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A GENERAL MODEL FOR THE QUALITATIVE ANALYSIS OF HYBRID DYNAMICAL SYSTEMS [1]

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1. Introduction

Hybrid systems which are capable of exhibiting simultaneously several kinds of dynamic behavior (e.g., continuous-time dynamics, discrete-time dynamics, jump phenomena, logic commands, and the like) in different parts of a system are of great current interest (see, e.g., [1]-[9]). Typical examples of such systems of varying degrees of complexity include computer disk drives [4], transmissions and stepper motors [3], constrained robotic systems [2], intelligent vehicle/highway systems [8], sampled-data systems [10]-[11], switched systems [12]-[13], and many other types of systems (refer, e.g., to the references included in [5]). Although some efforts have been made to provide a unified framework for describing such systems [9], most of the investigations in the literature focus on specific classes of hybrid systems. More to the point, at the present time, there does not appear to exist a satisfactory general model for hybrid dynamical systems which is suitable for the qualitative analysis of such systems.

In the present paper we first formulate a definition of hybrid dynamical system which covers a very large number of classes of hybrid systems and which is suitable for the qualitative analysis of such systems. Next, we present several specific examples of hybrid dynamical systems. In a companion paper [18] we develop a qualitative theory which is based on the model for hybrid dynamical systems developed herein.

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2. Notation

Let R denote the set of real numbers and let R^n denote real n-space. If $x \in R^n$, then $x^T = (x_1, \cdots, x_n)$ denotes the transpose of x. Let $R^{n\times m}$ denote the set of $n \times m$ real matrices. If $B = [b_{ij}]_{n\times m} \in R^{n\times m}$, then B^T denotes the transpose of B.

Let R^+ denote the set of nonnegative real numbers, i.e., $R^+ = [0, +\infty)$, and let N denote the set of nonnegative integers, i.e., $N = \{0, 1, \cdots, \}$. For any $r \in R^+$, $[r]$ denotes the greatest integer less or equal to r. Let X be a subset of R^n and let Y be a subset of R^m. We denote by $C[X, Y]$ the set of all continuous functions from X to Y, and we denote by $C^k[X, Y]$ the set of all functions from X to Y which have continuous derivatives up to and including order k.

A set T is said to be fully ordered with the order "$\prec$" if for any $t_1, t_2 \in T$ and $t_1 \neq t_2$, either $t_1 \prec t_2$, or $t_2 \prec t_1$. We will let (T, ρ) be a metric space where T represents the set of elements of the metric space and ρ denotes the metric.

We denote a mapping f of a set V into a set W by $f : V \to W$ and we denote the set of all mappings from V into W by $\{V \to W\}$.

3. Hybrid Systems

We require the following notion of time space.

Definition 1 (Time Space): A metric space (T, ρ) is called a *time space* if

i) T is fully ordered with order "$\prec$";

ii) T has a minimal element $t_{min} \in T$, i.e., for any $t \in T$ and $t \neq t_{min}$, it is true that $t_{min} \prec t$;

iii) for any $t_1, t_2, t_3 \in T$ such that $t_1 \prec t_2 \prec t_3$, it is true that

$$\rho(t_1, t_3) = \rho(t_1, t_2) + \rho(t_2, t_3);$$

iv) T is unbounded from above, i.e., for any $M > 0$, there exists a $t \in T$ such that $\rho(t, t_{min}) > M$. □

When ρ is clear from context, we will frequently write T in place of (T, ρ).

We can now introduce the concept of motion defined on a time space (T, ρ).

