P. J. Antsaklis and M. K. Sain, "Unity Feedback Compensation of Unstable Plants," P roc.of t he 2 0th I EEE C onf. o n D ecision a nd C ontro l , pp. 305-308, San Diego, CA, Dec. 16-18, 1981.

WP2 - 5:30

UNITY FEEDBACK COMPENSATION OF UNSTABLE PLANTS*

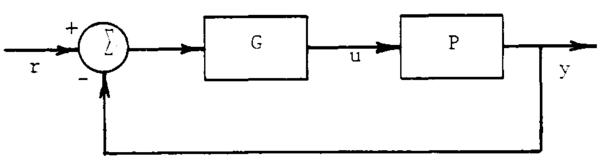
P.J. Antsaklis and M.K. Sain
Department of Electrical Engineering
University of Notre Dame
Notre Dame, Indiana 46556

Abstract

The compensation of a plant (possibly unstable and/or non-minimum phase) via unity feedback with internal stability is studied in this paper. The restrictions imposed on the designer by unstable poles and zeros of the plant are described and discussed, and the whole class of appropriate compensators needed for a particular design is given. The hidden modes which might be introduced in the closed loop system are fully characterized; and the sensitivity of the compensated system, as well as the stability and properness of the compensator and its relation to state observers, are also treated.

Summary

Assume that unity feedback is used to compensate a given plant described by a proper transfer matrix $P(p \times m)$:



If the feedback loop is well defined, the closed loop transfer matrix T is then given by

$$T = P(I + GP)^{-1} G. \qquad (1)$$

It is of interest in design to choose G so that, under internal stability in the loop, T is a desired stable and proper transfer matrix and

$$M \triangleq (I + GP)^{-1} G \qquad (2)$$

is also an acceptable (stable and proper) transfer matrix; note that $u = Mr$, that is M characterizes the control action u needed for the compensation of P.

For convenience, transfer functions here can be taken as elements of $R(s)$, the quotient field of the principal ideal domain $R[s]$ of polynomials with coefficients in R, the real numbers. However, everything can also be done over an arbitrary field k in place of R; in this case, the meaning of stability has to be adjusted appropriately.

It has been shown [1] that all pairs (T,M) which satisfy

$$T = PM, \qquad (3)$$

where P is given and the pair (T,M) is stable, are given by

$$\begin{bmatrix} T \\ M \end{bmatrix} = \begin{bmatrix} N \\ D \end{bmatrix} X, \qquad (4)$$

where X is any stable rational matrix, where the columns of

$$\begin{bmatrix} N \\ D \end{bmatrix}$$

form an $R[s]$-module basis for $\ker [I \quad -P]$, and where (N,D) is a right prime factorization of P. Cases in which the pair (T,M) are to be proper as well are a special case of the results above.

With respect to feedback realization, various general results are known. For example, if P is strictly proper, Bengtsson [2] has shown that (T,M) stable and proper is a sufficient condition for internally stable feedback realization, under suitable technicalities. This work has been generalized by Pernebo [3]. Notice, however, that here we are interested in a particular feedback realization, namely, unity feedback. $T = NX$ implies (under certain mild conditions) that all unstable zeros of the plant must also be zeros of T, independently of the particular realization used (compare with [4]). Equation (4) also implies that there exists a unique correspondence between realizable $\begin{bmatrix} T \\ M \end{bmatrix}$ and X; X is called the <u>design matrix</u>, as it is exactly what the designer must choose.

For unity feedback, if $G = \tilde{D}_G^{-1} \tilde{N}_G$ is a left-prime factorization, then in view of (2) and (4)

$$X = (\tilde{D}_G D + \tilde{N}_G N)^{-1} \tilde{N}_G. \qquad (5)$$

*This work was supported by the Office of Naval Research under Contracts N00014-79-C-0475, P00001, P00002.

0191-2216/81/0000-0305 $00.75 © 1981 IEEE

Internal Stability. The closed loop system is internally stable [5] if $\tilde{D}_G D + \tilde{N}_G N = A$, where A^{-1} exists and is a stable rational matrix, or equivalently, only if

$$\tilde{D}_G = Ax_1 - B\tilde{N} \quad , \quad \tilde{N}_G = Ax_2 + B\tilde{D} \tag{6}$$

where B is any polynomial matrix, $P = \tilde{D}^{-1}\tilde{N}$ is a left prime factorization and $x_1 D + x_2 N = I$. All internally stabilizing compensators are given by

$$G = (Ax_1 - B\tilde{N})^{-1}(Ax_2 + B\tilde{D}) \tag{7}$$

where A^{-1} is stable and B is such that $|Ax_1 - B\tilde{N}| \neq 0$.

