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EIGENSTRUCTURE ASSIGNMENT IN RECONFIGURABLE CONTROL SYSTEMS

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Ioannis K. Konstantopoulos and Panos J. Antsaklis
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556
email: antsakli@saturn.ee.nd.edu

Interdisciplinary Studies of Intelligent Systems

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Ioannis K. Konstantopoulos and Panos J. Antsaklis
Department of Electrical Engineering
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Notre Dame, IN 46556, U.S.A.
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Abstract

An optimization approach to control reconfiguration, based on eigenstructure assignment, for control systems with output feedback is presented. The emphasis is on the recovery of the nominal closed-loop performance, which is determined by the closed-loop eigenvalues and eigenvectors. The proposed scheme preserves the $\max(r, q)$ most dominant eigenvalues of the nominal closed-loop system and determines their associated closed-loop eigenvectors as close to the corresponding eigenvectors of the nominal closed-loop system as possible. Additionally, the stability of the remaining closed-loop eigenvalues is guaranteed by the satisfaction of an appropriate Lyapunov equation. The overall design is robust with respect to uncertainties in the state-space matrices of the reconfigured system. The cases of state feedback and dymanic output feedback are also studied. The approach is applied to two aircraft control examples, where it is shown to not only preserve the shape of the transient response but recover much of the characteristics of the steady-state response as well.

1 Introduction

Eigenstructure assignment is a powerful technique that has developed considerably over the last fifteen years or so. This technique is concerned with the placing of eigenvalues and their associated eigenvectors, via feedback control laws, to meet closed-loop design specifications. Specifically, the method alllows the designer to directly satisfy damping, settling time and mode decoupling specifications by appropriately selecting the closed-loop eigenvalues and eigenvectors. There are certain degrees of freedom associated with the use of state and output feedback; see for instance [2], [12], [24], [25]. Several approaches have been presented exercising these degrees of freedom to design closed-loop feedback systems using eigenstructure assignment.

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

The most popular approach to eigenstructure assignment has appeared in [1] where both cases of state and output feedback are studied and a design technique for eigenstructure assignment with output feedback is presented. This methodology has subsequently been used in [18], where eigenstructure assignment and command generator tracking are applied to the design of yaw pointing and lateral translation control laws for a flight propulsion control coupling (FPCC) vehicle. In [19], the above methodology is extended to the case of dynamic output compensators.

Eigenstructure assignment with constrained output feedback is also studied in [1], where it is shown that by suppressing certain entries of the output feedback matrix gain to zero, the designer reduces controller complexity and increases reliability. In [4], a method is proposed for choosing a priori which gains should be set to zero based on the sensitivities of the eigenvalues to changes in the feedback gains. In [20], an eigenstructure assignment design scheme for an aircraft and an helicopter are presented where both performance (in terms of mode decoupling) and eigenvalue sensitivity are considered. In [22], the results of [4] are extended to include eigenvector sensitivity to the feedback gains as well, and a systematic design method is presented for eigenstructure assignment with gain suppression by a priori selecting to eliminate (suppress to zero) those entries in the output feedback matrix that have the smallest impact on both the eigenvalues and eigenvectors.

In [17], [21], eigenstructure assignment results are extended to consider stability robustness in the frequency domain, which is measured by the minimum of the smallest singular value of the return difference matrix at the plant (aircraft) inputs. A time domain sufficient condition for stability robustness of linear time-invariant systems subject to structured state-space uncertainty is used in [27] to obtain a robust eigenstructure design method; this method yields a robust pitch pointing and vertical translation controller for an AFTI-16 aircraft. Optimization techniques based on eigenstructure assignment that give the desired closed-loop eigenvalues in a specified stability range or the closed-loop eigenvectors/eigenvalues close to the desired ones are presented in [13], [14] and [26] respectively. Finally note that a list of papers dealing with eigenstructure assignment can be found in [26] and the review paper of [23].

The interest here is in control reconfiguration and the main objective is the design of a feedback law that preserves the eigenstructure characteristics describing the nominal closed-loop system. In other words, we assume changes in the operating conditions or system component failures occurring in the nominal system whose performance is determined by the nominal closed-loop eigenvalues/eigenvectors. The new control law needs to be designed such that as much of this nominal performance is recovered as possible; this can be done by recovering as much of the nominal eigenstructure information as possible. Similar to the approach presented in [7], [9], the overall design needs to be robust with respect to the matrices of the impaired state-space model. Note that the design scheme here is different from the design scheme of [7], [9], where no information regarding the eigenstructure of the nominal closed-loop system was taken into account.

