}$ has $B^* = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, which has only rank 1. The first sufficient condition for diagonal decoupling is therefore violated. However, post-multiplying $T(s)$ by the matrix

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$$

gives

$$T(s)G = \frac{1}{s^2}\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix},$$

which clearly has nonsingular B^* matrix of $\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$. The role of the G matrix is basically to cancel those higher-power terms in s in $T(s)$ that give rise to linearly dependent rows in B^*; in the example, the first column of G, $(1\ 0\ -1)^T$, can be seen to be orthogonal to the repeated row vector $[1\ 0\ 1]$ in the original B^*. Lower-power terms in $T(s)$ then become the leading terms, so their coefficients contribute to the new B^*; these terms may well be independent of the first ones. An algorithm that goes into the details of constructing such a G, if it exists, for any right-invertible $T(s)$ is given by Williams [13].

Very similar results apply to the problem of block decoupling a system with more inputs than outputs ($p < m$) by means of state feedback. The more conservative sufficient condition states that the plant can be block decoupled if its interactor matrix has the desired block diagonal structure. This can then be tightened somewhat by proving that the plant $T(s)$ can be block decoupled if there exists some $(m \times p)$ constant matrix G that has interactor of the desired block diagonal form. Furthermore, the algorithm described previously for block decoupling of square plants can be applied equally in this case, either to $T(s)$ or $T(s)G$. The only distinction of significance between the square case and $p < m$ is that the algorithm was proved to use decoupling precompensation of lowest possible order in the square case; for nonsquare plants, minimality of this order cannot be proven.

In the case of plants with more outputs than inputs ($p > m$), the main complication is in modifying the definition of a "decoupled" closed-loop structure. Once this is done, the actual technical results are rather straightforward. As already noted, it is no longer possible to assign a single input to each individual output, as is required in the classical diagonal decoupling problem. The closest analog to this problem is one where the closed-loop system is decoupled into a set of m independent single-input/multi-output subsystems; each closed-loop control input influences a set of outputs, but does not affect any of the others. Similarly, it is not possible to assign equal numbers of independent inputs and outputs to each decoupled subsystem, as holds for square block decoupling. What we must do instead is to define decoupled subsystems that generally have more outputs than inputs; that is, they are of dimensions $p_i \times m_i$, where $p_i \leq m_i$; of course, $\sum p_i = p \leq \sum m_i = m$.

It can be shown that a very simple rank condition on the plant transfer matrix determines whether or not these decoupling problems have a solution. The simplest question to answer is whether the desired decoupled structure is achievable by means of a combination of state feedback and precompensation. The test is as follows:

Take the p_i rows of the plant transfer function corresponding to the outputs that are to be assigned to the i-th decoupled subsystem. If this $p_i \times m$ transfer matrix has rank m_i, and this holds for each i, then the plant can be decoupled into $p_i \times m_i$ subsystems by means of state feedback plus precompensation.

The significance of this result is easiest to see for the special case where $m_i = 1$ for each i, the closest analog to diagonal decoupling for systems with $p > m$. If decoupling is to be possible, we must have that each $p_i \times m$ transfer matrix of the i-th subsystem is of rank 1. This implies that the rows of this transfer matrix are all polynomial multiples of some common factor row vector. In other words, the p_i outputs of this subsystem are all made up of combinations of derivatives of a single "underlying" output variable. Similarly, decoupling into $p_i \times m_i$ subsystems is possible if and only if the p_i outputs making up the i-th subsystem are actually made up of some combinations of m_i "underlying" output variables.

In practice, applying these rank conditions to the plant transfer matrix dictates what block dimensions are possible as closed-loop block decoupled structure. They also show which outputs must be taken as members of the same decoupled subsystem. For instance, if we wish to achieve $p_i \times 1$ decoupling and two rows of the plant transfer matrix are linearly dependent, the corresponding outputs must clearly be placed in the same subsystem.

But this approach also has one further very important implication. Consider a system that satisfies these submatrix rank conditions. If we take the m_i "underlying" output variables for each of the r subsystems, write down the corresponding $m_i \times m$ transfer matrix, and then concatenate these, we obtain a new $m \times m$ transfer matrix, denoted by $T_m(s)$. It can then be shown (see [13]) that a controller will decouple $T(s)$ into $p_i \times m_i$ blocks if and only if it also decouples $T_m(s)$ into square $m_i \times m_i$ blocks. We can therefore take all of the decoupling results derived previously for square plants and use them to solve the problem of decoupling systems with more outputs than inputs. In particular, $T(s)$ can be decoupled into $p_i \times m_i$ blocks by state feedback if and only if it satisfies the submatrix rank conditions and the interactor matrix of $T_m(s)$ is $m_i \times m_i$ block diagonal. Also, it can be shown that $T_m(s)$ has precisely the same zeros as $T(s)$. The two systems therefore clearly also have the same coupling zeros, so the fixed poles when decoupling $T(s)$ are the same as the fixed poles when decoupling $T_m(s)$. Finally, decoupling by means of output feedback can also be studied by applying the existing results for square systems to the associated $T_m(s)$.

