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ON THE COMMON (LEFT,RIGHT) DIVISORS OF TWO POLYNOMIAL MATRICES

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Summary

The notions of common right and common left divisors of two polynomial matrices are well established. They can be used to characterize the unobservable and the uncontrollable modes of a system and to study polynomial matrix equations of the form $a_1V_1 + a_2V_2 = V_3$. The new notion of common (right,left) divisors of two polynomial matrices is introduced here and it is used to characterize the modes of the system which are both uncontrollable and unobservable; it is also used to study polynomial matrix equations of the form

$$Nx_1 + x_2D = V \tag{1}$$

where $N(qxp)$, $D(rxm)$ and $V(qxm)$ are given polynomial matrices; (1) has attracted recently considerable attention ([1] to [7]) mainly because of its importance in the output regulation problem with internal stability.

To introduce the common (right,left) divisors, consider the equation

$$Nx_1 + x_2D = 0 \tag{2}$$

which is important in solving (1), since the complete solution of (1) can be written as the sum of a particular solution of (1) and the general solution of (2). Assume that D is square and nonsingular which is the case of interest in the problem of regulation.

Theorem. The general solution of (2) is given by

$$(x_1,x_2) = ((W + M_2G_2^{-1})D\ ,\ -N(W + G_1^{-1}M_1)) \tag{3}$$

where W is any polynomial matrix, M_1 and M_2 polynomial matrices such that $M_2G_2^{-1} = G_1^{-1}M_1$ strictly proper with (M_2,G_2) right prime, (M_1,G_1) left prime and G_1,G_2 right, left divisors of N,D respectively.

Proof. Let (x_1,x_2) be a solution of (2). Then $Nx_1D^{-1} = -x_2$. Write $x_1D^{-1} = W + M_2G_2^{-1} = W + G_1^{-1}M_1$ (using division algorithm [8]) where W some polynomial matrix, (M_2,G_2) right prime, (M_1,G_1) left prime with $M_2G_2^{-1} = G_1^{-1}M_1$ strictly proper; note that $G_2^{-1}D$ is a polynomial matrix i.e. G_2 is a left divisor of D. Then $N(W + G_1^{-1}M_1) = -x_2$ which implies that NG_1^{-1} is a polynomial matrix i.e. G_1 is a right divisor of N. Therefore the solution (x_1,x_2) is of the form (3). Conversely x_1 and x_2 from (3) satisfy (2).

Definition. (G_1,G_2) is a pair of common (right, left) divisors of (N,D) (or (G_1,G_2) is $c(r,l)d$ of (N,D) if

$$N = \hat{N}G_1,\ D = G_2\hat{D} \tag{4}$$

and there exist (pxm) polynomial matrices M_1,M_2 such that

$$G_1^{-1}\,M_1 = M_2\,G_2^{-1} \tag{5}$$

with (M_1,G_1) left prime and (M_2,G_2) right prime polynomial matrices.□ (5) implies that $|G_1| = \alpha|G_2|$. G_1 and G_2 not only have the same invariant polynomials but they also contain certain structure common to N and D. G_1 is related to G_2 in the same way the denominators P_1, P_2 of two prime factorizations of a transfer matrix $T = P_1^{-1}\cdot Q_1 = R_2P_2^{-1}$ are related [9][10]. A greatest common (right,left) divisor $(gc(r,l)d)$ (G_1^*,G_2^*) can also be defined; consider it to be a $c(r,l)d$ (G_1^*,G_2^*) such that: degree $|G_1^*|$ = degree $|G_2^*|$ is maximum. Note that the notion of $c(r,l)d$ is unique to matrices, since if polynomials are considered then a $c(r,l)d$ is just a common divisor (i.e. a common factor) of the polynomials.

In view of the definition it is clear that the general solution (3) is written in terms of $c(r,l)d$ of (N,D). If all such divisors are unimodular i.e. the $gc(r,l)d$ of (N,D) unimodular, the general solution to (2) is: $(x_1,x_2) = (WD, -NW)$. An example of such case is when N is also square and $|N|$, $|D|$ are prime polynomials. (See [5, Theorem 4]).

