I.K. Konstantopoulos and P.J. Antsaklis, "Optimal Design of Robust Controllers for Uncertain Discrete-Time Systems," I ntern J ournal o f C ontro l , Vol.65, No.1, pp. 71-91, 1996.

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QS941 PD1Q17 1DN 21pp

INT. J. CONTROL, 1996, VOL. 0, NO. 0, 000–000

100526

Vol 65, No 1, pp 71-91, 1996

Optimal design of robust controllers for uncertain discrete-time systems

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This paper presents a fast algorithm for the design of robust output feedback controllers for linear uncertain discrete-time systems. The algorithm utilizes a version of the Broyden–Fletcher–Goldfarb–Shanno (BFGS) optimization method of conjugate directions and minimizes a performance index that includes an linear quadratic regulator (LQR) term to optimize performance and a robustness term based on recently developed bounds. The minimization of only the robustness term which corresponds to the maximization of the uncertainty bound is also studied. The case of unstructured perturbations in A has been the only one studied in the robust controller design literature; the present algorithm introduces a unified approach to cases of both unstructured and structured perturbations in the matrices of a state-space model. For the special case of unstructured perturbations in A only, the algorithm is shown to improve considerably the existing unstructured uncertainty bound. Several examples, including an aircraft control system and a paper-machine head box, are presented to illustrate the results.

1. Introduction

The problem of determining a linear feedback control law for uncertain linear systems has drawn considerable attention; for general information concerning static output feedback see Syrmos *et al.* (1994). Several criteria have been used to characterize the system uncertainties, so that the stability (asymptotic, quadratic or exponential) of the uncertain systems is guaranteed if these criteria are satisfied, and several robust controller design methods have been developed.

Kosmidou (1990) and Luo *et al.* (1994) used the guaranteed cost control approach for the design of robust feedback controllers that guarantee both the robust stability and the performance of continuous systems. Gu *et al.* (1991) presented a two-level optimization process that guarantees quadratic stabilizability of continuous systems, and Gu (1994) applied an algorithm consisting of a strictly quasiconvex minimization to the design of quadratically stabilizing output feedback controllers for both continuous and discrete-time systems. Tsay *et al.* (1991) proposed algorithms based on the Lyapunov stability criterion, to choose a set of weighting matrices for the quadratic cost function; these matrices were then used in the standard Riccati equation to give the linear quadratic optimal control law for the nominal continuous system, which was shown to stabilize quadratically the uncertain system. A similar combination of the Lyapunov stability criterion and the Riccati equation was used by Ni and Wu (1993), who presented a non-iterative procedure for the design of a robust state feedback controller that ensures the exponential stabilizability of uncertain continuous systems.

The linear quadratic regulator (LQR) formulation for continuous systems was used by Wang *et al.* (1987), where an upper bound on the cost incurred by state

Received 3 April 1995. Revised 22 August 1995.
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feedback law and parameter uncertainties was derived and the control law that minimizes this upper bound was found; conditions were presented under which the feedback system was stable for all admissible parameter variations. Another LQR-based control design which is robust to parametric uncertainties was developed by Douglas and Athans (1994) for continuous systems, where the resulting full-state controller was designed by solving a single Riccati-type equation. Luo and Johnson (1992) studied the robustness of the discrete-time linear quadratic gaussian (LQG) problem, where the system to be controlled is described by a state-space formulation that includes plant parameter perturbations and noise uncertainty.

Numerous synthesis results based on H_∞ techniques have also appeared in the literature. Fu *et al.* (1991) and de Souza *et al.* (1993), for instance derived conditions for quadratic stability with disturbance attenuation and for quadratic stabilization via dynamic output feedback respectively for uncertain continuous and discrete-time systems, whose uncertainty matrices are assumed to be of a specific structure. Then, an H_∞-based approach was described for the design of controllers that satisfy the aforementioned conditions. However, no specific information about the uncertainty bounds that describe the uncertainty matrices was provided. Zhou *et al.* (1992) presented a convex programming-based approach to the design of H_∞ controllers for uncertain systems. Specifically, they reduced the problem of controller design to a matrix inequalities problem and searched for a controller that satisfied the conditions for the strongly robust H_∞ performance criterion which they defined. Note that the systems which they studied are the typical H_∞ systems with exogenous disturbances included in the state-space model. Note also that no explicit way was presented to compute the uncertainty bounds, which were decided experimentally via the ellipsoidal method. Several other papers, some of those included in the references of the papers by de Souza *et al.* (1993) and Fu *et al.* (1991) have dealt with the problem of robust output feedback controller design in a fashion similar to that discussed in this paragraph.

All the above controller design approaches share the same general objective, which is to find a stabilizing controller that satisfies some stability conditions or is robust in some sense, without considering the maximization of any of the robust stability bounds existing in literature. This has been done by Yedavali (1986) for continuous systems with structured uncertainties in the system matrix A and by Kolla and Farison (1991) for discrete-time systems with unstructured uncertainties in A. Note that the design in the first paper relies on the selection of a weighting matrix not directly associated with the structured uncertainties and in the latter on the bound developed by Kolla *et al.* (1989). In both of these papers, the information about the uncertainty bound is a part of the minimizing quantity, which also includes the classical LQR cost. Therefore the controller design objective is twofold, that is to minimize the LQR cost and to maximize the perturbation bounds. A similar approach was used earlier by Menga and Dorato (1974) for continuous systems with structured uncertainties in all the state-space matrices, under some quite restrictive assumptions imposed on the perturbation matrices; although the maximization of some stability bound is not considered in the design process, the information about the structured uncertainty is directly included in the minimizing quantity. Finally note that a simple version of the approach of Kolla and Farison (1991) for time-invariant perturbations in A has recently appeared (Kolla 1995) and that a comprehensive survey on design methods for stability robustness of linear discrete-time systems can be found in the paper by Kolla and Farison (1994).

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From the above discussion it is clear that, for discrete-time systems, only the case of unstructured perturbations in A has been studied in the literature. Here, we present a unified output feedback controller design approach for both cases of unstructured and structured perturbations in A. Specifically, we study the general cases of unstructured perturbations in all state-space matrices and structured perturbations in any pair of these matrices and present all other cases such as the one of single perturbations in A as special cases. Our approach is based on new theorems for both the structured and the unstructurd cases, which were recently developed by Konstantopoulos and Antsaklis (1994a); these theorems have shown to provide bounds that improve the bounds obtained via the methodology suggested by Kolla *et al.* (1989) and used by Kolla and Farison (1991). Our design provides a stabilizing static output feedback controller that improves the unstructured bound for A derived by Kolla and Farison (1991). In all the cases studied here, the minimizing quantity consists of two terms: one is the robustness term, which is associated with the specific unstructured or structured bound that we wish to maximize and the other is the LQR term, which is associated with the specific control performance that we wish to maintain. Note that our optimization approach is also applied to a minimizing quantity consisted of only the robustness term, in order to find the controller that maximizes the stability bounds, without considering any control specifications. Note that only the case of static output feedback is studied, since the case of dynamic output feedback can be reduced to that of static feedback as well, as it is shown in Appendix C. Finally note that our minimization algorithm utilizes a version of the Broyden family method of conjugate directions, which is based on the Broyden–Fletcher–Goldfarb–Shanno (BFGS) rule (Bazaraa *et al.* 1993) and that the case of state feedback can be easily considered as a special case of the output feedback case for $C = I$.

The paper is organized as follows. In §2, we present the new theorems of Konstantopoulos and Antsaklis (1994a, 1995b) for the cases of unstructured and structured perturbations in discrete-time systems. In §3, we study the case of unstructured perturbations in all state-space matrices and present an algorithm based on the BFGS rule that solves the minimization problem. Note that several special cases such as the ones of unstructured perturbations in A only or in (A, B) only are also discussed. In §4, we study the case of structured perturbations in (A, B); the case of single structured perturbations in any of the state-space matrices is also discussed. In §5, we provide several illustrative examples for some of the cases mentioned above; these examples include an aircraft longitudinal control system and a paper-machine head box. Finally, in §6, concluding remarks are included.