EXAMPLE 50.8:

The state-space model

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad C = I_3$$

has transfer matrix

$$T(s) = \frac{1}{(s+1)(s^2-s-1)} \begin{pmatrix} s(s+1) & s \\ (s+1)^2 & s+1 \\ (s+1)^2 & s^2 \end{pmatrix}.$$

Clearly, the first two rows are linearly dependent, so this system can be decoupled into the block diagonal form $\begin{pmatrix} \star & 0 \\ \star & 0 \\ 0 & \star \end{pmatrix}$ by state feedback plus precompensation. In fact, the associated invertible transfer function for this system is

$$T_m(s) = \frac{1}{(s+1)(s^2-s-1)} \begin{pmatrix} s+1 & 1 \\ (s+1)^2 & s^2 \end{pmatrix},$$

which has interactor $\begin{pmatrix} s^2 & 0 \\ 0 & s \end{pmatrix}$ diagonal [with $K_T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$]. Thus, block decoupling is actually possible for this system using state feedback alone.

As a final point on general block decoupling, note that this problem can also be studied using the geometric approach; see [19]. This state-space technique is based on considering the supremal (A, B)-invariant subspaces contained in the kernels of the various subsystems formed by deleting the outputs corresponding to each desired block in turn. The ranges of these subspaces determine whether decoupling is possible by state feedback. If it is not, the related "efficient extension" approach allows a precompensator of relatively low order to be found that will produce the desired block diagonal structure. This approach is somewhat involved, and the interested reader is referred to Morse and Wonham [8] for further details.

50.1.5 Triangular Decoupling

There is a form of "partially decoupled" system that can be of particular value for certain plants. This is the triangularized form, where all entries of the closed-loop transfer matrix above its leading diagonal are made zero. The first closed-loop output, y_1, is therefore affected only by the first input r_1; the second, y_2, is influenced only by inputs r_1 and r_2; etc. This type of transfer matrix can be used in the following sequential control scheme. First, input r_1 is adjusted until output y_1 is as desired, and the control is then frozen. Output variable y_2 is then affected only by r_2 and the fixed r_1, so r_2 can be adjusted until this output is also as desired. The third input, r_3, can then be used to set output y_3, etc. This scheme can be seen to be less powerful than diagonal decoupling, as the outputs must be adjusted sequentially rather than fully independently. However, it has one strong point in its favor: *any right-invertible plant can be triangularized by state feedback alone,* regardless of whether additional precompensation is required to make it diagonally decoupled. Proof of this follows directly from the fact that there always exists some state feedback gains F, G for which $T_{F,G}(s) = X_T^{-1}(s)$, and the interactor is, by definition, lower triangular. Of course, similar results apply for generalized rather than standard interactors also. Therefore, it can be shown, as originally proved by Morse and Wonham [9], that all closed-loop poles of the triangularly decoupled system can be arbitrarily assigned.

Finally, it may also be possible to triangularize a system by means of the simpler constant output feedback. If the original plant is square and strictly proper ($D = 0$), it can be shown that this is possible if and only if all entries of the modified inverse transfer matrix $K_T T^{-1}(s)$ that lie above the leading diagonal are constant. This is quite a simple condition to test and is very similar to the test for diagonal decoupling by output feedback. The required gain matrix H is given from the fact that the upper triangular part of $K_T H$ is precisely the upper triangular constant part of $K_T T^{-1}(s)$. It can be noted that there is therefore some non-uniqueness in the choice of the gain H: in particular, we can add a term of the form $K_T^{-1} Z$ to H, where Z is any lower triangular constant matrix, and still get a suitable output gain matrix. If it is possible to triangularize a given system by output feedback, there is consequently some freedom to assign closed-loop poles also. However, it is difficult to quantify this freedom in any concrete way.

