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OPTIMAL DESIGN OF ROBUST CONTROLLERS FOR UNCERTAIN DISCRETE-TIME SYSTEMS

Ioannis K. Konstantopoulos and Panos J. Antsaklis
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556, U.S.A.
antsakli@saturn.ee.nd.edu
ikonstan@elgreco.helios.nd.edu

Abstract

This paper presents an algorithm for the design of robust output feedback controllers for linear uncertain discrete-time systems. The algorithm utilizes a version of the Broyden-Fletcher-Goldfarb-Shanno (BFGS) optimization method of conjugate directions and minimizes a performance index that includes an LQR term to optimize performance and a robustness term which is based on recently developed bounds. The minimization of only the robustness term which corresponds to the maximization of the uncertainty bound is also studied. The case of unstructured perturbations in A has been the only one studied in the literature; the present algorithm not only introduces a unified approach to both unstructured and structured perturbations in A but also is shown to improve considerably the existing unstructured uncertainty bound. Several other cases involving unstructured/structured perturbations in all the state-space matrices are also presented and examples, including an aircraft control system, are used to illustrate the results.

1. Introduction

The problem of determining a linear feedback control law for uncertain linear systems has drawn considerable attention. Several criteria have been used to characterize the system uncertainties, so that the stability (asymptotic, quadratic, exponential) of the uncertain systems is guaranteed if these criteria are satisfied, and several robust controller design methods have been developed. These methods are based on H_∞ techniques (as in [3], [5], [24]), the LQR/LQG formulation (as in [4], [16], [22]), the Guaranteed Cost Control approach (as in [14], [17]), optimization techniques (as in [6], [7]) and the theory of Lyapunov stability/Riccati equations (as in [20], [21]).

All the above controller design approaches share the same general objective, which is to find a stabilizing controller that satisfies some stability conditions or is robust in some sense, without considering the maximization of any of the robust stability bounds existing in literature. This has been done in [23] for continuous systems with structured uncertainties in the system matrix A and in [11] for discrete-time systems with unstructured uncertainties in A. The design in the first paper relies on the selection of a weighting matrix not directly associated with the structured uncertainties and in the latter on the bound developed in [10]. In both papers, the information about the uncertainty bound is part of the minimizing quantity, which also includes the classical LQR cost. Therefore, the controller design objective is twofold, that is to minimize the LQR cost and maximize the perturbation bounds. A similar approach was used earlier in [19] for continuous systems with structured uncertainties in all the state-space matrices, under some quite restrictive assumptions imposed on the perturbation matrices. Although the maximization of some stability bound is not considered in the design process, the information about the structured uncertainty is directly included in the minimizing quantity.

Here, we present a unified output feedback controller design approach for both cases of unstructured and structured perturbations in A. Note that from the above discussion it is clear that for discrete-time systems, only the case of unstructured perturbations in A has been studied in the literature. Our approach is based on new results for both the structured and unstructured cases, which were recently developed in [12]; these theorems have shown to provide bounds that improve the ones obtained via the methodology suggested in [10] and used in [11]. Our design not only provides a stabilizing static output feedback controller that improves the unstructured bound for A derived in [11], but is also capable of finding another such controller that maximizes the bound for the case that A is perturbed by known uncertainty matrices. In addition, several other interesting cases are studied. In all these cases, the minimizing quantity consists of two terms; one is the robustness term, which is associated with the specific unstructured/structured bound we wish to maximize

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and the other is the LQR term, which is associated with the specific control performance we wish to maintain. Our approach is also applied to a minimizing quantity consisted of only the robustness term, in order to find the controller that maximizes the stability bounds, without considering any control specifications. Note that only the case of static output feedack is studied, since it can easily be shown that the case of dynamic output feedback can be reduced to that of static feedback as well. Finally note that our minimization algorithm utilizes a version of the Broyden family method of conjugate directions, which is based on the BFGS rule, [2], and that the case of state feedback can be easily considered as a special case of the output feedback case for $C = I$.

The paper is organized as follows. In section 2, we present without proofs the new theorems of [12] for the cases of unstructured and structured perturbations in discrete-time systems. In section 3, we study the case of unstructured/structured perturbations in the system matrix A, and present an algorithm based on the BFGS rule that solves the minimization problem. In section 4, we consider unstructured/structured perturbations in either the input matrix B or the output matrix C and in section 5, unstructured/structured perturbations in either (A, B) or (A, C). In section 6, we study the case of unstructured perturbations in all state-space matrices. In section 7, we provide illustrative examples for some of the cases studied above and finally in section 8, concluding remarks are included.