2. Preliminaries

We consider the linear discrete-time system with the state-space description

$$x(k+1) = Ax(k) \tag{2.1}$$

where $x \in \Re^n$ is the state vector and A an asymptotically stable matrix. Then, for every symmetric positive definite matrix Q, we can find a symmetric positive definite matrix P, which is the unique solution of the Lyapunov equation

$$A^T PA - P + Q = 0 \tag{2.2}$$

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When A is perturbed by the matrix ΔA, then for the perturbed system

$$y(k+1) = (A + \Delta A) y(k) \tag{2.3}$$

the following theorem holds. First define

$$\Omega_1 = A^T P Z^{-1} P A \tag{2.4}$$

and note that the notation $A < B$ for two symmetric square matrices A and B means that the matrix $A - B$ is negative definite, that is all the eigenvalues of $A - B$ lie to the left of the imaginary axis in the complex plane.

Theorem 2.1: *Consider the linear discrete-time system* (2.1) *where A is an asymptotically stable matrix that satisfies* (2.2). *Suppose that $A \to A + \Delta A$, then the perturbed system* (2.3) *remains asymptotically stable, if*

$$(\Delta A)^T (\alpha Z + P)(\Delta A) + \frac{1}{\alpha}\Omega_1 < Q \tag{2.5}$$

or

$$\sigma_{\max}(\Delta A) < \left(\frac{\sigma_{\min}(Q) - \sigma_{\max}[(1/\alpha)\Omega_1]}{\sigma_{\max}(\alpha Z + P)}\right)^{1/2} \tag{2.6}$$

where P, Q are defined in (2.2), *Ω_1 in* (2.4), *Z can be any positive definite matrix of appropriate dimensions, and α any positive number that satisfies*

$$\alpha > \frac{\sigma_{\max}(\Omega_1)}{\sigma_{\min}(Q)} \tag{2.7}$$

Proof: The proof is given in Appendix A. □

When the perturbation matrix ΔA is described by

$$\Delta A = \sum_{i=1}^{m} \theta_i A_i = (\Theta \times I_n)^T \tilde{A} \tag{2.8}$$

where $\theta_i, i = 1, \ldots, m$ denote real uncertain parameters and $A_i, i = 1, \ldots, m$ are constant known matrices, the following theorem holds. Obviously, the following definitions have been used:

$$\Theta = [\theta_1 \quad \theta_2 \quad \ldots \quad \theta_m]^T, \quad \tilde{A} = [A_1^T \quad A_2^T \quad \ldots \quad A_m^T]^T \tag{2.9}$$

Theorem 2.2: *The linear discrete-time system* (2.3) *with A an asymptotically stable matrix and structured perturbations of the form of* (2.8) *remains asymptotically stable, if the uncertainty parameters satisfy*

$$\sum_{i=1}^{m} \theta_i^2 < \frac{\sigma_{\min}(Q) - \sigma_{\max}[(1/\alpha)\Omega_1]}{\sigma_{\max}^2(\tilde{A})\,\sigma_{\max}(\alpha Z + P)} \tag{2.10}$$

where Ω_1, θ_i and $\tilde{A}$ are defined in (2.4), (2.9) *and* (2.9) *respectively, Z can be any positive definite matrix of appropriate dimensions, and α any positive number that satisfies* (2.7).

Proof: It follows easily from Theorem 2.1; details have been given in the paper by Konstantopoulos and Antsaklis (1995b).

The main point of the approach used for the theorems above is the appropriate

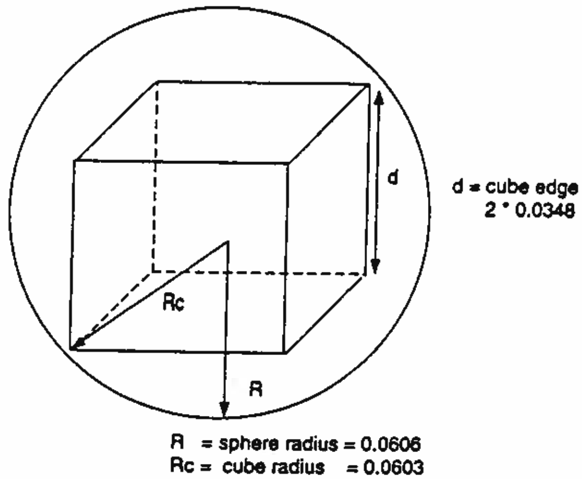

Figure 1. Example 2.2.

selection of a positive definite matrix Z and a positive number α that maximize the stability region within which the uncertain parameters vary. A major advantage of this approach is that it is unified with respect to both unstructured and structured perturbations, owing to the similarity between (2.6) and (2.10); this appears to be very convenient in the design methodology that we present in § 3 that follows. Note that the above approach has also been extended, by Konstantopoulos and Antsaklis (1994a, 1995b) to the case of structured perturbations in all the state-space matrices.

The above theorems provide bounds that imprive the bounds obtained via the methodology suggested by Kolla *et al.* (1989) and used by Kolla and Farison (1991) for the case of unstructured perturbations in A. Note that here we compare our results with those of Kolla *et al.* (1989) and Kolla and Farison (1991) only, because these are the only results in the literature for studies similar to those that we present here. Since the emphasis in this paper is on the optimal controller design, further discussion of analysis results (derivation of robust stability bounds, and comparison with previous results) is not of interest here. Note, however, that a detailed discussion of the present approach and its application to other cases that are not of interest here, as well as a comparative study of other literature analysis methods and results for both continuous and discrete-time systems, can be found in the papers by Halicka and Rosinova (1994) and Konstantopoulos and Antsaklis (1994a, 1994c, 1995b).

Example 2.1: Consider the following uncertain discrete-time system (2.3) from Kolla *et al.* (1989) with

$$A = \begin{pmatrix} 0{\cdot}20 & 0{\cdot}30 \\ 0{\cdot}10 & -0{\cdot}15 \end{pmatrix} \tag{2.11}$$

Using (2.6) of Theorem 2.1 for $Q = I$, $\alpha = 0{\cdot}2702$, and

$$Z = \begin{pmatrix} 2{\cdot}0399 & -0{\cdot}2037 \\ -0{\cdot}2037 & 1{\cdot}4586 \end{pmatrix}$$

we obtain $\sigma_{\max}(\Delta A) < 0{\cdot}6787$, which compares favourably with the result of Kolla *et al.* (1989), which is $\sigma_{\max}(\Delta A) < 0{\cdot}6373$. Note that the same result was derived with the method proposed by Su and Fong (1993). □

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Example 2.2: Consider the same nominal system as before, but now with structured perturbations of the form of (2.8), with $m = 3$ and

$$A_1 = \begin{pmatrix} 10 & 0{\cdot}1 \\ -1 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -0{\cdot}5 & 9 \\ 0 & -3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0{\cdot}6 \\ 1 & 0{\cdot}3 \end{pmatrix} \tag{2.12}$$

Using (2.10) of Theorem 2.2 for $Q = I$, $\alpha = 0{\cdot}40$, and

$$Z = \begin{pmatrix} 1{\cdot}3462 & -0{\cdot}1184 \\ -0{\cdot}1184 & 0{\cdot}8786 \end{pmatrix}$$

we obtain $\kappa_1^2 + \kappa_2^2 + \kappa_3^2 < (0{\cdot}0606)^2$, that is a sphere with radius $R = 0{\cdot}0606$, whereas the method suggested by Kolla *et al.* (1989) gives $|\kappa_i| < 0{\cdot}0348$ for $i = 1, 2, 3$. Note that, as we can see in Fig. 1, the defined cube is completely included in the sphere found above, which shows that our bound is less conservative than that of Kolla *et al.* (1989).