Definition 2 (Motion): Let (X, d) be a metric space and let $A \subset X$. Let (T, ρ) be a time space, and let $T_0 \subset T$. For any fixed $a \in A, t_0 \in T_0$, we call a mapping $p(\cdot, a, t_0) : T_{a,t_0} \to X$ a *motion* if $p(t_0, a, t_0) = a$ where $T_{a,t_0} = \{t \in T : t_0 \preceq t, \rho(t, t_0) < l\}$ and $l > 0$ is finite or infinite. □

We are now in a position to define *hybrid system.*

Definition 3.3 (Hybrid System): Let S be a family of motions, i.e.,

$$S \subset \{p(\cdot, a, t_0) \in \Lambda : p(t_0, a, t_0) = a\},$$

where

$$\Lambda = \cup_{(a,t_0) \in A \times T_0} \{T_{a,t_0} \times \{a\} \times \{t_0\} \to X\}.$$

The five-tuple $\{T, X, A, S, T_0\}$ is called a *hybrid dynamical system.* □

In the existing literature, several variants for dynamical system definitions are considered (see, e.g., [14]-[17]). Typically, in these definitions time is either $T = R^+$ or $T = N$, but not both simultaneously, $T_0 = T$, and depending on the particular definition, various continuity requirements are imposed on the motions which comprise the dynamical system. It is important to note that these system definitions are not general enough to accommodate even the simplest types of hybrid systems, such as, for example, sampled-data systems of the type considered in the example below. In the vast literature on sampled-data systems, the analysis and/or synthesis usually proceeds by replacing the hybrid system by an equivalent system description which is valid only at discrete points in time. This may be followed by a separate investigation to determine what happens to the plant to be controlled between samples.

4. Examples of Hybrid Dynamical Systems

We now elaborate on the above concepts by considering some specific examples of dynamical systems.

Example 1 (Nonlinear sampled-data feedback control system) We consider systems described by equation of the form

$$\begin{cases} \dot{x}(r) = f(x(r)) + Bu(k), & r \in [k, k+1) \\ u(k+1) = Cu(k) + Dx(k) \end{cases} \tag{1}$$

where $x \in R^n$, $f \in C^1[R^n, R^n]$, $f(0) = 0$, $u \in R^m$, $B \in R^{n \times m}$, $C \in R^{m \times m}$, $D \in R^{m \times n}$, $r \in R^+$, and $k \in N$. This system is a special case of the hybrid dynamical system specified in Definition 3. In the present case the time space is given by

$$T \triangleq \{(r, k) \in R^2 : r \geq 0, k = [r]\}. \tag{2}$$

This space T is equipped with a metric ρ which has the property that for any $t_1 = (r_1, k_1) \in T$ and $t_2 = (r_2, k_2) \in T$, $\rho(t_1, t_2) = |r_1 - r_2|$. The set T is a fully ordered space in such a way that $t_1 \prec t_2$ if and only if $r_1 < r_2$. It is easily verified that T defined in (2), along with the metric ρ given above, is a time space in the sense of Definition 1. The set T_0 is given by $T_0 = \{(k, k) \in R^2 : k \in N\}$. In Fig.1, we provide the "graph" for T.

The motion p determined by system (1) are of the form

$$p(t) = [x(r)^T, u(k)^T]^T \tag{3}$$

where in (3) $t = (r, k) \in T$, and the initial condition $a \in A$ and the initial time $t_0 \in T_0$ have been suppressed.

The system (1) may be viewed as an interconnection of two subsystems: a *plant* which is described by a system of first order ordinary differential equations, and as such, is defined on "*continuous-time*", R^+, and a *digital controller* which is described by a system of first order ordinary difference equations, and as such, is defined on "*discrete-time*", N. The entire system (1) is then defined on $T \subset R^+ \times N$.

In our considerations of the above sampled-data system, we did not include explicitly a description of the interface between the plant and the digital controller (a sample element) and between the digital controller and the plant (a sample and hold element). □

Example 2 (Motion control system) Several different kinds of motion control systems are considered in [3]. These systems satisfy Definition 3 of hybrid dynamical system. In the following, we consider Example 4 of [3] for purpose of demonstration.