Theorem [6] $\begin{bmatrix} T \\ M \end{bmatrix} = \begin{bmatrix} N \\ D \end{bmatrix} X$ can be realized via unity feedback with internal stability if and only if

a) rank $(I - NX)$ is full $\quad$ (8)

and

b) there exist polynomial matrices A, B (A^{-1} stable) such that

$$X = A^{-1}(Ax_2 + B\tilde{D}). \tag{9}$$

If a solution exists, the compensator G is unique and is given by

$$G = (Ax_1 - B\tilde{N})^{-1} AX. \tag{10}$$

Notice that (8) guarantees the existence of G in (10) ($|Ax_1 - B\tilde{N}| \neq 0$ with A, B from (9)); note also that (8) is necessary for any unity feedback compensation to be well defined since $I - T = (I + PG)^{-1}$. Condition (9) guarantees internal stability.

Observe that if only $T = NX$ is of interest (model matching via unity feedback), (8) and (9) can be expressed in terms of T; furthermore, if internal stability is the only design objective, so that X need only be stable, then one may always select (A,B) so that (8) and (9) are satisfied. This is a generalization of [2], since P is not required to be strictly proper.

There are alternative ways to state a theorem which do not involve A and B. First notice that, from (9),

$$(Ax_2 + B\tilde{D}) = AX \tag{11a}$$

so that from (6)

$$\tilde{D}_G = A(I - XN)D^{-1}, \tag{11b}$$

that is, internal stability implies X and $(I - XN)D^{-1}$ stable. Sufficiency can also be shown, therefore condition (9) is equivalent to

$$X \text{ and } (I - XN)D^{-1} \text{ stable.} \tag{12}$$

Now (2) and (4) imply

$$G = M(I - PM)^{-1} = DX(I - NX)^{-1} = [(I - XN)D^{-1}]^{-1} X \tag{13}$$

which can be used to determine internally stabilizing G provided that X satisfies conditions (8) and (12). It should be noted that (9) is also equivalent to the condition

$$D X \tilde{D}^{-1} = M\tilde{D}^{-1} \text{ stable} \tag{14a}$$

and

$$(I - NX)\tilde{D}^{-1} = S\tilde{D}^{-1} \text{ stable} \tag{14b}$$

where $S \triangleq (I + PG)^{-1} = I - T$ the comparison sensitivity matrix; this can be shown using the dual factorization $N_G D_G^{-1}$ of G.

Stable and Unstable P. Condition (9) can be written as:

$$(X - x_2)\tilde{D}^{-1} = A^{-1}B \tag{15}$$

which clearly shows (also (12, (14)) that the requirement for internal stability implies restrictions on the achievable X only when the plant is unstable. If the plant is stable, it is clear that condition (9) is always satisfied i.e. any stable $T = NX$ satisfying (8) can be realized via unity feedback with internal stability (see also [7]). In this case $M = DX$ can be almost any (subject to (8)) stable proper matrix, which implies for example that G evaluated from (13) with M any stable strictly proper rational matrix will internally stabilize the system (see also [8]), giving $T = NX$ as the closed loop transfer matrix. Notice however that in this case the poles of the plant are likely to become hidden modes of the system thus unnecessarily increasing the order of G. This point will be clarified below. If the plant is unstable, X is not free any longer but it must have certain structural properties to satisfy (12) or (15). Furthermore in this case the part of D which corresponds to the unstable poles must be known exactly to guarantee that the condition will be satisfied since exact cancellation is required.

Hidden modes. It is known that if a feedback design is carried out using transfer matrices, the order of the resulting compensator can be unnecessarily high or more significantly, bring about unwanted stable modes inside the loop. This is due to the (unintentional) introduction of "hidden modes" which appear as modes of the closed loop system but not as poles of the closed loop transfer matrix. The polynomial matrix system description is now used to characterize fully this phenomenon and to suggest ways to avoid it. Clearly, the hidden modes are those zeros of $|A|$ which do not appear as poles of T. Assume that an X is given which satisfies conditions (8) and (12). In view of (11), A (also $\tilde{N}_G, \tilde{D}_G$) can be determined from

$$[(I - XN)D^{-1}, X] = A^{-1}[\tilde{D}_G, \tilde{N}_G] \tag{16}$$

P. J. Antsaklis and M. K. Sain, "Unity Feedback Compensation of Unstable Plants," P roc.of t he 2 0th I EEE C onf. o n D ecision a nd C ontro l , pp. 305-308, San Diego, CA, Dec. 16-18, 1981.