3. Unstructured perturbations

We consider the linear discrete-time system with the state-space description

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) \tag{3.1}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^r$ is the input vector and $y \in \Re^q$ is the output vector. We assume unstructured perturbations in all system matrices, that is

$$A = A_0 + \Delta A, \quad B = B_0 + \Delta B, \quad C = C_0 + \Delta C \tag{3.2}$$

With the static output feedback law

$$u(k) = Ky(k) = KCx(k) \tag{3.3}$$

the closed-loop system is described by

$$x(k+1) = [\bar{A}_0 + \Delta A + (\Delta B)KC_0 + B_0 K(\Delta C) + (\Delta B)K(\Delta C)]x(k) \tag{3.4}$$

where obviously the following definition has been used:

$$\bar{A}_0 = A_0 + B_0 KC_0 \tag{3.5}$$

For the closed-loop system of (3.4), (2.2) and (2.4) can be translated into

$$\bar{A}_0^T P \bar{A}_0 - P + Q = 0, \quad \Omega_1 = \bar{A}_0^T P Z^{-1} P \bar{A}_0 \tag{3.6}$$

It can easily be shown that the stability of the closed-loop system is maintained, if the perturbation matrices ΔA, ΔB and ΔC satisfy the following sufficient condition:

$$\begin{aligned} &\sigma_{\max}(\Delta A) + \sigma_{\max}(\Delta B)\sigma_{\max}(K)\sigma_{\max}(C_0) + \sigma_{\max}(B_0)\sigma_{\max}(K)\sigma_{\max}(\Delta C) \\ &\quad + \sigma_{\max}(\Delta B)\sigma_{\max}(K)\sigma_{\max}(\Delta C) < \left(\frac{\sigma_{\min}(Q) - \sigma_{\max}[(1/\alpha)\Omega_1]}{\sigma_{\max}(\alpha Z + P)}\right)^{1/2} \end{aligned} \tag{3.7}$$

where P, Q, Ω_1 are defined in (3.6) above. Note that the above inequality defines a region in $\Re^3$ for $\sigma_{\max}(\Delta A)$, $\sigma_{\max}(\Delta B)$ and $\sigma_{\max}(\Delta C)$.

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3.1. *Design without performance specifications*

Our objective is to find a static stabilizing output feedback gain K that maximizes the region specified in (3.7). As shown in Appendix B, in order to maximize the volume of this region, we need

(i) to minimize $\sigma_{\max}(K)$

and also to maximize the right-hand side of (3.7). Since Q in (3.6) is selected beforehand, in order to maximize the right-hand side of (3.7), we need

(ii) to minimize $\sigma_{\max}(\alpha Z+P)$ and

(iii) to minimize $\sigma_{\max}[(1/\alpha)\,\Omega_1]$.

For (i) we choose to minimize the quantity $J_1 = \mathrm{Tr}\,(K^{\mathrm{T}}K)$, since $\sigma^2_{\max}(A) \leqslant \|A\|^2_{\mathrm{F}} = \mathrm{Tr}\,(A^{\mathrm{T}}A)$, where $\|A\|_{\mathrm{F}}$ denotes the Frobenius norm and $\mathrm{Tr}\,(A)$ the trace of a matrix A. Similarly, for (ii), we choose to minimize the quantity $J_2 = \mathrm{Tr}\,[(\alpha Z+P)^{\mathrm{T}}(\alpha Z+P)] = \mathrm{Tr}\,(\alpha^2Z^2+2\alpha PZ+P^2)$. For (iii), we have $\sigma_{\max}[(1/\alpha)\,\Omega_1) \leqslant (1/\alpha)\,\sigma^2_{\max}(\bar{A}_0)\,\sigma^2_{\max}(P)\,\sigma_{\max}(Z^{-1})$. Since Z is selected beforehand and an upper bound of $\sigma_{\max}(P)$, that is Tr (P^2), is already minimized in J_2, for (iii) we simply choose to minimize $J_3 = (1/\alpha)\,\mathrm{Tr}\,(\bar{A}_0^{\mathrm{T}}\bar{A}_0)$, which is an upper bound of $(1/\alpha)\,\sigma^2_{\max}(\bar{A}_0)$. Note that the minimization of the sum of $\mathrm{Tr}\,(\bar{A}_0^{\mathrm{T}}\bar{A}_0)$ and $\mathrm{Tr}\,(P^2)$ is an indirect and harder way to minimize their product; in other words, we impose a more demanding task on the minimizing process. On the other hand, note that α is included in (iii), because we need to satisfy the positiveness of the numerator, as indicated in (2.7). Therefore the minimizing quantity is given as $J'_{ABC} = J_1+J_2+J_3$, under the condition that (3.6) holds. This is clearly a constrained minimization problem. By including (3.6) in J'_{ABC}, we finally reduce the problem to an unconstrained minimization problem, with the minimizing quantity finally given by

$$J_{ABC} = \mathrm{Tr}\,[K^{\mathrm{T}}K+\alpha^2Z^2+2\alpha PZ+P^2+\frac{1}{\alpha}\bar{A}_0^{\mathrm{T}}\bar{A}_0+L_1(\bar{A}_0^{\mathrm{T}}P\bar{A}_0-P+Q)] \qquad (3.8)$$

where $L_1 \in \Re^{n\times n}$ is the Lagrange multiplier matrix. Next, we need the following properties from the work of Athans (1967):

$$\frac{\partial}{\partial X}\mathrm{Tr}\,(X^2) = 2X^{\mathrm{T}} \qquad (3.9)$$

$$\frac{\partial}{\partial Y}\mathrm{Tr}\,(A_1\,YB_1) = A_1^{\mathrm{T}}B_1^{\mathrm{T}} \qquad (3.10)$$

$$\frac{\partial}{\partial Y}\mathrm{Tr}\,(A_2\,Y^{\mathrm{T}}B_2) = B_2A_2 \qquad (3.11)$$

$$\frac{\partial}{\partial Y}\mathrm{Tr}\,(A_2\,YB_3\,Y^{\mathrm{T}}) = A_3\,YB_3+A_3^{\mathrm{T}}\,YB_3^{\mathrm{T}} \qquad (3.12)$$

for any $X\in\Re^{n\times n}$, $Y\in\Re^{n\times m}$, $A_1\in\Re^{l\times n}$, $B_1\in\Re^{m\times l}$, $A_2\in\Re^{l\times m}$, $B_2\in\Re^{n\times l}$, $A_3\in\Re^{n\times n}$, $B_3\in\Re^{m\times m}$. With these properties, we have

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$$\frac{\Delta J^{\mathrm{r}}_{ABC}}{\partial L_1} = \Delta^{\mathrm{r}}_{L_1} = \bar{A}_0^{\mathrm{T}} P \bar{A}_0 - P + Q \tag{3.13}$$

$$\frac{\partial J^{\mathrm{r}}_{ABC}}{\partial \alpha} = \Delta^{\mathrm{r}}_{\alpha} = 2\alpha \operatorname{Tr}(Z^2) + 2\operatorname{Tr}(PZ) - \frac{1}{\alpha^2}\operatorname{Tr}(\bar{A}_0^{\mathrm{T}} \bar{A}_0) \tag{3.14}$$

$$\frac{\partial J^{\mathrm{r}}_{ABC}}{\partial P} = \Delta^{\mathrm{r}}_{P} = 2P + 2\alpha Z + \bar{A}_0 L_1^{\mathrm{T}} \bar{A}_0^{\mathrm{T}} - L_1^{\mathrm{T}} \tag{3.15}$$

$$\begin{aligned}\frac{\partial J^{\mathrm{R}}_{ABC}}{\partial K} = \Delta^{\mathrm{r}}_{K} &= 2K + \frac{2}{\alpha} B_0^{\mathrm{T}} B_0 K C_0 C_0^{\mathrm{T}} + \frac{2}{\alpha} B_0^{\mathrm{T}} A_0 C_0^{\mathrm{T}} \\ &\quad + B_0^{\mathrm{T}} P B_0 K C_0 (L_1 + L_1^{\mathrm{T}}) C_0^{\mathrm{T}} + B_0^{\mathrm{T}} P A_0 (L_1 + L_1^{\mathrm{T}}) C_0^{\mathrm{T}}\end{aligned} \tag{3.16}$$

To minimize (3.8), we use a version of the Broyden family method of conjugate directions, which is based on the BFGS update rule; details have been given by Bazaraa *et al.* (1993). The proposed algorithm is presented next.

Algorithm:

Initialization step

Let $\varepsilon > 0$ be the termination scalar. Choose an initial stabilizing gain

$$K_1 = \begin{pmatrix} (\tau_1^1)^{\mathrm{T}} \\ \vdots \\ (\tau_r^1)^{\mathrm{T}} \end{pmatrix} \tag{3.17}$$

where $(\tau_l^1)^{\mathrm{T}}$, $l = 1, \ldots, r$ are the $1 \times q$ rows of K_1, which stabilizes (A_0, B_0, C_0), that is $\bar{A}_0$ stable. Also, choose an initial symmetric positive definite matrix $D_1 \in \Re^{rq \times rq}$. Let

$$\phi_1 = \chi_1 = \begin{pmatrix} (\tau_1^1) \\ \vdots \\ (\tau_r^1) \end{pmatrix} \tag{3.18}$$

be a column vector consisting of the transposes of the rows of K_1. Also let $k_N = j = 1$ and go to the main step.