2. Preliminaries

We consider the linear discrete-time system

$$x(k+1) = A\,x(k) \tag{1}$$

where $x \in \Re^n$ is the state vector and A an asymptotically stable matrix. Then, for every symmetric positive definite matrix Q, we can find a symmetric positive definite matrix P, which is the unique solution of the Lyapunov equation

$$A^T P A - P + Q = 0 \tag{2}$$

When A is perturbed by the matrix ΔA, then for the perturbed system

$$y(k+1) = (\ A + \Delta A\ )\ y(k) \tag{3}$$

the following theorem has been proven in [12]. First define

$$\Omega_1 = A^T P Z^{-1} P A \tag{4}$$

Theorem 2.1 *Consider the linear discrete-time system (1) where A is an asymptotically stable matrix that satisfies (2). Suppose that $A \rightarrow A + \Delta A$, then the perturbed system (3) remains asymptotically stable, if $(\Delta A)^T\,(\alpha Z + P)\,(\Delta A) + \frac{1}{\alpha}\,\Omega_1 < Q$ or*

$$\sigma_{max}(\Delta A) < \sqrt{\frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\,\Omega_1)}{\sigma_{max}(\alpha Z + P)}} \tag{5}$$

where P, Q are defined in (2), Ω_1 in (4), Z can be any positive definite matrix of appropriate dimensions, and α any positive number that satisfies

$$\alpha > \frac{\sigma_{max}(\Omega_1)}{\sigma_{min}(Q)} \tag{6}$$

When the perturbation matrix ΔA is described by

$$\Delta A = \sum_{i=1}^{m} \kappa_i A_i = (\tilde{K} \otimes I_n)^T\,\tilde{A} \tag{7}$$

where $\kappa_i, i = 1,..,m$ denote real, uncertain parameters and $A_i, i = 1,..,m$ are constant, known matrices, the following theorem has been proven in [12]. Obviously, we have used the definitions $\tilde{K} = [\kappa_1\ \kappa_2\ \cdots\ \kappa_m]^T$, and $\tilde{A} = [A_1^T\ A_2^T\ \cdots\ A_m^T]^T$.

Theorem 2.2 *The linear discrete-time system (3) with structured perturbations of the form of (7) remains asymptotically stable, if the uncertainty parameters satisfy*

$$\sum_{i=1}^{m} \kappa_i^2 < \frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\,\Omega_1)}{\sigma_{max}^2(\tilde{A})\ \sigma_{max}(\alpha Z + P)} \tag{8}$$

where Ω_1, k_i, $\tilde{A}$ are defined in (4), (7), (7) respectively, Z can be any positive definite matrix of appropriate dimensions, and α any positive number that satisfies (6).

The above theorems provide bounds that improve the ones obtained via the methodology of [10]; this is demonstrated next. The main point of the approach used for the theorems above is the appropriate selection of a positive definite matrix Z and a positive number α that maximize the stability region within which the uncertain parameters vary.

Example 2.1 Consider the following uncertain discrete-time system (3) from [10] with

$$A = \begin{pmatrix} 0.20 & 0.30 \\ 0.10 & -0.15 \end{pmatrix} \tag{9}$$

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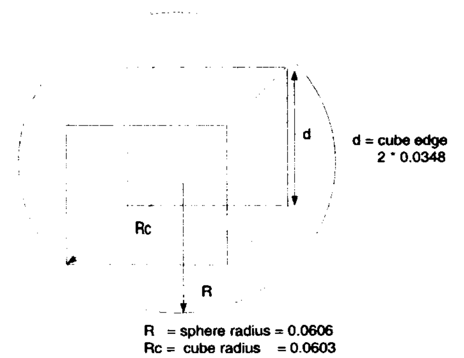

Figure 1: Example 2.2

We choose $Q = I$, $\alpha = 0.2702$ and $Z = \begin{pmatrix} 2.0399 & -0.2037 \\ -0.2037 & 1.4586 \end{pmatrix}$. Using (5) of Theorem 2.1, we obtain $\sigma_{max}(\Delta A) < 0.6787$, which compares favorably to the result of [10], which is $\sigma_{max}(\Delta A) < 0.6373$.

Example 2.2 Consider the same nominal system, but now with structured perturbations of (7), that is

$$A_1 = \begin{pmatrix} 10 & 0.1 \\ -1 & 5 \end{pmatrix},\ A_2 = \begin{pmatrix} -0.5 & 9 \\ 0 & -3 \end{pmatrix},\ A_3 = \begin{pmatrix} 1 & 0.6 \\ 1 & 0.3 \end{pmatrix} \quad (10)$$

We choose $Q = I$, $\alpha = 0.40$, $Z = \begin{pmatrix} 1.3462 & -0.1184 \\ -0.1184 & 0.8786 \end{pmatrix}$. Using (8) of Theorem 2.2, we obtain $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 < (0.0606)^2$, that is a sphere with radius $R = 0.0606$, whereas the method of [10] gives $|\kappa_i| < 0.0348$ for $i = 1, 2, 3$. As we can see in Fig. 1, the defined cube is completely included in the sphere found above, which shows that our bound is less conservative than the one of [10].