In section 2, results on eigenstructure assignment are discussed; the characterization of

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

the closed-loop eigenvectors and the computation of the output feedback matrix that achieves a specific desired eigenstructure (closed-loop eigenvalues and eigenvectors) are presented. In section 3, the control reconfiguration problem is formulated and the algorithmic approach that determines the new output feedback control law based on the eigenstructure assignment results is presented. The cases of state feedback and dynamic output feedback are also discussed. In section 4, the proposed methodology is applied to two aircraft control systems, and in section 5, concluding remarks are included.

2 Eigenstructure assignment using output feedback

We consider the linear multivariable continuous system with the state-space description

$$\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u(t) && (1)\\ y(t) &= C\,x(t) && (2) \end{aligned}$$

where $x \in \Re^n$ is the state vector, $u \in \Re^r$ the input vector, and $y \in \Re^q$ the output vector; $A \in \Re^{n \times n}$, $B \in \Re^{n \times r}$, $C \in \Re^{q \times n}$ are the system matrices. The above system is assumed to be both controllable and observable, that is

$$\begin{aligned} \text{rank}\,[\,B \;\; AB \;\; \cdots \;\; A^{n-1}B\,] &= n && (3)\\ \text{rank}\,[\,C^T \;\; A^TC^T \;\; \cdots \;\; (A^T)^{n-1}C^T\,] &= n && (4) \end{aligned}$$

We also assume that the input and output matrices are of full rank, that is $\text{rank}(B) = r$ and $\text{rank}(C) = q$. Also, as is usually the case in aircraft problems, it is assumed that $r < q < n$. We consider static output feedback of the form

$$u(t) = K\,y(t) = KC\,x(t) \tag{5}$$

The freedom that characterizes the placing of the closed-loop poles using output feedback has extensively been studied; see for instance [2], [25]. For the additional freedom that characterizes the selection of the associated closed-loop eigenvectors, the following theorem has been proven in [25]

Theorem 2.1 *Consider the controllable and observable system of (1)-(2) with the output feedback law of (5) and the assumption that the matrices B and C are of full rank. Then, there exists a matrix $K \in \Re^{r \times q}$ such that*

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

(i) max(r, q) closed-loop eigenvalues can be assigned;

(ii) max(r, q) eigenvectors can be partially assigned with min(r, q) entries in each vector arbitrarily chosen.

Note that the above theorem also applies to the general case where the closed-loop eigenvalues can be repeated or in complex-conjugate pairs.

It should be stated that eigenvalue assignment for the state feedback case has thoroughly been investigated as well; see for instance [12], [24]. For controllable systems with the state-feedback law $u(t) = Kx(t)$, it has been shown in [24] that (i) all n closed-loop eigenvalues and a maximum of nr eigenvector entries can be arbitrarily assigned, and (ii) no more than r entries of any one eigenvector can be chosen arbitrarily. In other words, a maximum of r entries in each of the n closed-loop eigenvectors can be arbitrarily chosen. It is apparent that state-feedback compared to output feedback offers a greater flexibility with regard to eigenstructure assignment. Note, however, that from a practical point of view, state-feedback is quite undesirable, since for large systems it requires measuring and feeding back all states of the system. This can be quite expensive, not to mention the fact that several states are usually not available for measurement. This is the reason we usually prefer feeding back only the measured states, which makes output feedback considerably attractive. Note that an extensive discussion of eigenstructure assignment with respect to both state and output feedback can be found in [1].

2.1 Eigenvector characterization

In view of (1), (2), (5), the closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= (A + BKC)\, x(t) && (6) \\ y(t) &= C\, x(t) && (7) \end{aligned}$$

For any pair of desired closed-loop eigenvalues and their associated closed-loop eigenvectors $\{(\lambda_i,\, v_i), i = 1, .., q\}$, the following equations hold

$$\begin{aligned} (A + BKC)\, v_i &= \lambda_i v_i && (8) \\ v_i &= (\lambda_i I_n - A)^{-1} BKC v_i && (9) \end{aligned}$$

Note that the above inverse $(\lambda_i I_n - A)^{-1}$ exists under the assumption that the closed-loop eigenvalues do not belong to the set of the open-loop eigenvalues, that is the set of eigenvalues of A. Defining the $(r \times 1)$ vector

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

$$m_i = KCv_i \tag{10}$$

we rewrite (9) as

$$v_i = (\lambda_i I_n - A)^{-1} B m_i \tag{11}$$

From the above equation, we easily conclude that all achievable eigenvectors v_i that correspond to the desired closed-loop eigenvalue λ_i must lie in the subspace spanned by the columns of $(\lambda_i I_n - A)^{-1}B$. Therefore, if the desired eigenvectors $\{v_i, i = 1, .., q\}$ belong to the subspaces spanned by the columns of $\{(\lambda_i I_n - A)^{-1}B, i = 1, .., q\}$ respectively, then they are achievable. In other words, there exists an output feedback matrix K such that the desired eigenvectors can be achieved exactly. The computation of the output feedback matrix K is outlined next.