a left prime factorization. It can now be shown that if $X = \tilde{D}_x^{-1}\tilde{N}_x$ then $\tilde{N}_G = AX = A_1\tilde{N}_x$, $A = A_1\tilde{D}_x$ where the zeros of $|A_1|$ are exactly those stable poles of P which do not cancel in $(I - XN)D^{-1}$. The polynomial matrix description of the closed loop, namely $Az = \tilde{N}_G r$, $y = Nz$, clearly implies that the zeros of $|A_1|$ are uncontrollable modes and therefore hidden modes. Furthermore, there might be additional, unobservable hidden modes which are exactly those poles of X which cancel in $NX = T$ i.e. $A = A_1\tilde{D}_x = A_1\hat{\tilde{D}}_x A_2$, $N = \hat{N}A_2$. Note the uncontrollable hidden modes (zeros of $|A_1|$) will appear in $\tilde{N}_G = A_1\tilde{N}_x$, while the unobservable ones (zeros of $|A_2|$) will appear in $|\tilde{D}_G|$ i.e. as poles of G. All hidden modes will cancel in the product PG. It is thus clear that, in order to avoid hidden modes choose stable X such that: first, no cancellations take place in NX and secondly, all poles of P are cancelled in $(I - XN)D^{-1}$. The second requirement is of course more difficult to satisfy and it is the main cause for the hidden modes in feedback designs.

Sensitivity. The comparison sensitivity matrix is

$$S \triangleq (I + PG)^{-1} = I - T = I - NX \qquad (17)$$

and (14) directly implies that the unstable poles of the plant must be zeros of S if internal stability is to be achieved. That is, if the plant is unstable then restrictions on the possible choices for S are imposed. Furthermore, if the plant is nonminimum phase, additional restrictions are imposed on S via N (see also [9-10]). Note also that all of the poles of P must be zeros of S if no (stable) hidden modes are desirable. In general, when internal stability is present, the zeros of S are: All of the poles of P except the uncontrollable hidden modes (zeros of $|A_1|$) and all of the poles of G except the unobservable hidden modes (zeros of $|A_2|$); the poles of S are the poles of T.

The Compensator G. The numerator $\tilde{N}_G$ and the denominator $\tilde{D}_G$ of G satisfy (16). In view of the discussion on hidden modes $\tilde{N}_G = A_1\tilde{N}_x$, while the zeros of $|\tilde{D}_G|$ (poles of G) are: the zeros of $|A_2|$ and the zeros of $S\tilde{D}^{-1}$. This implies that if given X is to be realized with stable G then S should not have unstable zeros other than the unstable poles of the plant. So X must be such that $(I - XN)D^{-1}$ is stable with stable zeros as well. This is more difficult to achieve when the plant is nonminimum phase. If internal stabilization is the only objective then X is any stable rational ($M = DX$ strictly proper); for G to be stable then again X must be chosen so that $(I - XN)D^{-1}$ is stable with stable zeros (see also [9-10]).

For G proper it is sufficient to choose X such that $DX(=M)$ is proper and $NX(=T)$ strictly proper as it can be seen from (13). Note that the order of G, that is $\partial|\tilde{D}_G| \triangleq$ degree of $|\tilde{D}_G|$ is $\partial|A| - \partial|D| = \partial|A_1| + \partial|\tilde{D}_x| - \partial|D|$ which shows that for G to be of low order, no hidden modes must be present. It can now be shown that for internal stability, properness and no hidden modes $X = \tilde{D}_x^{-1}\tilde{N}_x$ must satisfy $LD + \tilde{N}_x N = \tilde{D}_x$ with $D\tilde{D}_x^{-1}\tilde{N}_x$ proper and $N\tilde{D}_x^{-1}\tilde{N}_x$ strictly proper where L is some invertible polynomial matrix. Note that this equation is exactly $\tilde{D}_G D + \tilde{N}_G N = A$ with $DA^{-1}\tilde{N}_G$ proper and $NA^{-1}\tilde{N}_G$ strictly proper which is obtained when working with polynomial matrices. It can now be shown that these equations have always a solution as long as $\partial|\tilde{D}_x|$ (or $\partial|A|$) is higher than $\partial|D|$ by an amount which depends on the structure of N and D. In general when P is stable any X (subject to (8) and DX str. proper) can be realized via G proper as long as we are willing to accept stable hidden modes (stable poles and zeros of P). If P unstable and/or no stable hidden modes are desirable X must satisfy certain restrictions; $\partial|\tilde{D}_x|$ must be high enough for properness, $\tilde{N}_x$ must be appropriately chosen. So in this case our choices for X are certainly restricted and if they are not satisfactory, more complicated compensators should be introduced.