3. Perturbations in A

We consider the linear discrete-time system

$$x(k+1) = Ax(k) + B_0u(k), \quad y(k) = C_0x(k) \quad (11)$$

where $x \in \Re^n$ is the state vector, $u \in \Re^r$ is the input vector and $y \in \Re^q$ is the output vector. Both unstructured and structured perturbations for the system matrix A are of interest here, that is $A = A_0 + \Delta A$ and $A = A_0 + \sum_{i=1}^m \kappa_i A_i$. We apply the static output feedback law $u(k) = Ky(k)$. Defining $\bar{A}_0 = A_0 + B_0KC_0$, the closed-loop systems are described respectively by $x(k+1) = (\bar{A}_0 + \Delta A)x(k)$, and $x(k+1) = (\bar{A}_0 + \sum_{i=1}^m \kappa_i A_i)x(k)$.

3.1. Design without performance specifications

Our objective is to find a stabilizing output feedback gain K that maximizes the bounds of (5) and (8). Due to the similarity between these two relations, we present here a unified approach for both the unstructured and structured cases. For the closed-loop systems above, relations (2) and (4) can be translated into

$$\bar{A}_0^T P \bar{A}_0 - P + Q = 0, \qquad \Omega_1 = \bar{A}_0^T P Z^{-1} P \bar{A}_0 \quad (12)$$

Since Q in (12) is selected beforehand, in order to maximize the bounds of (5), (8), we need to

- *(A.1) : minimize* $\sigma_{max}(\alpha Z + P)$
- *(A.2) : minimize* $\sigma_{max}(\frac{1}{\alpha}\,\Omega_1)$

For *(A.1)*, we choose to minimize the quantity -see [13]- $J_{11} = Tr(\alpha^2 Z^2 + 2\alpha ZP + P^2)$. For *(A.2)*, we have $\sigma_{max}(\frac{1}{\alpha}\,\Omega_1) \leq \frac{1}{\alpha}\,\sigma_{max}^2(\bar{A}_0)\,\sigma_{max}^2(P)\,\sigma_{max}(Z^{-1})$. Since Z is selected beforehand and an upper bound of $\sigma_{max}(P)$, that is $Tr(P^2)$, is already minimized in J_{11}, for *(A.2)* we simply choose to minimize $J_{12} = \frac{1}{\alpha}\,Tr(\bar{A}_0^T\,\bar{A}_0)$, which is an upper bound of $\frac{1}{\alpha}\sigma_{max}^2(\bar{A}_0)$. Note that the minimization of the sum of $Tr(\frac{1}{\alpha}\,\bar{A}_0^T\bar{A}_0)$ and $Tr(P^2)$ is an indirect and harder way to minimize their product; in other words, we impose a more demanding task on the minimizing process. Note also that α is included in the minimizing quantity, because we need to satisfy the positiveness of the numerator, as indicated in (6). Therefore, the minimizing quantity is given as $J_1' = J_{11} + J_{12}$ under the condition that (12) holds. This is clearly a constrained minimization problem. By including (12) in J_1', we finally reduce the problem to an unconstrained minimization one, with the minimizing quantity finally given by

$$\begin{aligned} J_1 &= Tr[\,\alpha^2 Z^2 + 2\alpha PZ + P^2 + \frac{1}{\alpha}\,\bar{A}_0^T\bar{A}_0 \\ &\quad + L_1\,(\bar{A}_0^T P \bar{A}_0 - P + Q)\,] \end{aligned} \quad (13)$$

where $L_1 \in \Re^{n\times n}$ is the Lagrange multiplier matrix. Next, we need the following properties from [1]

$$\frac{\partial}{\partial X}\,Tr(X^2) = 2X^T \quad (14)$$

$$\frac{\partial}{\partial Y}\,Tr(A_1YB_1) = A_1^TB_1^T \quad (15)$$

$$\frac{\partial}{\partial Y}\,Tr(A_2Y^TB_2) = B_2A_2 \quad (16)$$

$$\frac{\partial}{\partial Y}\,Tr(A_3YB_3Y^T) = A_3YB_3 + A_3^TYB_3^T \quad (17)$$