Main step

(M1) Substitute the gain matrix K_j in the gradients of (3.13)–(3.15), set them to zero, that is $\Delta^{\mathrm{r}}_{L_1} = 0$, $\Delta^{\mathrm{r}}_{\alpha} = 0$, $\Delta^{\mathrm{r}}_{P} = 0$, and solve respectively for P, α, L_1, in that specific order.

(M2) Substitute these parameters in (3.16) and compute

$$\Delta^{\mathrm{r}}_{K_j} = \begin{pmatrix} (\sigma_1^j)^{\mathrm{T}} \\ \vdots \\ (\sigma_r^j)^{\mathrm{T}} \end{pmatrix} \tag{3.19}$$

where $(\sigma_l^j)^{\mathrm{T}}$, $l = 1, \ldots, r$ are the $1 \times q$ rows of $\Delta^{\mathrm{r}}_{K_j}$.

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(M3) Define

$$\nabla J^{r}_{ABC}(\phi_j) = \begin{pmatrix} \sigma^j_1 \\ \vdots \\ \sigma^j_r \end{pmatrix} \tag{3.20}$$

If $\|\nabla J^{r}_{ABC}(\phi_j)\| < \varepsilon$, stop. The optimal gain is K_j. Otherwise, go to (M4).

(M4) If $j > 1$, update the positive definite matrix D_j as follows:

$$D_j = D_{j-1} + \frac{p_{j-1} p^{\mathrm{T}}_{j-1}}{p^{\mathrm{T}}_{j-1} q_{j-1}} \left(1 + \frac{q^{\mathrm{T}}_{j-1} D_{j-1} q_{j-1}}{p^{\mathrm{T}}_{j-1} q_{j-1}} \right) - \frac{D_{j-1} q_{j-1} p^{\mathrm{T}}_{j-1} + p_{j-1} q^{\mathrm{T}}_{j-1} D_{j-1}}{p^{\mathrm{T}}_{j-1} q_{j-1}} \tag{3.21}$$

where

$$p_{j-1} = \lambda_{j-1} d_{j-1} = \phi_j - \phi_{j-1} \tag{3.22}$$

$$q_{j-1} = \nabla J^{r}_{ABC}(\phi_j) - \nabla J^{r}_{ABC}(\phi_{j-1}) \tag{3.23}$$

(M5) Define

$$d_j = -D_j \nabla J^{r}_{ABC}(\phi_j) \tag{3.24}$$

and let λ_j be an optimal solution to the problem of minimizing $J^{r}_{ABC}(\phi_j + \lambda d_j)$ subject to $\lambda \geqslant 0$. Let

$$\phi_{j+1} = \phi_j + \lambda_j d_j = \begin{pmatrix} (\tau^{j+1}_1) \\ \vdots \\ (\tau^{j+1}_r) \end{pmatrix} \tag{3.25}$$

which implies that

$$K_{j+1} = \begin{pmatrix} (\tau^{j+1}_1)^{\mathrm{T}} \\ \vdots \\ (\tau^{j+1}_r)^{\mathrm{T}} \end{pmatrix} \tag{3.26}$$

where obviously $(\tau^{j+1}_l), l = 1, \ldots, r$ are $q \times 1$ column vectors.

(M6) If $j < rq$, replace j by $j+1$ and repeat the main step. Otherwise, if $j = rq$, then let $\phi_1 = \chi k_{N+1} = \phi_{rq+1}$, replace k_N by $(k_N + 1)$, let $j = 1$ and repeat the main step.

Several issues need to be discussed here. First, note that the line search in (M5) is restricted to stabilizing gain matrices. Therefore the selected new gain matrix needs first to stabilize the closed-loop system (3.4) and then minimize J^{r}_{ABC}. Since our algorithm is an indirect version of the BFGS algorithm, as an alternative to the stopping criterion of (M3), we could use another quite practical criterion. Specifically, we may consider monitoring J^{r}_{ABC} and stop when we reach a plateau or when we see that J^{r}_{ABC} is sufficiently small and the associated bound derived is satisfactorily large. From (3.14), we can easily see that there is at least one real positive solution for α. For our algorithm, we choose to keep the largest value of α, since we also need to satisfy

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the positiveness of the numerator of (2.6) and (2.10), as discussed before. Finally, note that for optimization problems similar to the problem that we study here, alternative methods based on gradient-type and non-gradient-type algorithms have been proposed by Horisberger and Belanger (1974) and Mendel and Feather (1975) respectively.

3.2. *Design with performance specifications*

In the previous subsection, we focused on finding a stabilizing output feedback gain K that maximizes the volume of the region in (3.7). If, in addition to this objective, we also wish to attain a specific control performance, then we need to include in our minimizing quantity a term that evaluates this control performance. Therefore we consider the familiar LQR cost (Franklin *et al.* 1990) which is given as follows:

$$J'_{\mathrm{LQR}} = \sum_{k=0}^{\infty} x^{\mathrm{T}}(k)\, Q_1\, x(k) + u^{\mathrm{T}}(k)\, R_1\, u(k) \tag{3.27}$$

where Q_1, R_1 are positive definite matrices of appropriate dimensions. For the nominal system (A_0, B_0, C_0) with the output feedback law (3.3), we rewrite (3.27) as $J'_{\mathrm{LQR}} = \sum_{k=0}^{\infty} x^{\mathrm{T}}(0)\,(\bar{A}_0^{\mathrm{T}})^k\, \bar{Q}(\bar{A}_0)^k\, x(0)$, where obviously $\bar{Q} = Q_1 + C_0^{\mathrm{T}} K^{\mathrm{T}} R_1 K C_0$. The following equivalence has been shown by Ogata (1987) for the solution of the discrete-time Lyapunov equation (2.2)

$$A^{\mathrm{T}}PA - P + Q = 0 \Leftrightarrow P = \sum_{k=0}^{\infty} (A^{\mathrm{T}})^k\, Q A^k \tag{3.28}$$

With the above relation, we have $J'_{\mathrm{LQR}} = x^{\mathrm{T}}(0)\, P_2\, x(0)$, where P_2 is the solution of the Lyapunov equation

$$\bar{A}_0^{\mathrm{T}} P_2 \bar{A}_0 - P_2 + \bar{Q} = 0 \tag{3.29}$$

As we see, J'_{LQR} depends on the initial state $x(0)$, which implies that the optimal gain matrix K will also depend on $x(0)$. To eliminate this dependence, we may assume (Levine *et al.* 1971, O'Reilly 1980) that $x(0)$ is a random vector with expected value and second-order moment given respectively as $E[x(0)] = x_0$ and $E[x(0)\, x^{\mathrm{T}}(0)] = X_0 > 0$. The most widely used method (Levine and Athans 1970) is to consider $x(0)$ uniformly distributed on a sphere of radius σ, that is $X_0 = \sigma I_n$, with $\sigma = 1$ being the obvious choice. Note that alternative methods to deal with this dependence can be found in the work of Dabke (1970) and Man (1970). We choose the following modified cost (O'Reilly 1980):

$$\begin{aligned} J_{\mathrm{LQR}} &= E[\mathrm{Tr}\,(J'_{\mathrm{LQR}})] \\ &= E\{\mathrm{Tr}\,[x^{\mathrm{T}}(0)\, P_2\, x(0)]\} \\ &= E\{\mathrm{Tr}\,[P_2\, x(0)\, x^{\mathrm{T}}(0)]\} \\ &= \mathrm{Tr}\,\{E[P_2\, x(0)\, x^{\mathrm{T}}(0)]\} \\ &= \mathrm{Tr}\,(P_2 X_0) \end{aligned} \tag{3.30}$$

In view of (3.8) and (3.30), we finally define the overall minimizing quantity, which