The motion control system in Example 4 of [3] concerns an engine-drive system for an automobile with an automatic transmission. It is described by a system of equations of the form

$$\begin{cases} \dot{x}_1(r) = x_2(r) \\ \dot{x}_2(r) = [-a(x_2(r)) + u(r)]/[1 + z([p])] \\ \dot{p}(r) = l \\ z([p]+1) = f(z([p]), x_1(r_{[p]}), x_2(r_{[p]})), \end{cases} \tag{4}$$

where $x_1, x_2 \in R^n$ denote vehicle ground speed and engine rpm, respectively, $u(r) \in R^n$ denotes the external input as the throttle position, the $a(\cdot)$ term describes the decrease in the ability of the system to produce torque at high rpms, $z \in N$ represents the shift position of the transmission, and $f : N \times R^n \times R^n \to N$ determines the shifting rule. The variable $p \in C^1(R^+, R^+)$ represents a special "clock" or "counter", where $l \in R^+$ in (4) depends implicitly on x_1, x_2, u, and z. The notation $r_{[p]}$ denotes the most recent time at which p passes an integer, i.e.,

$$r_{[p]} = \{r \in R^+ : p(r) = [p]\}.$$

Notice that in the above definition, $r_{[p]}$ is uniquely determined because $l > 0$.

In the present example the abstract time space T is defined as

$$T = \{(r, r_{[p]}) \in R^2 : r \geq 0, p = p(r)\}.$$

This space is equipped with a metric ρ such that for any $t = (r, r_{[p]}) \in T$, $\tilde{t} = (\tilde{r}, \tilde{r}_{[p]}) \in T$, we have $\rho(t, \tilde{t}) = |r - \tilde{r}|$. Furthermore, T is fully ordered in such a way that $t \prec \tilde{t}$ if and only if $r < \tilde{r}$. It is now easily verified that T is a time space in the sense of Definition 1.

The motions determined by system (4) are of the form

$$q(t) = [x_1(r), x_2(r), z([p])]^T$$

where $t = (r, r_{[p]}) \in T$. □

Example 3 (Systems with Impulse Effects) There are numerous examples of evolutionary systems which at certain instants of time are subjected to rapid changes. In the simulations of such processes it is frequently convenient and valid to neglect the durations of the rapid changes and to assume that the changes can be represented by state jumps. Examples of such systems arise in mechanics (e.g., the behavior of a buffer subjected to a shock effect, the behavior of clock mechanisms, the change of velocity of a rocket at the time of separation of a stage, and so forth), in radio engineering and communication systems (where the generation of impulses of various forms is common), in biological systems (where, e.g., sudden population changes due to external effects occur frequently), in control theory (e.g., impulse control, robotics, etc), and the like. For additional specific examples, refer to [19].

Appropriate mathematical models for processes of the type described above are so-called *systems with impulse effects.* The qualitative behavior of such systems has been investigated extensively in the literature (refer to [19] and the references cited in [19]).

The class of systems with impulse effects under investigation can be described by equations of the form

$$\begin{cases} \frac{dx}{dt} = f(x,t), & t \neq \tau_k \\ \Delta x = I_k(x), & t = \tau_k \end{cases}, \tag{5}$$

where $x \in X \subset R^n$ denotes the state, $f \in C[R^n \times R \to R^n]$ satisfies a Lipschitz condition with respect to x which guarantees the existence and uniqueness of solutions of system (5) for given initial conditions, $E = \{\tau_1, \tau_2, \cdots : \tau_1 < \tau_2 < \cdots\} \subset R^+$ is an unbounded discrete subset of R^+ which denotes the set of times when jumps occur, and $I_k \in C[R^n, R^n]$ denotes the incremental change of the state at the time τ_k. It should be pointed out that in general E depends on a specific motion and that for different motions, the corresponding sets $E = \{\tau_1, \tau_2, \cdots : \tau_1 < \tau_2 < \cdots\} \subset R^+$ are in general different. The function $\phi : [t_0, \infty) \to R^n$ is said to be a *solution* of the system with impulse effects (5) if i) $\phi(t)$ is left continuous on $[t_0, \infty)$ for some $t_0 \geq 0$ ii) $\phi(t)$ is differentiable and $\frac{d\phi}{dt}(t) = f(t, \phi(t))$ everywhere on (t_0, ∞) except on an unbounded discrete subset $E = \{\tau_1, \tau_2, \cdots : \tau_1 < \tau_2 < \cdots\} \subset R^+$; and iii) for any $t = \tau_k \in E$, $\phi(t^+) = \phi(t) + I_k(\phi(t))$, where $\phi(t^+)$ denotes the right limit of ϕ at t, i.e., $\phi(t^+) = \lim_{r \to t^+} \phi(r)$.