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for any $X \in \Re^{n\times n}$, $Y \in \Re^{n\times m}$, $A_1 \in \Re^{l\times n}$, $B_1 \in \Re^{m\times l}$, $A_2 \in \Re^{l\times m}$, $B_2 \in \Re^{n\times l}$, $A_3 \in \Re^{n\times n}$, $B_3 \in \Re^{m\times m}$. With these properties, we have

$$\frac{\partial J_1}{\partial L_1} = \Delta^1_{L_1} = \bar{A}_0^T P \bar{A}_0 - P + Q \quad (18)$$

$$\frac{\partial J_1}{\partial \alpha} = \Delta^1_{\alpha} = 2\alpha\, Tr(Z^2) + 2\, Tr(PZ) - \frac{1}{\alpha^2}\, Tr(\bar{A}_0^T \bar{A}_0) \quad (19)$$

$$\frac{\partial J_1}{\partial P} = \Delta^1_{P} = 2P + 2\alpha Z + \bar{A}_0 L_1^T \bar{A}_0^T - L_1^T \quad (20)$$

$$\frac{\partial J_1}{\partial K} = \Delta^1_{K} = \frac{2}{\alpha} B_0^T B_0 K C_0 C_0^T + \frac{2}{\alpha} B_0^T A_0 C_0^T + B_0^T P B_0 K C_0\,(L_1 + L_1^T) C_0^T + B_0^T P A_0 (L_1 + L_1^T)\, C_0^T \quad (21)$$

To minimize (13), we use a version of the Broyden family method of conjugate directions, which is based on the Broyden-Fletcher-Goldfarb-Shanno (BFGS) update rule; details in [2]. The proposed algorithm is presented next.

Initialization Step Let $\epsilon > 0$ be the termination scalar. Choose an initial stabilizing gain

$$K_1 = \begin{pmatrix} (\tau_1^1)^T \\ \vdots \\ (\tau_r^1)^T \end{pmatrix} \quad (22)$$

where $(\tau_l^1)^T, l = 1,..,r$ are the $1 \times q$ rows of K_1, which stabilizes (A_0, B_0, C_0), that is $\bar{A}_0$ stable. Also, choose an initial symmetric positive definite matrix $D_1 \in \Re^{rq\times rq}$. Let

$$y_1 = x_1 = (\,(\tau_1^1)^T \;\cdots\; (\tau_r^1)^T\,)^T \quad (23)$$

be a column vector consisting of the transposes of the rows of K_1. Also let $k = j = 1$ and go to the *Main Step.*

Main Step

M1. Substitute the gain matrix K_j in the gradients of (18)-(20), set them to zero, that is $\Delta^1_{L1} = 0$, $\Delta^1_\alpha = 0$, $\Delta^1_P = 0$, and solve for P, α, L_1 respectively, in that order.

M2. Substitute these parameters in (21) and compute

$$\Delta^1_{K_j} = \begin{pmatrix} (\sigma_1^j)^T \\ \vdots \\ (\sigma_r^j)^T \end{pmatrix} \quad (24)$$

where $(\sigma_l^j)^T, l = 1,..,r$ are the $1 \times q$ rows of $\Delta^1_{K_j}$.

M3. Define $\nabla J_1(y_j) = (\,(\sigma_1^j)^T \cdots (\sigma_r^j)^T\,)^T$. If $\|\nabla J_1(y_j)\| < \epsilon$, *STOP.* The optimal gain is K_j. Otherwise, go to M4.

M4. If $j > 1$, update the positive definite matrix D_j

$$D_j = D_{j-1} + \frac{p_{j-1}p_{j-1}^T}{p_{j-1}^T q_{j-1}}\,[1 + \frac{q_{j-1}^T D_{j-1} q_{j-1}}{p_{j-1}^T q_{j-1}}] - \frac{[D_{j-1}q_{j-1}p_{j-1}^T + p_{j-1}q_{j-1}^T D_{j-1}]}{p_{j-1}^T q_{j-1}} \quad (25)$$

where $p_{j-1} = \lambda_{j-1} d_{j-1} = y_j - y_{j-1}$, and $q_{j-1} = \nabla J_1(y_j) - \nabla J_1(y_{j-1})$.

M5. Define $d_j = -D_j \nabla J_1(y_j)$, and let λ_j be an optimal solution to the problem of minimizing $J_1(y_j + \lambda d_j)$ subject to $\lambda \geq 0$. Let $y_{j+1} = y_j + \lambda_j d_j = (\,(\tau_1^{j+1})^T \cdots (\tau_r^{j+1})^T\,)^T$, which implies that

$$K_{j+1} = \begin{pmatrix} (\tau_1^{j+1})^T \\ \vdots \\ (\tau_r^{j+1})^T \end{pmatrix} \quad (26)$$

where obviously $(\tau_l^{j+1}), l = 1,..,r$ are $q \times 1$ column vectors.

M6. If $j < rq$, replace j by $j+1$ and repeat the *Main Step.* Otherwise, if $j = rq$, let $y_1 = x_{k+1} = y_{rq+1}$, replace k by $k+1$, let $j = 1$ and repeat the *Main Step.* □□

Several issues need to be discussed here. First, note that the line search in *(M5)* is restricted to stabilizing gain matrices. Therefore, the selected new gain matrix needs first to stabilize the closed-loop system and then minimize J_1. Since our algorithm is an indirect version of the BFGS algorithm, as an alternative to the stopping criterion of *(M3)*, we could use another quite practical criterion. Specifically, we may consider monitoring J_1 and stop when we see that J_1 is sufficiently small and the derived bound derived is satisfactorily large. From (19), we can easily see that there is at least one real positive solution for α. For our algorithm, we choose to keep the largest value of α, since we also need to satisfy the positiveness of the numerator of (5) and (8), as discussed before. Finally, note that for optimization problems similar to the one we study here, alternative methods based on gradient-type and nongradient-type algorithms have been proposed in [8] and [18] respectively.