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is associated with both the robustness of the matrix $\bar{A}_0$ and the control performance of the closed-loop system

$$\begin{aligned} J^{\mathrm{r,p}}_{ABC} &= J^{\mathrm{r}}_{ABC} + J_{\mathrm{LQR}} \\ &= \mathrm{Tr}\Big(K^{\mathrm{T}}K + \alpha^2 Z^2 + 2\alpha PZ + P^2 + \frac{1}{\alpha}\bar{A}_0^{\mathrm{T}}\bar{A}_0 + L_1(\bar{A}_0^{\mathrm{T}}P\bar{A}_0 - P + Q) \\ &\quad + P_2X_0 + L_2(\bar{A}_0^{\mathrm{T}}P_2\bar{A}_0 - P_2 + \bar{Q})\Big) \end{aligned} \tag{3.31}$$

where, similar to (3.8), we have reduced the problem to an unconstrained minimization one by including (3.29) in the minimizing quantity via the Langrange multiplier matrix L_2. Owing to the introduction of P_2 and L_2 in the new cost $J^{\mathrm{r,p}}_{ABC}$, we need to consider its partial derivatives with respect to these new matrix variables as well. For the same reason, we have some additional terms in Δ^{r}_K of (3.16). Therefore the partial derivatives of the final cost $J^{\mathrm{r,p}}_{ABC}$ with respect to all the matrix variables entailed are as follows:

$$\frac{\partial J^{\mathrm{r,p}}_{ABC}}{\partial L_1} = \Delta^{\mathrm{r,p}}_{L_1} = \bar{A}_0^{\mathrm{T}}P\bar{A}_0 - P + Q \tag{3.32}$$

$$\frac{\partial J^{\mathrm{r,p}}_{ABC}}{\partial L_2} = \Delta^{\mathrm{r,p}}_{L_2} = \bar{A}_0^{\mathrm{T}}P_2\bar{A}_0 - P_2 + \bar{Q} \tag{3.33}$$

$$\frac{\partial J^{\mathrm{r,p}}_{ABC}}{\partial \alpha} = \Delta^{\mathrm{r,p}}_{\alpha} = 2\alpha\,\mathrm{Tr}(Z^2) + 2\,\mathrm{Tr}(PZ) - \frac{1}{\alpha^2}\mathrm{Tr}(\bar{A}_0^{\mathrm{T}}\bar{A}_0) \tag{3.34}$$

$$\frac{\partial J^{\mathrm{r,p}}_{ABC}}{\partial P} = \Delta^{\mathrm{r,p}}_{P} = 2P + 2\alpha Z + \bar{A}_0L_1^{\mathrm{T}}\bar{A}_0^{\mathrm{T}} - L_1^{\mathrm{T}} \tag{3.35}$$

$$\frac{\partial J^{\mathrm{r,p}}_{ABC}}{\partial P_2} = \Delta^{\mathrm{r,p}}_{P_2} = X_0^{\mathrm{T}} + \bar{A}_0L_2^{\mathrm{T}}\bar{A}_0^{\mathrm{T}} - L_2^{\mathrm{T}} \tag{3.36}$$

$$\begin{aligned} \frac{\partial J^{\mathrm{r,p}}_{ABC}}{\partial K} = \Delta^{\mathrm{r,p}}_{K} &= 2K + \frac{2}{\alpha}B_0^{\mathrm{T}}B_0KC_0C_0^{\mathrm{T}} + \frac{2}{\alpha}B_0^{\mathrm{T}}A_0C_0^{\mathrm{T}} + R_1KC_0(L_2 + L_2^{\mathrm{T}})C_0^{\mathrm{T}} \\ &\quad + B_0^{\mathrm{T}}PB_0KC_0(L_1 + L_1^{\mathrm{T}})C_0^{\mathrm{T}} + B_0^{\mathrm{T}}PA_0(L_1 + L_1^{\mathrm{T}})C_0^{\mathrm{T}} \\ &\quad + B_0^{\mathrm{T}}P_2B_0KC_0(L_2 + L_2^{\mathrm{T}})C_0^{\mathrm{T}} + B_0^{\mathrm{T}}P_2A_0(L_2 + L_2^{\mathrm{T}})C_0^{\mathrm{T}} \end{aligned} \tag{3.37}$$

To minimize $J^{\mathrm{r,p}}_{ABC}$, the algorithm of §3.1 can be used again, the only difference being that steps (M1) and (M2) have to be replaced by the following

(M1*a*) Substitute the gain matrix K_j in the gradients of (3.32)–(3.36), set them to zero, that is $\Delta^{\mathrm{r,p}}_{L_1} = 0$, $\Delta^{\mathrm{r,p}}_{L_2} = 0$, $\Delta^{\mathrm{r,p}}_{\alpha} = 0$, $\Delta^{\mathrm{r,p}}_{P} = 0$, $\Delta^{\mathrm{r,p}}_{P_2} = 0$ and solve respectively for P, P_2, α, L_1, L_2, in that specific order.

(M2*a*) Substitute these parameters in (3.37) and compute

$$\Delta^{\mathrm{r,p}}_{K_j} = \begin{pmatrix} (\sigma_1^j)^{\mathrm{T}} \\ \vdots \\ (\sigma_r^j)^{\mathrm{T}} \end{pmatrix} \tag{3.38}$$

where $(\sigma_l^j)^{\mathrm{T}}$, $l = 1, \ldots, m$ are the $1 \times q$ rows of $\Delta^{\mathrm{r,p}}_{K_j}$.

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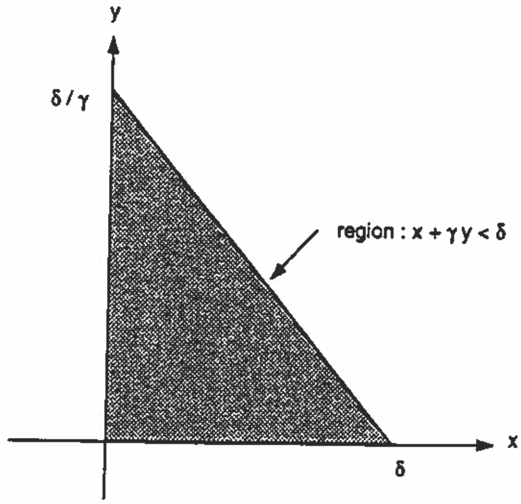

Figure 2. Stability region for unstructured perturbations in (A, B).

All the other steps of the algorithm remain the same, but we now refer to $J^{r,p}_{ABC}$, instead of J^{r}_{ABC}.

3.3. *Special cases*

3.3.1. *Perturbations in A*. When we have perturbations in A only, then the stability of the closed-loop system $x(k+1) = (\bar{A}_0 + \Delta A)\, x(k)$ is maintained, if (2.6) holds for P, Q, Ω_1 as defined in (3.6). Therefore it suffices to consider objectives (ii) and (iii). Hence, the algorithms of §§3.1 and 3.2 can be used again for $J^r_A = J_2 + J_3$ and $J^{r,p}_A = J^r_A + J_{LQR}$, the only difference being the omission of the term $\partial J_1/\partial K = 2K$ from (3.16) and (3.37). Note that single perturbations in B or C can be handled similarly (Konstantopoulos and Antsaklis 1995 a).

3.3.2. *Perturbations in* (A, B). We consider perturbations in A and B only. It can easily be shown (Konstantopoulos and Antsaklis 1995 a) that the stability of the closed-loop system is maintained, if the perturbation matrices ΔA, ΔB satisfy the following sufficient condition:

$$\sigma_{\max}(\Delta A) + \sigma_{\max}(\Delta B)\, \sigma_{\max}(KC_0) < \left(\frac{\sigma_{\min}(Q) - \sigma_{\max}[(1/\alpha)\,\Omega_1]}{\sigma_{\max}(\alpha Z + P)} \right)^{1/2} \tag{3.39}$$

where again P, Q, Ω_1 are defined in (3.6). The region that satisfies the inequality $x + \gamma y < \delta$ for positive x, y, γ, δ is the shaded triangle shown in Fig. 2. Obviously, this region gets larger for larger δ and smaller γ. Therefore, in order to maximize the stability region that is defined by (3.39), we need

(iv) to minimize $\sigma_{\max}(KC_0)$

and to maximize the right-hand side of (3.39), which corresponds to objectives (ii) and (iii). For (iv), we choose to minimize $J_4 = \mathrm{Tr}\,[(KC_0)^T (KC_0)]$. Therefore, our algorithm

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can be used again for $\tilde{J}^r_{AB} = J^r_A + J_4$ and $\tilde{J}^{r,p}_{AB} = J^{r,p}_A + J_4$ as well; the only difference is that the term $2K$ needs to be replaced by $\partial J_4/\partial K = 2KC_0C_0^T$ in (3.16) and (3.37). Finally, note that the same approach can be used for perturbations in (A, C) only.