There is an observer of the state of P uniquely defined by any given internally stabilizing compensator G. This can be seen from $(A^{-1}\tilde{D}_G)D + (A^{-1}\tilde{N}_G)N = I$ which in view of $Dz = u$, $Nz = y$ implies that

$$(A^{-1}\tilde{D}_G)u + (A^{-1}\tilde{N}_G)y = z$$

Note that $G = (\tilde{D}_G^{-1}A)(A^{-1}\tilde{N}_G)$ which shows that any internally stabilizing compensator consists of two factors one square invertible with stable zeros and the other stable. When G proper, it can also be written as

$$G = ((\tilde{D}_G D_1)^{-1}A)(D_1 A^{-1}\tilde{N}_G) = G_1 G_2$$

where D_1^{-1} stable and $\partial|D_1| = \partial|D|$

This implies that given P, $\hat{P} \triangleq PG_1$ can always be stabilized via a proper stable compensator G_2. This is also an alternative proof to the fact that any given plant P, can be stabilized via an observer (see also [10]).

References

1. M.K. Sain, B.F. Wyman, R.R. Gejji, P.J. Antsaklis, and J.L. Peczkowski, "The Total Synthesis Problem of Linear Multivariable Control---Part I: Nominal Design", Proceedings 1981 Joint Automatic Control Conference, Paper WP-4A.

P. J. Antsaklis and M. K. Sain, "Unity Feedback Compensation of Unstable Plants," P roc.of t he 2 0th I EEE C onf. o n D ecision a nd C ontro l , pp. 305-308, San Diego, CA, Dec. 16-18, 1981.

2. G. Bengtsson, "Feedback Realizations in Linear Multivariable Systems", IEEE Trans. on Automatic Control, Vol. 22, No. 4, pp. 576-585, August 1977.

3. L. Pernebo, "An Algebraic Theory for the Design of Controllers for Linear Multivariable Systems---Parts I, II", IEEE Trans. on Automatic Control, Vol. 26, No. 1, pp. 171-193, February 1981.

4. V.H.L. Cheng and C.A. Desoer, "Limitations on the Closed-Loop Transfer Function due to Right-Half Plane Transmission Zeros of the Plant", IEEE Trans. on Automatic Control, Vol. AC-25, pp. 1218-1220, December 1980.

5. P.J. Antsaklis, "Some Relations Satisfied by Prime Polynomial Matrices and Their Role in Linear Multivariable System Theory", Technical Report No. 7806, Department of Electrical Engineering, Rice University, June 1978 and IEEE Trans. on Automatic Control, Vol. AC-24, pp. 611-616, August 1979.

6. M.K. Sain, P.J. Antsaklis, B.F. Wyman, R.R. Gejji, and J.L. Peczkowski, "The Total Synthesis Problem of Linear Multivariable Control, Part II: Unity Feedback and the Design Morphism", Proceedings 20th IEEE Conference on Decision and Control, December 1981.

7. C.A. Desoer and M.J. Chen, "Design of Multivariable Feedback Systems with Stable Plants", IEEE Trans. on Automatic Control, Vol. AC-26, pp. 408-415, April 1981.

8. G. Zames, "Feedback and Optimal Sensitivity: Model Reference Transformations, Weighted Seminorms and Approximate Inverses", IEEE Trans. on Automatic Control, Vol. AC-26, pp. 301-320, April 1981.

9. J.J. Bongiorno, Jr. and D.C. Youla, "On the Design of Single Loop Single-Input-Output Feedback Control Systems in the Complex-Frequency Domain", IEEE Trans. on Automatic Control, Vol. AC-22, pp. 416-423, June 1977.

10. D.C. Youla, J.J. Bongiorno, Jr. and C.N. Liu, "Single Loop Feedback Stabilization of Linear Multivariable Dynamical Plants", Automatica, Vol. 10, pp. 159-173, 1974.