3.2. Design with performance specifications

In the previous subsection, we focused on finding an output feedback gain K that maximizes the bounds of (5) and (8). If, in addition to this objective, we also wish to attain a specific control performance, then we need to include in our minimizing quantity a term that evaluates this control performance. Therefore, we consider the familiar LQR cost $J_2' = \sum_{k=0}^{\infty} x^T(k) Q_1 x(k) + u^T(k) R_1 u(k)$,

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where Q_1, R_1 are positive definite matrices of appropriate dimensions. For the nominal system (A_0, B_0, C_0) with the output feedback law $u(k) = Ky(k)$, we finally choose, see [13], the following modified cost $J_2 = Tr(P_2\ X_0)$, where P_2 is the solution of the Lyapunov equation

$$\bar{A}_0^T P_2 \bar{A}_0 - P_2 + \bar{Q} = 0 \tag{27}$$

We have defined $\bar{Q} = Q_1 + C_0^T K^T R_1 K C_0$, $X_0 = E[x(0)x^T(0)] > 0$. Usually, [15], we consider $x(0)$ uniformly distributed on a sphere of radius σ, that is $X_0 = \sigma I_n$, with $\sigma = 1$ the obvious choice. Therefore, the overall minimizing quantity, which is associated with both the robustness of $\bar{A}_0$ and the control performance of the closed-loop system is given by

$$\begin{aligned} J_A &= Tr[\alpha^2 Z^2 + 2\alpha P Z + P^2 + \frac{1}{\alpha}\bar{A}_0^T \bar{A}_0 + L_1(\bar{A}_0^T P \bar{A}_0 \\ &\quad - P + Q) + P_2 X_0 + L_2(\bar{A}_0^T P_2 \bar{A}_0 - P_2 + \bar{Q})] \end{aligned} \tag{28}$$

where, similarly to (13), we have reduced the problem to an unconstrained minimization one by including (27). Due to the introduction of P_2 and L_2 in the new cost J_A, we need to consider its partial derivatives with respect to these new matrix variables as well. For the same reason, we have some additional terms in Δ_K^1 of (21).

$$\frac{\partial J_A}{\partial L_1} = \Delta_{L_1}^A = \Delta_{L_1}^1 \tag{29}$$

$$\frac{\partial J_A}{\partial L_2} = \Delta_{L_2}^A = \bar{A}_0^T P_2 \bar{A}_0 - P_2 + \bar{Q} \tag{30}$$

$$\frac{\partial J_A}{\partial \alpha} = \Delta_\alpha^A = \Delta_\alpha^1 \tag{31}$$

$$\frac{\partial J_A}{\partial P} = \Delta_P^A = \Delta_P^1 \tag{32}$$

$$\frac{\partial J_A}{\partial P_2} = \Delta_{P_2}^A = X_0^T + \bar{A}_0 L_2^T \bar{A}_0^T - L_2^T \tag{33}$$

$$\begin{aligned} \frac{\partial J_A}{\partial K} = \Delta_K^A &= \Delta_K^1 + R_1 K C_0\,(L_2 + L_2^T)\,C_0^T \\ &\quad + B_0^T P_2 B_0 K C_0\,(L_2 + L_2^T)\,C_0^T \\ &\quad + B_0^T P_2 A_0\,(L_2 + L_2^T)\,C_0^T \end{aligned} \tag{34}$$

To minimize J_A, the algorithm of the previous subsection can be used again, the only difference being that steps *(M1)*, *(M2)* have to be replaced by the following

M1a. Substitute the gain matrix K_j in the gradients of (29)-(33), set them to zero, that is $\Delta_{L1}^A = 0$, $\Delta_{L2}^A = 0$, $\Delta_\alpha^A = 0$, $\Delta_P^1 = 0$, $\Delta_{P_2}^A = 0$ and solve for P, P_2, α, L_1, L_2 respectively, in that order.

M2a. Substitute these parameters in (34) and compute

$$\Delta_{K_j}^A = \begin{pmatrix} (\sigma_1^j)^T \\ \vdots \\ (\sigma_r^j)^T \end{pmatrix} \tag{35}$$

where $(\sigma_l^j)^T, l = 1,..,m$ are the $1 \times q$ rows of $\Delta_{K_j}^A$.