4. Structured perturbations

We consider the linear discrete-time system with the state-space description

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = C_0 x(k) \tag{4.1}$$

and assume structured perturbations in (A, B), that is

$$A = A_0 + \sum_{i=1}^{m_A} \kappa_i A_i, \quad B = B_0 + \sum_{j=1}^{m_B} \lambda_j B_j \tag{4.2}$$

We define

$$\hat{\Theta} = [\kappa_1 \dots \kappa_{m_A} | \lambda_1 \dots \lambda_{m_B}]^T = [\hat{\theta}_1 \dots \hat{\theta}_{m_A+m_B}]^T \tag{4.3}$$

$$\hat{\Pi} = [A_1^T \dots A_{m_A}^T | (B_1 KC_0)^T \dots (B_{m_B} KC_0)^T]^T \tag{4.4}$$

With the output feedback law $u(k) = Ky(k) = KC_0 x(k)$, the closed-loop system is given by

$$x(k+1) = [\bar{A}_0 + (\hat{\Theta} \otimes I_n)^T \hat{\Pi}] x(k) \tag{4.5}$$

and remains asymptotically stable if, in view of (2.10), the uncertain parameters satisfy

$$\sum_{i=1}^{m_A+m_B} \hat{\theta}_i^2 < \frac{\sigma_{\min}(Q) - \sigma_{\max}[(1/\alpha)\Omega_1]}{\sigma^2_{\max}(\hat{\Pi})\,\sigma_{\max}(\alpha Z + P)} \tag{4.6}$$

In order to maximize (4.6), in addition to objectives (ii) and (iii), we also need

(v) to minimize $\sigma_{\max}(\hat{\Pi})$.

For (v) we minimize $J_5 = \mathrm{Tr}[\hat{\Pi}^T\hat{\Pi}]$, so that the minimizing quantities are now $\tilde{J}^r_{AB} = J^r_A + J_5$ for the case of robustness only specifications, and $\tilde{J}^{r,p}_{AB} = J^{r,p}_A + J_5$ for the case of both robustness and performance specifications. Our algorithm can be used again, with the term $2K$ now being replaced by the term $\partial J_5/\partial K = 2(B^*_{m_B})^T B^*_{m_B} KC_0 C_0^T$ in (3.16) and (3.37), where $B^*_{m_B} = (B_1^T \dots B_{m_B}^T)^T$. Note that a similar approach can be used for the case of structured perturbations in (A, C). Finally note that owing to the similarity between (2.6) and (2.10), the discussion of §3.3 concerning the case of unstructured perturbations in A also applies to the case of structured perturbations in A only.

5. Illustrative examples

Example 5.1: Consider the scalar system

$$x(k+1) = 0{\cdot}5x(k) + u(k), \quad x(0) = 1{\cdot}0 \tag{5.1}$$

with state feedback $u(k) = Kx(k)$. This system was studied by Kolla and Farison (1991), where the LQR cost $J'_{LQR} = \sum_{k=0}^{\infty} x^2(k) + u^2(k)$ was used, that is $Q_1 = R_1 = 1$.

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The derived bound for unstructured perturbations in the system matrix A was $\sigma_{\max}(\Delta A) < 0{\cdot}8436$ for a gain of $K = -0{\cdot}3436$. We apply our method for the same LQR term. Choosing $Q = 1{\cdot}30$, $Z = 0{\cdot}60$, initial stabilizing gain $K_1 = 0{\cdot}1$ and positive definite matrix $D_1 = 0{\cdot}001$, we obtain a stabilizing gain of $K = -0{\cdot}49998$, which corresponds to $\sigma_{\max}(\Delta A) < 0{\cdot}99998$, which compares favourably with the result of Kolla and Farison (1991) given above. The components of $J_A^{r,p}$ that are associated with the robustness and the performance objectives are respectively $J_A^r = 1{\cdot}69$, $J_{LQR} = 1{\cdot}25$.

Now, if we neglect the performance specifications, as indicated by the above LQR cost and focus on just the maximization of the robustness bound, we obtain a stabilizing gain of $K = -0{\cdot}49999$, which corresponds to $\sigma_{\max}(\Delta A) < 0{\cdot}99999$ and $J_A^r = 1{\cdot}69$. Note that Q, Z, K_1 and D_1 are the same as before. As we see, in this scalar case, we obtain the same results for the final stabilizing gain K, the uncertainty bound and the robustness component J_A^r of the minimizing quantity, no matter whether the LQR term is included or not in the minimizing quantity. Note, however, that this is not the case, in general, for multiple-input multiple-output systems, as we can clearly see in the examples that follow. □

Example 5.2: Consider an aircraft longitudinal control system from Jiang (1994), whose linearized continuous dynamic model is given by

$$\begin{pmatrix} \dot\alpha(t) \\ \dot\beta(t) \\ \dot\psi(t) \\ \dot\theta(t) \end{pmatrix} = \begin{pmatrix} -0{\cdot}0582 & 0{\cdot}0651 & 0 & -0{\cdot}171 \\ -0{\cdot}303 & -0{\cdot}685 & 1{\cdot}109 & 0 \\ -0{\cdot}0715 & -0{\cdot}658 & -0{\cdot}947 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha(t) \\ \beta(t) \\ \psi(t) \\ \theta(t) \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -0{\cdot}0541 & 0 \\ -1{\cdot}11 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta(t) \\ \tau(t) \end{pmatrix}$$

$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha(t) \\ \beta(t) \\ \psi(t) \\ \theta(t) \end{pmatrix} \tag{5.2}$$

where $\alpha(t)$ and $\beta(t)$ are the forward and vertical speeds, $\psi(t)$ is the pitch rate and $\theta(t)$ is the pitch angle. The control inputs $\eta(t)$ and $\tau(t)$ are the elevator angle and throttle position respectively. Note that all states are assumed available for measurement. We consider the discrete-time model for $T = 0{\cdot}5$ s. The state-space matrices are given by

$$A_d = \begin{pmatrix} 0{\cdot}9692 & 0{\cdot}0283 & -0{\cdot}0112 & -0{\cdot}0842 \\ -0{\cdot}1302 & 0{\cdot}6469 & 0{\cdot}3584 & 0{\cdot}0059 \\ -0{\cdot}0086 & -0{\cdot}2126 & 0{\cdot}5644 & 0{\cdot}0007 \\ -0{\cdot}0041 & -0{\cdot}0621 & 0{\cdot}3873 & 1{\cdot}0001 \end{pmatrix},$$

$$B_d = \begin{pmatrix} 0{\cdot}0017 & 0{\cdot}4924 \\ -0{\cdot}1385 & -0{\cdot}0344 \\ -0{\cdot}4266 & -0{\cdot}0041 \\ -0{\cdot}1170 & -0{\cdot}0009 \end{pmatrix}, \quad C_d = I_4 \tag{5.3}$$

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We study the case of structured perturbations in the system matrix A_d; specifically we assume that

$$\Delta A_d = \kappa_1 \begin{pmatrix} 0\cdot1 & 0\cdot15 & 0 & 0 \\ 0\cdot05 & 0 & 0\cdot1 & 0 \\ 0 & 0 & 0 & 0 \\ 0\cdot1 & 0 & 0 & 0\cdot05 \end{pmatrix} + \kappa_2 \begin{pmatrix} 0 & 0 & 0 & 0\cdot05 \\ 0 & -0\cdot1 & 0 & 0 \\ 0 & 0 & 0 & 0\cdot05 \\ 0 & 0 & 0\cdot05 & 0 \end{pmatrix} \tag{5.4}$$