50.1.6 Static Decoupling

Static decoupling, as opposed to dynamic decoupling already described, is much easier to achieve. A system is statically decoupled if a step change in the (static) steady-state level of the i-th input is reflected by a change in the steady-state level of the i-th output and only that output. To derive the conditions for static decoupling, assume that the system is described by a $p \times p$ transfer matrix $T(s)$ that is bounded-input/bounded-output stable; that is, all of its poles are in the open left half of the s-plane and none is on the imaginary axis. Note that stability is necessary for the steady-state values of the outputs to be well defined. Assume now that the p inputs are step functions described by $u_i(s) = \frac{k_i}{s}$, $i = 1, \ldots, p$. The steady-state value of the output vector y, y_{ss}, can then be found using the final value theorem, as follows:

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$$y_{ss} = \lim_{s\to\infty} y(t) = \lim_{s\to 0} sT(s)\frac{1}{s}\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_p \end{pmatrix} = T(0)\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_p \end{pmatrix} \tag{50.10}$$

It is now clear that $T(s)$ is statically decoupled if and only if $T(0)$ is a diagonal nonsingular matrix; that is, all the off-diagonal entries of $T(s)$ must be divisible by s, while the entries on the diagonal should not be divisible by s. It can be shown easily that a system described by a $p \times p$ transfer matrix $T(s)$ that is bounded-input/bounded-output stable can be statically decoupled, via $u = Gr$, if and only if

$$rank T(0) = p \tag{50.11}$$

that is, if and only if there is no transmission zero at $s = 0$. Note that this condition, if a controllable and observable state-space description is given, is

$$rank \begin{pmatrix} A & B \\ C & D \end{pmatrix} = n + p \tag{50.12}$$

If this is the case, any feedforward constant gain G, in $u = Gr$, such that $T(0)G$ is a diagonal and nonsingular matrix will statically decouple the system. To illustrate, consider the following example:

EXAMPLE 50.9:

$$T(s) = \begin{pmatrix} \frac{s+2}{s+1} & \frac{2}{s+3} \\ \frac{s(s+1)}{(s+3)^2} & \frac{1}{s+1} \end{pmatrix}$$

Here $T(0) = \begin{pmatrix} 2 & \frac{2}{3} \\ 0 & 1 \end{pmatrix}$, which has full rank; therefore, it can be statically decoupled. Let $T(0)G = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$; then $G = \begin{pmatrix} 1 & \frac{-1}{3} \\ 0 & 1 \end{pmatrix}$. Note that

$$T(s)G = \begin{pmatrix} \frac{s+2}{s+1} & \frac{-s(s-1)}{3(s+1)(s+3)} \\ \frac{s(s+1)}{(s+3)^2} & \frac{-s^3+s^2+17s+27}{3(s+1)(s+3)^2} \end{pmatrix}$$

where all the off-diagonal entries of $T(s)$ are divisible by s, while the entries on the diagonal are not divisible by s. If now the input $\frac{1}{s}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ is applied to $T(s)G$, the steady-state output is $T(0)G\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 2k_1 \\ k_2 \end{pmatrix}$.

50.2 Defining Terms

Decoupling: Separating the system into a number of independent subsystems.

Non-interacting control: The control inputs and the outputs can be partitioned into disjoint subsets; each subset of outputs is controlled by only one subset of inputs, and each subset of inputs affects only one subset of outputs. From an input/output viewpoint, the system is split into independent subsystems; it is called decoupled.

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Further Reading

Making the system diagonally dominant is a powerful design approach. Details on how to achieve diagonal dominance using Rosenbrock's Inverse Nyquist Array method can be found in Rosenbrock, H.H. 1974. *Computer-Aided Control System Design*, Academic Press, New York.

A good introduction to the geometric approach and to the decoupling problem using that approach can be found in Wohnam, W.M. 1985. *Linear Multivariable Control: A Geometric Approach*, Springer-Verlag, New York. The problem of disturbance decoupling or disturbance rejection, where a disturbance in the state equations must become unobservable from the output, is also studied there using the geometric approach.

A geometric approach has also been used to study noninteracting control in nonlinear systems; see, for example, Battilotti, S. 1994. *Noninteracting Control with Stability for Nonlinear Systems*, Springer-Verlag, New York.

For the decoupling of singular systems see, for example Paraskevopoulos, P.N. and Koumboulis, F.N. 1992. The decoupling of generalized state-space systems with state feedback, *IEEE Trans. Autom. Control*, pp. 148–152, vol.37.

The following journals report advances in all areas of decoupling including diagonal, block and triangular decoupling: *IEEE Transactions on Automatic Control*, *International Journal of Control* and *Automatica*.