4. Perturbations in B or C

We consider perturbations in either the input matrix B or the output matrix C. Since both cases are similar, we study the case of perturbations in B. Therefore, we consider the linear discrete-time system

$$x(k+1) = A_0 x(k) + Bu(k), \qquad y(k) = C_0 x(k) \tag{36}$$

with static output feedback for both unstructured and structured perturbations in the input matrix B, that is $B = B_0 + \Delta B$, $B = B_0 + \sum_{i=1}^m \lambda_i B_i$ respectively.

4.1. Unstructured perturbations

It can easily be shown, [13], that the stability of the closed-loop system is maintained, if $\sigma_{max}(\Delta B) < \frac{\sqrt{\frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\Omega_1)}{\sigma_{max}(\alpha Z + P)}}}{\sigma_{max}(KC_0)}$. Hence, in addition to the minimization objectives of (A.1) and (A.2), we also need to

- *(A.3) : minimize* $\sigma_{max}(KC_0)$

Instead of *(A.3)*, we choose to minimize its upper bound, that is $J_3 = Tr[(KC_0)^T(KC_0)]$. The minimizing quantity is $J_{B_u}^r = J_1 + J_3$ when no performance specifications are considered, where J_1 is defined in (13). When performance specifications are considered, then $J_{B_u}^{r,p} = J_A + J_3$, where J_A is defined in (28). The algorithm of the previous section can be used here for $J_{B_u}^r$ and $J_{B_u}^{r,p}$ as well, the only difference being that the term $\frac{\partial J_3}{\partial K} = 2\ KC_0C_0^T$ needs to be added to (21) and (34).

4.2. Structured perturbations

It can easily be shown, [13], that the stability of the closed-loop system is maintained, if $\sum_{i=1}^m \lambda_i^2 < \frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\Omega_1)}{\sigma_{max}^2(B^\star KC_0)\,\sigma_{max}(\alpha Z + P)}$, where $B^\star = (B_1^T \cdots B_m^T)^T$. Now, in addition to *(A.1)* and *(A.2)*, we also need to

- *(A.4) : minimize* $\sigma_{max}(B^\star KC_0)$

We choose to minimize $J_4 = Tr[(B^\star KC_0)^T(B^\star KC_0)]$ so that the minimizing quantities are now $J_{B_s}^r = J_1 + J_4$, $J_{B_s}^{r,p} = J_A + J_4$. The algorithm of the previous section

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applies here for $J^r_{B_s}$ and $J^{r,p}_{B_s}$ as well, the only difference now being that the term $\frac{\partial J_4}{\partial K} = 2\,(B^*)^T B^* K C_0 C_0^T$ needs to be added to (21) and (34).

5. Perturbations in (A, B) or (A, C)

We consider perturbations in the system matrix A and the input matrix B or in A and the output matrix C. Since both cases are similar, we study the case of perturbations in (A, B).

5.1. Unstructured perturbations

We consider the linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \qquad y(k) = C_0 x(k) \qquad (37)$$

where $A = A_0 + \Delta A$, $B = B_0 + \Delta B$ and static output feedback is applied. Here, the stability of the closed-loop system is maintained, if $\sigma_{max}(\Delta A) + \sigma_{max}(\Delta B)\,\sigma_{max}(KC_0) < \sqrt{\frac{\sigma_{min}(Q)-\sigma_{max}(\frac{1}{\alpha}\Omega_1)}{\sigma_{max}(\alpha Z+P)}}$. It can be shown, [13], that in order to maximize the stability region defined above, we need to satisfy the objectives *(A.1)*, *(A.2)*, *(A.3)*. We see that the present case has the same objectives with the case of unstructured perturbations in B or C we studied before.

5.2. Structured perturbations

We consider again the system of (37) with $A = A_0 + \sum_{i=1}^{m_A} \kappa_i A_i$, $B = B_0 + \sum_{j=1}^{m_B} \lambda_j B_j$. Here, the closed-loop system remains asymptotically stable, if $\sum_{i=1}^{m_A+m_B} \hat{\theta}_i^2 < \frac{\sigma_{min}(Q)-\sigma_{max}(\frac{1}{\alpha}\Omega_1)}{\sigma_{max}^2(\hat{\Pi})\,\sigma_{max}(\alpha Z+P)}$, where $\hat{\theta}_i = \kappa_i, i = 1,..,m_A$ and $\hat{\theta}_i = \lambda_{i-m_A}, i = m_A+1,..,m_A+m_B$, and $\hat{\Pi} = (A_1^T \;\cdots\; A_{m_A}^T \;\; (B_1KC_0)^T \;\cdots\; (B_{m_B}KC_0)^T)^T$

Therefore, in addition to *(A.1)*, *(A.2)*, we also need to

- *(A.5) : minimize* $\sigma_{max}(\hat{\Pi})$

Similarly to before, we choose to minimize $J_5 = Tr[\hat{\Pi}^T\hat{\Pi}]$, so that the minimizing quantities are $J^r_{AB} = J_1 + J_5$, $J^{r,p}_{AB} = J_A + J_5$. Hence, our algorithm can be used again, the only diffrence being that the term $\frac{\partial J_5}{\partial K} = \sum_{j=1}^{m_B} 2\, B_j^T B_j K C_0 C_0^T$ needs to be added to (21) and (34).