If for system (5) we assume further that $f(0,t) = 0$ for all $t \in R^+$, and $I_k(0) = 0$ for all $k \in N$, then it is clear that $x = 0$ is an equilibrium. For the qualitative behavior of this equilibrium, some Lyapunov type theorems have been established in [19].

System (5) also satisfies Definition 3 for hybrid systems. The state space is $X \subset R^n$, and the time space for system (5) is R^+. In fact, system (5) also satisfies the general definitions for dynamical systems (see, e.g., [14], [15]). However the state of system (5) is discontinuous. In [20], we will regard system (5) as a hybrid system and then, we apply our stability results for hybrid systems to system (5). This enables us to obtain better qualitative results for the stability of the equlibrium $x = 0$ of system (5).

Example 4 (Switched Systems)

Switched systems, which constitute a special class of *hybrid dynamical systems*, include ***multimodal*** systems or ***systems with variable structure***. In the present example we concentrate on switched systems which are combinations of finitely many continuous dynamical systems. These systems can be described by equations of the form

$$\dot{x}(t) = f_i(x(t)), \qquad i \in \{1, \cdots, K\} \qquad (6)$$

where $x(t) \in R^n$ denotes the state of the system, $f_i \in C(R^n, R^n)$ is Lipschitz continuous, and the $i's$ are picked in such a way that there are ***finite switchings in finite time***.

The qualitative behavior of systems (6) has been analyzed in [12] by utilizing "multiple Lyapunov functions". A special class of system (6), where $f_i(x) = A_i x$ for some constant matrices A_i $(i = 1, \cdots, K)$, has been studied in detail in [13]. It is well known that even if we have individual ***candidate Lyapunov functions*** for each system f_i which ensure certain desired stability properties, we still need to impose restrictions on the switchings to guarantee the desired stability property for the entire switched system. This is demonstrated, e.g., in [12].

We can view the switched systems (6) as special hybrid dynamical systems which satisfy our definition. The state space is $X \subset R^n$ and the time space is R^+. In [20], we are able to obtain a stability criterion for system (6) which is less conservative than the results in [13] and [12].

5. Qualitative Analysis of Hybrid Dynamical Systems

In a companion paper [18], we develop a qualitative theory for the class of hybrid dynamical systems developed herein. Items which we address in [18] include Lyapunov stability and asymptotic stability of an invariant set (such as, e.g., an equilibrium). Specifically, we establish sufficient conditions and also necessary conditions (i.e., converse theorems) for stability and asymptotic stability of an invariant set. Furthermore, to demonstrate the applicability of these results, we conduct in [18] a stability analysis of the hybrid dynamical system described in Example 1 of the present paper.

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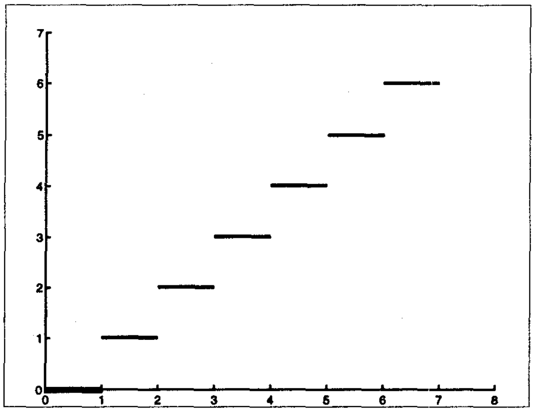

Fig. 1 Time space T for Example 1