The $c(r,l)d$ plays a role in characterizing the modes of the system

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$$Pz = Qu; \; y = Rz + Wu \qquad (6)$$

which are both uncontrollable and unobservable. In particular, let G_L and G_R be greatest common left (gcl) and right (gcr) divisors of P,Q and P,R respectively. It is known that the roots of $|G_L|$ are all the uncontrollable ($\bar{c}$) modes [8] (the input decoupling zeros [11]) of the system. Similarly, the roots of $|G_R|$ are all the unobservable ($\bar{o}$) modes (the output decoupling zeros) of the system. Furthermore, G_L and G_R also contain certain structural information about the system which can be translated into information about the corresponding uncontrollable and unobservable subspaces. Write $P = G_L P_L = P_R G_R$ and let $\hat{G}_R$ be a gcr divisor of P_L, R. Then $\hat{G}_R$ will also be a gcr divisor of G_R and P_L. That is:

$$G_R = G_2 \hat{G}_R, \; P_L = P_2 \hat{G}_R \qquad (7)$$

with G_2, P_2 right prime. $\hat{G}_R$ contains those $\bar{o}$ modes which are c, while G_2 contains those $\bar{o}$ modes which are also $\bar{c}$. (Note that similar approach is described in [11]) Similarly, let $\hat{G}_L$ be a gcl divisor of P_R, Q. Then

$$G_L = \hat{G}_L G_1, \; P_R = \hat{G}_L P_1 \qquad (8)$$

with G_1, P_1 left prime. $\hat{G}_L$ contains those $\bar{c}$ modes which are o while G_1 contains those $\bar{c}$ modes which are also $\bar{o}$. Note that

$$G_L^{-1} P G_R^{-1} = P_L G_R^{-1} = P_2 G_2^{-1} = G_L^{-1} P_R = G_1^{-1} P_1 \qquad (9)$$

The zeros of $|G_2| = \alpha |G_1|$ are exactly the $\bar{c}\bar{o}$ modes. Furthermore G_2 and G_1 contain structural information which can be translated into information about the uncontrollable part of the unobservable subspace (G_2) or about the unobservable part of the uncontrollable subspace (G_1).

$$P = \underset{(\bar{c})}{G_L} \underset{(c)}{P_L} = \underset{(\bar{c}o)}{\hat{G}_L} \underset{(\bar{c}\bar{o})}{G_1} \underset{(co)}{P_2} \underset{(c\bar{o})}{\hat{G}_R} = \underset{(o)}{P_R} \underset{(\bar{o})}{G_R}$$
$$= \underset{(o\bar{c})}{\hat{G}_L} \underset{(oc)}{P_1} \underset{(\bar{o}\bar{c})}{G_2} \underset{(\bar{o}c)}{\hat{G}_R} \qquad (10)$$

which corresponds to the (Kalman) state-space decomposition into co, $\bar{c}$o, co, $\bar{c}$o subspaces.

Note that, from (9), $P_2 G_2^{-1} = G_1^{-1} P_1$ where (P_2, G_2) right prime, (P_1, G_1) left prime with G_1 and G_2 right and left divisors of G_L and G_R respectively. i.e. (G_1, G_2) is a c(r,l)d of (G_L, G_R); it is also a c(l,r)d of $\hat{G}_L^{-1} P \hat{G}_R^{-1}$ (= $G_1 P_2 = P_1 G_2$).

The new notion of c(r,l)d appears to be a natural extension of the notions of cr and cl divisors. Further investigation is needed to study the full potential of this new idea and its implications.

References

1. G. Bengtsson, "Output Regulation and Internal Models. A Frequency Domain Approach", Automatica, Vol. 13, pp. 333-345, 1977.

2. L. Cheng and J.B. Pearson, "Frequency Domain Synthesis of Multivariable Linear Regulators", IEEE Trans. Autom. Contr., Vol. AC-23, pp. 3-15, February 1978.

3. W.A. Wolovich and P. Ferreira, "Output Regulation and Tracking in Linear Multivariable Systems", IEEE Trans. Autom. Contr., Vol. AC-24, pp. 460-465, June 1979.

4. P.J. Antsaklis, "Some Relations Satisfied by Prime Polynomial Matrices and Their Role in Linear Multivariable System Theory", Technical Report No. 7806, Department of Electrical Engineering, Rice University, June 1978, and IEEE Trans. Autom. Contr., Vol. AC-24, pp. 611-616, August 1979.

5. W.A. Wolovich, "Skew Prime Polynomial Matrices", IEEE Trans. Autom. Contr., Vol. AC-23, pp. 880-887, October 1978.

6. P.J. Antsaklis, "On Multivariable Regulators. An Explicit Description", Proceedings of the 19th IEEE Conference on Decision and Control, Albuquerque, New Mexico, December 1980.

7. E. Emre and L.M. Silverman, "The Polynomial Equation XR + QY = Φ: A Characterization of Solutions", Proceedings of the 19th Conference on Decision and Control, Albuquerque, New Mexico, December 1980.

8. W.A. Wolovich, Linear Multivariable Systems, Springer-Verlag, 1974.

9. P.A. Fuhrmann, "On Strict System Equivalence and Similarity", Int. J. Control, Vol. 25, pp. 5-10, 1977.

10. W.A. Wolovich, "On the Numerators and Zeros of Rational Transfer Matrices", IEEE Trans. Autom. Contr., Vol. AC-18, pp. 544-546, Oct. 1973.

11. H.H. Rosenbrock, State-Space and Multivariable Theory, Wiley, 1970.