6. Perturbations in (A, B, C)

We consider unstructured perturbations in all system matrices. Specifically, we consider the linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \qquad y(k) = Cx(k) \qquad (38)$$

where $A = A_0 + \Delta A$, $B = B_0 + \Delta B$, $C = C_0 + \Delta C$. Here, the stability of the closed-loop system is maintained, if

$$\sigma_{max}(\Delta_{ABC}) < \sqrt{\frac{\sigma_{min}(Q) - \sigma_{max}(\frac{1}{\alpha}\Omega_1)}{\sigma_{max}(\alpha Z + P)}} \qquad (39)$$

where

$$\begin{aligned} \sigma_{max}(\Delta_{ABC}) &= \sigma_{max}(\Delta A) \\ &+ \sigma_{max}(\Delta B)\sigma_{max}(K)\sigma_{max}(C_0) \\ &+ \sigma_{max}(B_0)\sigma_{max}(K)\sigma_{max}(\Delta C) \\ &+ \sigma_{max}(\Delta B)\sigma_{max}(K)\sigma_{max}(\Delta C) \end{aligned} \qquad (40)$$

This inequality defines a region in $\Re^3$ for $\sigma_{max}(\Delta A)$, $\sigma_{max}(\Delta B)$ and $\sigma_{max}(\Delta C)$. It can be shown, see [13], that in order to maximize the volume of this region, we need to satisfy objectives *(A.1)*, *(A.2)* above, and also

- *(A.6) : minimize* $\sigma_{max}(K)$

which corresponds to the minimization of $J_6 = Tr(K^TK)$. Therefore, the minimizing quantities are $J^r_{ABC} = J_1 + J_6$, $J^{r,p}_{ABC} = J_A + J_6$. Our algorithm can be used again, with the addition of the term $\frac{\partial J_6}{\partial K} = 2K$ to (21) and (34).

7. Illustrative examples

Example 7.1 Consider the scalar system

$$x(k+1) = 0.5\,x(k) + u(k), \qquad x(0) = 1.0 \qquad (41)$$

with state feedback $u(k) = Kx(k)$. This system was studied in [11], where the LQR cost $J_2' = \sum_{k=0}^{\infty} x^2(k) + u^2(k)$ was used, that is $Q_1 = R_1 = 1$. The derived bound for unstructured perturbations in the system matrix A was $\sigma_{max}(\Delta A) < 0.8436$, for a gain of $K = -0.3436$. We apply our method for the same LQR term. Choosing $Q = 1.30$, $Z = 0.60$, initial stabilizing gain $K_1 = 0.1$ and positive definite matrix $D_1 = 0.001$, we obtain a stabilizing gain of $K = -0.49998$, which corresponds to $\sigma_{max}(\Delta A) < 0.99998$, which compares favorably to the result of [11] given above. The components of the minimizing quantity (28) that are associated with the robustness and the performance objectives are $J_1 = 1.69$ and $J_2 = 1.25$ respectively.

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Neglecting the performance specifications, as indicated by the LQR cost, and focusing on just the maximization of the robustness bound, that is the minimization of (13), we obtain a stabilizing gain of $K = -0.49999$, which corresponds to $\sigma_{max}(\Delta A) < 0.99999$ and $J_1 = 1.6900$. Note that the same Q, Z, K_1 and Q_1 have been used. As we see, in this scalar case, we obtain almost the same results for the final stabilizing gain K, the uncertainty bound and the robustness component J_1 of the minimizing quantity, no matter whether the LQR term is included or not in the minimizing quantity. Note, however, that this is not the case, in general, for MIMO systems, as we can see in the examples of [13] and the example that follows.

Example 7.2 Consider an aircraft longitudinal control system from [9], whose the linearized continuous dynamic model is given by

$$\dot{x}(t) = \begin{pmatrix} -0.0582 & 0.0651 & 0 & -0.171 \\ -0.303 & -0.685 & 1.109 & 0 \\ -0.0715 & -0.658 & -0.947 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 & 1 \\ -0.0541 & 0 \\ -1.11 & 0 \\ 0 & 0 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x(t) \tag{42}$$

where $x(t) = (\alpha(t)\ \ \beta(t)\ \ \psi(t)\ \ \theta(t))^T$, $u(t) = (\eta(t)\ \ \tau(t))^T$, $\alpha(t)$ and $\beta(t)$ are the forward and vertical speeds, $\psi(t)$ is the pitch rate and $\theta(t)$ is the pitch angle. The control inputs $\eta(t)$ and $\tau(t)$ are the elevator angle and throttle position respectively. Note that all states are assumed available for measurement. We consider the discrete-time model for $T = 0.5$ sec. The state-space matrices are given by