First we need to find the static output feedback matrix that maximizes the stability bound (2.10), without considering performance specifications. We choose $Q = 10^{-2} I_4$, $D_1 = I_8$,

$$Z = \begin{pmatrix} 0\cdot0110 & 0 & 0 & 0 \\ 0 & 0\cdot0148 & -0\cdot0007 & -0\cdot0110 \\ 0 & -0\cdot0007 & 0\cdot0199 & -0\cdot0028 \\ 0 & -0\cdot0110 & -0\cdot0028 & 0\cdot0047 \end{pmatrix}$$

and an initial gain that places the closed-loop poles at $(0\cdot20,\ 0\cdot70,\ -0\cdot50 \pm 0\cdot25j)$,

$$K_1 = \begin{pmatrix} -0\cdot0264 & -0\cdot1722 & 3\cdot0531 & 10\cdot2700 \\ -1\cdot6068 & 0\cdot2706 & 0\cdot0224 & -0\cdot0742 \end{pmatrix}$$

Our algorithm converges to the stabilizing gain

$$K = \begin{pmatrix} -0\cdot1089 & -0\cdot5016 & 2\cdot2162 & 1\cdot5402 \\ -1\cdot8981 & 0\cdot0748 & 0\cdot0317 & -0\cdot1342 \end{pmatrix}$$

which corresponds to the $\tilde{J}^{r}_{A} = 0\cdot4468$. With this gain, the stability region for the uncertain parameters is defined by the circle $\kappa_1^2 + \kappa_2^2 < (0\cdot6161)^2$.

Next, we include the LQR cost (3·27) in the minimizing quantity for $Q_1 = 0\cdot1 I_4$, $R_1 = 0\cdot01 I_2$, $X_0 = 0\cdot1 I_4$. We consider the same Q, Z, D_1, K_1 as before and obtain the output gain

$$K = \begin{pmatrix} -0\cdot0672 & -0\cdot1988 & 1\cdot4512 & 0\cdot9723 \\ -1\cdot9253 & 0\cdot0895 & 0\cdot0590 & 0\cdot0641 \end{pmatrix}$$

which corresponds to $\tilde{J}^{r,p}_{A} = \tilde{J}^{r}_{A} + J_{LQR} = 0\cdot3985 + 0\cdot0932 = 0\cdot4917$. This gain defines the circle $\kappa_1^2 + \kappa_2^2 < (0\cdot5269)^2$. Note that the uncertainty radius here is more conservative than the case of only the robustness specifications studied above. □

Example 5.3: Consider the paper-machine head box from Franklin *et al.* (1990, p. 788), whose continuous equations of motion are given by

$$\begin{pmatrix} \dot{H}(t) \\ \dot{h}(t) \\ \dot{u}_a(t) \end{pmatrix} = \begin{pmatrix} -0\cdot2 & 0\cdot1 & 1 \\ -0\cdot05 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} H(t) \\ h(t) \\ u_a(t) \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0\cdot7 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_c(t) \\ u_s(t) \end{pmatrix} \tag{5.5}$$

where $H(t)$ is the total head perturbation, $h(t)$ the stock-level perturbation, $u_a(t)$ the perturbation in the air-valve opening, $u_c(t)$ the command value to the air valve, and

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$u_s(t)$ the perturbation in the stock-valve opening. We consider the discrete-time model for $T = 0{\cdot}2$ s (Franklin *et al.* 1990, p. 462)

$$A_d = \begin{pmatrix} 0{\cdot}9607 & 0{\cdot}0196 & 0{\cdot}1776 \\ -0{\cdot}0098 & 0{\cdot}9999 & -0{\cdot}0009 \\ 0 & 0 & 0{\cdot}8187 \end{pmatrix}, \quad B_d = \begin{pmatrix} 0{\cdot}0185 & 0{\cdot}1974 \\ -0{\cdot}0001 & 0{\cdot}1390 \\ 0{\cdot}1813 & 0 \end{pmatrix} \tag{5.6}$$

We consider the state-feedback case and assume unstructured perturbations in both matrices A, B. We need to find the gain matrix K that maximizes the stability region, without first considering any performance specifications. We choose as initial stabilizing gain the one indicated by Franklin *et al.* (1990), that is

$$K_1 = \begin{pmatrix} -6{\cdot}81 & 9{\cdot}79 & -3{\cdot}79 \\ -0{\cdot}95 & -4{\cdot}94 & -0{\cdot}10 \end{pmatrix}, \quad Z = \begin{pmatrix} 0{\cdot}0009 & 0{\cdot}0001 & 0 \\ 0{\cdot}0001 & 0{\cdot}0009 & 0 \\ 0 & 0 & 0{\cdot}0013 \end{pmatrix}$$

$D_1 = I_3$ and $Q = 10^{-6} I_3$. Our algorithm converges very rapidly to the stabilizing gain

$$K = \begin{pmatrix} -0{\cdot}00024 & 0{\cdot}00004 & -0{\cdot}00207 \\ -0{\cdot}00265 & -0{\cdot}00215 & -0{\cdot}00048 \end{pmatrix}$$

which defines the stability region of Fig. 2 for $x = \sigma_{\max}(\Delta A)$, $y = \sigma_{\max}(\Delta B)$, and $\delta = 0{\cdot}0027$, $\delta/y = 0{\cdot}7772$.

Next, we introduce performance specifications in the minimizing quantity. Specifically, we include the LQR cost of (3.27) for $Q_1 = \operatorname{diag}(0{\cdot}25, 1, 1)$, $R_1 = 0{\cdot}05 I_2$ and $X_0 = 10^{-6}$. We keep Z, Q, D_1, K_1 as before and obtain a stabilizing gain of

$$K = \begin{pmatrix} -0{\cdot}0001 & 0{\cdot}0005 & -0{\cdot}0021 \\ -0{\cdot}0030 & -0{\cdot}0033 & -0{\cdot}0005 \end{pmatrix}$$

which defines the stability region of Fig. 2 for $\delta = 0{\cdot}0027$, $\delta/y = 0{\cdot}6110$. Note that for this specific case the crucial parameter was X_0 since, the smaller the value of X_0 was selected, the larger the stability region that the algorithm defined. □

Note that, for the examples presented above and the numerous examples of Konstantopoulos and Antsaklis (1994 b), our algorithm proved to be quite fast. The algorithm, written in MATLAB code, converged in just several iterations of the main step; this took approximately a minute on a Sun SPARCstation 10.

6. Conclusions

A fast optimization algorithm for the design of robust output feedback controllers for linear uncertain discrete-time systems has been presented. This algorithm utilizes a version of the Broyden family method of conjugate directions which is based on the BFGS rule. The minimizing quantity reflects the twofold optimization objective, which is the simultaneous maximization of established uncertainty bounds and the minimization of the typical LQR performance criterion. The first objective is based on the improved bounds of Konstantopoulos and Antsaklis (1995 b). The algorithm has also been applied to the case where the only objective is the design of a stabilizing output feedback controller that maximizes the uncertainty bounds. In that case, the derived stability bounds are, in general, larger than those derived in the case of the

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robustness–LQR minimizing quantity. This was expected, since the inclusion of the LQR term in the minimizing quantity added an additional requirement to the optimization task.

Previous related work was restricted to the case of unstructured perturbations in the system matrix A. Here, a unified approach to cases of both unstructured and structured perturbations in the matrices of a state-space model has been presented. Specifically, the general cases of unstructured perturbations in all state-space matrices (A, B, C) and structured perturbations in any pair of the above set of matrices have been studied. It has been shown that, for the special case of single unstructured perturbations in A, the present design process improves significantly the unstructured bound derived by Kolla and Farison (1991). Examples have been provided to illustrate the results. A case that remains to be addressed is that of structured perturbations in all the state-space matrices. The recently developed bound of Konstantopoulos and Antsaklis (1995b) does not appear very convenient for that case, and therefore alternative bounds need to be investigated. Finally, the continuous counterpart of the present discrete-time case remains to be further investigated.