Since B has been assumed to be of full rank, the dimension of the above subspace is equal to rank$(B) = r$, that is the number of independent input variables. Finally note that the subspace for each desired closed-loop eigenvalue depends upon the matrices A, B and the closed-loop eigenvalue.

2.2 Computation of output feedback matrix

We assume that the set of desired closed-loop eigenvalues/eigenvectors is specified and given by the set of pairs $\{(\lambda_i, v_i),\ i = 1, .., q\}$. Note that each desired eigenvector v_i needs to belong to the corresponding subspace spanned by the columns of $(\lambda_i I_n - A)^{-1}B$ as discussed above. We need to compute the output feedback gain that achieves the above set of eigenvalues/eigenvectors.

We consider the state-trasformation defined by

$$x(t) = T\,\tilde{x}(t) \tag{12}$$

and select the state-transformation matrix

$$T = \begin{pmatrix} B & S \end{pmatrix} \tag{13}$$

where $S \in \Re^{n \times (n-r)}$ is any matrix such that rank$(T) = n$. Note that the selection of S is not unique. This change of state-variables transforms the system of (1)-(2) to

$$\dot{\tilde{x}}(t) = \tilde{A}\,\tilde{x}(t) + \tilde{B}\,u(t) \tag{14}$$

$$y(t) = \tilde{C}\,\tilde{x}(t) \tag{15}$$

where

$$\tilde{A} = T^{-1}AT \tag{16}$$

$$\tilde{B} = T^{-1}B = \begin{pmatrix} I_r \\ O_{n-r,r} \end{pmatrix} \tag{17}$$

$$\tilde{C} = CT \tag{18}$$

where $O_{n-r,r}$ is defined as an $[(n-r) \times r]$ zero matrix. The structure of the new input matrix $\tilde{B}$ is quite convenient, as it will be shown next, and was the reason that motivated the above state-transformation. With (12), (16)-(18), the closed-loop system of (6) is transformed to

$$\dot{\tilde{x}}(t) = (\tilde{A} + \tilde{B}K\tilde{C})\,\tilde{x}(t) \tag{19}$$

As we see, the state-transformation does not affect the output feedback matrix. This is also true for the eigenvalues of the transformed system, which remain the same with the eigenvalues of the original system. Therefore, for the transformed system $(\tilde{A}, \tilde{B}, \tilde{C})$ we have the same set of desired eigenvalues $\{\tilde{\lambda}_i = \lambda_i,\ i = 1,..,q\}$, whereas the corresponding desired eigenvectors are given by

$$\tilde{v}_i = T^{-1}v_i, \qquad i = 1,..,q \tag{20}$$

Similar to (8), each of the above pairs of desired eigenvalues/eigenvectors of the transformed system satisfies

$$(\tilde{A} + \tilde{B}K\tilde{C})\,\tilde{v}_i = \lambda_i\tilde{v}_i \tag{21}$$

$$(\lambda_i I_n - \tilde{A})\,\tilde{v}_i = \tilde{B}K\tilde{C}\,\tilde{v}_i \tag{22}$$

Define

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{11} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} \tag{23}$$

$$\tilde{v}_i = \begin{pmatrix} \tilde{f}_i \\ \tilde{l}_i \end{pmatrix} \tag{24}$$

where $\tilde{A}_{11} \in \Re^{r\times r}$, $\tilde{A}_{22} \in \Re^{(n-r)\times(n-r)}$, $\tilde{f}_i \in \Re^{r\times 1}$, $\tilde{l}_i \in \Re^{(n-r)\times 1}$. In view of (17) and the above definitions, we rewrite (22) as

$$\begin{pmatrix} \lambda_i I_r - \tilde{A}_{11} & -\tilde{A}_{12} \\ -\tilde{A}_{21} & \lambda_i I_{n-r} - \tilde{A}_2 \end{pmatrix} \begin{pmatrix} \tilde{f}_i \\ \tilde{l}_i \end{pmatrix} = \begin{pmatrix} I_r \\ O_{n-r,r} \end{pmatrix} K\tilde{C} \begin{pmatrix} \tilde{f}_i \\ \tilde{l}_i \end{pmatrix} \tag{25}$$