$$A_d = \begin{pmatrix} 0.9692 & 0.0283 & -0.0112 & -0.0842 \\ -0.1302 & 0.6469 & 0.3584 & 0.0059 \\ -0.0086 & -0.2126 & 0.5644 & 0.0007 \\ -0.0041 & -0.0621 & 0.3873 & 1.0001 \end{pmatrix}$$

$$B_d = \begin{pmatrix} 0.0017 & 0.4924 \\ -0.1385 & -0.0344 \\ -0.4266 & -0.0041 \\ -0.1170 & -0.0009 \end{pmatrix}, \quad C_d = I_4 \tag{43}$$

We study the case of structured perturbations in the system matrix A_d; specifically we assume that

$$\Delta A_d = \kappa_1 \begin{pmatrix} 0.1 & 0.15 & 0 & 0 \\ 0.05 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 \\ 0.1 & 0 & 0 & 0.05 \end{pmatrix} + \kappa_2 \begin{pmatrix} 0 & 0 & 0 & 0.05 \\ 0 & -0.1 & 0 & 0 \\ 0 & 0 & 0 & 0.05 \\ 0 & 0 & 0.05 & 0 \end{pmatrix} \tag{44}$$

First we need to find the static output feedback matrix that maximizes the stability bound (8), without considering performance specifications. We choose $Q = 10^{-2} I_4$, $Z = \begin{pmatrix} 0.0110 & 0 & 0 & 0 \\ 0 & 0.0148 & -0.0007 & -0.0110 \\ 0 & -0.0007 & 0.0199 & -0.0028 \\ 0 & -0.0110 & -0.0028 & 0.0047 \end{pmatrix}$, initial gain $K_1 = \begin{pmatrix} -0.0264 & -0.1722 & 3.0531 & 10.2700 \\ -1.6068 & 0.2706 & 0.0224 & -0.0742 \end{pmatrix}$, that places the closed-loop poles at $(0.20, 0.70, -0.50 \pm 0.25j)$, and $D_1 = I_8$. Our algorithm converges to the stabilizing gain $K = \begin{pmatrix} -0.1089 & -0.5016 & 2.2162 & 1.5402 \\ -1.8981 & 0.0748 & 0.0317 & -0.1342 \end{pmatrix}$ which corresponds to the $J_1 = 0.4468$. With this gain, the stability region for the uncertain parameters is defined by the circle $\kappa_1^2 + \kappa_2^2 < (0.6161)^2$.

Next, we include the LQR cost in the minimizing quantity (28) for $Q_1 = 0.1 I_4$, $R_1 = 0.01 I_2$, $X_0 = 0.1 I_4$. We consider the same Q, Z, D_1, K_1 as before and obtain the output gain $K = \begin{pmatrix} -0.0672 & -0.1988 & 1.4512 & 0.9723 \\ -1.9253 & 0.0895 & 0.0590 & 0.0641 \end{pmatrix}$, which corresponds to $J_1 = 0.3985$ and $J_2 = 0.0932$. This gain defines the circle $\kappa_1^2 + \kappa_2^2 < (0.5269)^2$. Note that the uncertainty radius here is more conservative compared to the case of only the robustness specifications studied above.

Note that for the examples presented above and the numerous examples of [13], our algorithm proved to be quite fast. The algorithm, written in MATLAB code, converged in just several iterations of the *Main Step*; this took approximately a minute on a Sun SPARCstation 10.

8. Conclusions

A fast optimization algorithm for the design of robust output feedback controllers for linear uncertain discrete-time systems based on the BFGS rule has been presented. The minimizing quantity reflects the twofold optimization objective, which is the simultaneous maximization of established uncertainty bounds and the minimization of an LQR performance criterion. The first objective is based on recently established improved bounds that were developed in [12]. Note that the algorithm has also been applied to the case where the LQR term is not included in the minimizing

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quantity, so that the only objective is the design of a stabilizing controller that maximizes the uncertainty bounds. In that case, the derived stability bounds are, in general, larger than the ones derived in the case of the robustness/LQR minimizing quantity. This was expected, since the inclusion of the LQR term in the minimizing quantity added an additional requirement to the optimization task.

Previous work was restricted to the case of unstructured perturbations in A. Here, a unified approach to both unstructured and structured perturbations in A has been presented. It has been shown that the present design process improves significantly the unstructured bound of [11]. Additionally, the cases of unstructured/structured perturbations in B or C, in (A, B) or (A, C), together with the case of unstructured perturbations in (A, B, C) have also been studied. Note that the case of structured perturbations in all the state-space matrices remains to be addressed.

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