Appendix A: Proof of Theorem 2.1

We rewrite (2.2) as follows:

$$(A+\Delta A)^{\mathrm{T}} P(A+\Delta A)-P+Q-(\Delta A)^{\mathrm{T}} P(\Delta A)-A^{\mathrm{T}} P(\Delta A)-(\Delta A)^{\mathrm{T}} PA=0 \quad \text{(A 1)}$$

Using the direct method of Lyapunov, we see that $A+\Delta A$ remains an asymptotically stable matrix, if

$$\tilde{Q}=Q-(\Delta A)^{\mathrm{T}} P(\Delta A)-A^{\mathrm{T}} P(\Delta A)-(\Delta A)^{\mathrm{T}} PA>0 \quad \text{(A 2)}$$

We know that the following inequality holds for any positive definite matrix Z and positive number α:

$$X^{\mathrm{T}}\Psi+\Psi^{\mathrm{T}}X \leqslant \alpha X^{\mathrm{T}}ZX+\frac{1}{\alpha}\Psi^{\mathrm{T}}Z^{-1}\Psi \quad \text{(A 3)}$$

Applying (A 3) for $X=\Delta A$ and $\Psi=PA$, we have

$$\left.\begin{aligned} (\Delta A)^{\mathrm{T}} PA+A^{\mathrm{T}} P(\Delta A) &\leqslant \alpha(\Delta A)^{\mathrm{T}} Z(\Delta A)+\frac{1}{\alpha}\Omega_1 \\ (\Delta A)^{\mathrm{T}} P(\Delta A)+(\Delta A)^{\mathrm{T}} PA+A^{\mathrm{T}} P(\Delta A) &\leqslant (\Delta A)^{\mathrm{T}}(\alpha Z+P)(\Delta A)+\frac{1}{\alpha}\Omega_1 \end{aligned}\right\} \quad \text{(A 4)}$$

where Ω_1 is defined in (2.4). In view of (A 2) and (A 4), we see that

$$\tilde{Q} \geqslant Q-(\Delta A)^{\mathrm{T}}(\alpha Z+P)(\Delta A)-\frac{1}{\alpha}\Omega_1 \quad \text{(A 5)}$$

Therefore a sufficient condition for $\tilde{Q}$ to be positive definite is that the right-hand side of (A 5) is positive definite, from which (2.5) follows easily. Note that α can be any positive number that satisfies (2.5). Next a sufficient lower bound for α is derived. We know that for any two positive definite matrices A, B

$$A<B \Leftrightarrow \sigma_{\max}(A)<\sigma_{\min}(B) \quad \text{(A 6)}$$

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We have

$$\sigma_{\max}\left((\Delta A)^{\mathrm{T}}(\alpha Z+P)(\Delta A)+\frac{1}{\alpha}\Omega_1\right) \leqslant \sigma_{\max}[(\Delta A)^{\mathrm{T}}(\alpha Z+P)(\Delta A)] + \sigma_{\max}\left(\frac{1}{\alpha}\Omega_1\right)$$
$$\leqslant \sigma^2_{\max}(\Delta A)\,\sigma_{\max}(\alpha Z+P)+\sigma_{\max}\left(\frac{1}{\alpha}\Omega_1\right) \tag{A 7}$$

In view of (A 6) and (A 7), we see that a sufficient condition for (2.5) to hold is

$$\sigma^2_{\max}(\Delta A)\,\sigma_{\max}(\alpha Z+P)+\sigma_{\max}\left(\frac{1}{\alpha}\Omega_1\right) < \sigma_{\min}(Q) \tag{A 8}$$

Now, (2.6) follows easily. To maintain the right-hand side of (2.6) positive, α has to satisfy (2.7). □

Appendix B: Computation of volume defined in (3.7)

In this appendix, we are interested in computing the volume that is confined by the following inequalities:

$$x, y, z > 0 \tag{B 1}$$

$$x+ay+bz+cyz < d, \quad a, b, c, d > 0 \tag{B 2}$$

This volume has been computed by Konstantopoulos and Antsaklis (1994 b):

$$V = \frac{a^2b^2}{4c^3}\left\{\left(1+\frac{cd}{ab}\right)^2\left[\ln\left(1+\frac{cd}{ab}\right)^2-1\right]+1\right\}-\frac{d^2}{2c} \tag{B 3}$$

If a, b can be written in terms of c, that is

$$a=\omega_1 c, \quad b=\omega_2 c, \quad \omega_1, \omega_2 > 0 \tag{B 4}$$

then (B 3) can be written as

$$V(c,d) = \frac{\omega_1^2\omega_2^2 c}{4}\left\{\left(1+\frac{d}{\omega_1\omega_2 c}\right)^2\left[\ln\left(1+\frac{d}{\omega_1\omega_2 c}\right)^2-1\right]+1\right\}-\frac{d^2}{2c} \tag{B 5}$$

Since we need to know how c and d affect $V(c,d)$, we compute next the partial derivatives of V with respect to c and d. First, we see how $V(c,d)$ is affected by d

$$\frac{1}{\omega_1\omega_2}\frac{\partial V(c,d)}{\partial d} = F_1\left(\frac{d}{\omega_1\omega_2 c}\right) \tag{B 6}$$

where $F_1(x)=(1+x)\ln(1+x)-x$ (Konstantopoulos and Antsaklis 1994 b). We can easily verify that $F_1(x) > 0$, for $x > 0$. Therefore we have

$$\frac{\partial V(c,d)}{\partial d} > 0 \tag{B 7}$$

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Next, we see how $V(c,d)$ is affected by c

$$\frac{1}{\omega_1^2\omega_2^2}\frac{\partial V(c,d)}{\partial c} = F_2\left(\frac{d}{\omega_1\omega_1 c}\right) \tag{B 8}$$

where $F_2(x) = \frac{1}{4}x(x-2) - \frac{1}{2}(x^2-1)\ln(1+x)$ (Konstantopoulos and Antsaklis 1994b). Again, we can easily verify that $F_2(x) < 0$, for $x > 0$. Therefore we have

$$\frac{\partial V(c,d)}{\partial c} < 0 \tag{B 9}$$

Therefore, from (B 7) and (B 9), we conclude that, in order to maximize the volume $V(c,d)$ of (B 5), we simply need to maximize d and minimize c.

Appendix C: Dynamic output feedback case

In this appendix, we demonstrate that the dynamic output feedback case can be reduced to the static output feedback case, so that the approach described above for static feedback can be used for dynamic feedback as well. We consider the linear discrete-time system with the state-space description

$$x(k+1) = Ax(k) + Bu(k), \quad y(k) = Cx(k) \tag{C 1}$$

and apply dynamic output feedback

$$x_c(k+1) = A_c x_c(k) + B_c y(k), \quad u(k) = C_c x_c(k) + D_c y(k) \tag{C 2}$$

The closed-loop system is then described by

$$\begin{pmatrix} x(k+1) \\ x_c(k+1) \end{pmatrix} = \begin{pmatrix} A + BD_c C & BC_c \\ B_c C & A_c \end{pmatrix} \begin{pmatrix} x(k) \\ x_c(k) \end{pmatrix} = A_{cl} \begin{pmatrix} x(k) \\ x_c(k) \end{pmatrix} \tag{C 3}$$

We can easily show that

$$A_{cl} = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & B \\ I & 0 \end{pmatrix} \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix} \begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix} = \tilde{A} + \tilde{B} K_D \tilde{C} \tag{C 4}$$

Now consider perturbations in A, that is $A = A_0 + \Delta A$ or $A = A_0 + \sum_{i=1}^m \kappa_i A_i$. Then we write the state matrix $\tilde{A}$ of the closed-loop system above (C 4) respectively as

$$\tilde{A} = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \Delta A & 0 \\ 0 & 0 \end{pmatrix} = \tilde{A}_0 + \Delta\tilde{A} \tag{C 5}$$

$$\tilde{A} = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{i=1}^m \kappa_i \begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix} = \tilde{A}_0 + \sum_{i=1}^m \kappa_i \tilde{A}_i \tag{C 6}$$

Similarly, we can define the nominal and perturbation matrices (structured and unstructured) for $\tilde{B}$ and $\tilde{C}$. Therefore the problem of dynamic output feedback

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(A_c, B_c, C_c, D_c) for the system (A, B, C) reduces to the problem of static output feedback

$$K_D = \begin{pmatrix} A_c & B_c \\ C_c & D_c \end{pmatrix}$$

for the system $(\tilde{A}, \tilde{B}, \tilde{C})$, as indicated in (C 4), and all the results presented above can readily be used.

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