Considering the first r rows of the above equation, we obtain

$$\begin{pmatrix} \lambda_i I_r - \tilde{A}_{11} & -\tilde{A}_{12} \end{pmatrix} \begin{pmatrix} \tilde{f}_i \\ \tilde{l}_i \end{pmatrix} = K\tilde{C} \begin{pmatrix} \tilde{f}_i \\ \tilde{l}_i \end{pmatrix} \tag{26}$$

or equivalently

$$\begin{aligned} \lambda_i \tilde{f}_i - \tilde{A}_1 \tilde{v}_i &= K\tilde{C}\tilde{v}_i \\ (\tilde{A}_1 + K\tilde{C})\,\tilde{v}_i &= \lambda_i \tilde{f}_i \end{aligned} \tag{27}$$

where $\tilde{A}_1 = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \end{pmatrix}$ contains the first r rows of $\tilde{A}$. Considering (27) for the q pairs $(\lambda_i, \tilde{v}_i)$ and defining

$$\tilde{V} = \begin{pmatrix} \tilde{v}_1 & \tilde{v}_2 & \cdots & \tilde{v}_q \end{pmatrix} \tag{28}$$

$$\tilde{F} = \begin{pmatrix} \lambda_1 \tilde{f}_1 & \lambda_2 \tilde{f}_2 & \cdots & \lambda_q \tilde{f}_q \end{pmatrix} \tag{29}$$

we finally obtain

$$(\tilde{A}_1 + K\tilde{C})\,\tilde{V} = \tilde{F} \tag{30}$$

The above equation can be used to solve for the output feedback matrix K. Before we do so, we need to consider matrices $\tilde{V}$, $\tilde{F}$. If all desired eigenvalues are real, then the corresponding desired eigenvectors $\{\tilde{v}_i, i = 1, .., q\}$ and therefore matrices $\tilde{V}$, $\tilde{F}$ are real as well. This is not the case, however, when there is at least one pair of complex-conjugate eigenvalues/eigenvectors. In that case, both $\tilde{V}$, $\tilde{F}$ are complex matrices and the following procedure has to be followed to transform them to real matrices.

Without loss of generality, we assume that the set of desired eigenvalues consists of a complex conjugate pair, that is $\lambda_1 = \lambda_2^\star \in \mathcal{C}$, and $(q-2)$ real eigenvalues, that is $\{\lambda_i \in \Re, i = 3, .., q\}$. Then, $\tilde{v}_1 = \tilde{v}_2^\star$, where $v^\star$ denotes the complex conjugate of a vector v. Defining

$$\tilde{v}_1 = \tilde{v}_1^R + j\tilde{v}_1^I \tag{31}$$

$$\lambda_1 \tilde{f}_1 = \hat{f}_1^R + j\hat{f}_1^I \tag{32}$$

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

where v^R and v^I denote the real and imaginary part of a vector v respectively. We rewrite (30) as

$$\begin{aligned}(\tilde{A}_1 + K\tilde{C})\ (\, \tilde{v}_1^R + j\tilde{v}_1^I \quad \tilde{v}_1^R - j\tilde{v}_1^I \quad \cdots \quad \tilde{v}_q \,) = \\ (\, \hat{f}_1^R + j\hat{f}_1^I \quad \hat{f}_1^R - j\hat{f}_1^I \quad \cdots \quad \lambda_q \tilde{f}_q \,)\end{aligned} \tag{33}$$

Multiplying both sides of (33) with the nonsingular $(q \times q)$ matrix

$$\Lambda = \left(\begin{array}{cc|c} 0.5 & -0.5j & O_{1,q-2} \\ 0.5 & 0.5j & O_{1,q-2} \\ \hline O_{q-2,1} & O_{q-2,1} & I_{rq-2} \end{array} \right) \tag{34}$$

we finally have

$$\begin{aligned}(\tilde{A}_1 + K\tilde{C})\ (\, \tilde{v}_1^R \quad \tilde{v}_1^I \quad \cdots \quad \tilde{v}_q \,) &= (\, \hat{f}_1^R \quad \hat{f}_1^I \quad \cdots \quad \lambda_q \tilde{f}_q \,) & (35)\\ (\tilde{A}_1 + K\tilde{C})\, \hat{V} &= \hat{F} & (36)\end{aligned}$$

where obviously

$$\begin{aligned}\hat{V} &= (\, \tilde{v}_1^R \quad \tilde{v}_1^I \quad \cdots \quad \tilde{v}_q \,) & (37)\\ \hat{F} &= (\, \hat{f}_1^R \quad \hat{f}_1^I \quad \cdots \quad \lambda_q \tilde{f}_q \,) & (38)\end{aligned}$$

The generalization to the case of more complex conjugate pairs is straightforward. Note that $\hat{V}$, $\hat{F}$ are real matrices. Therefore, the output feedback matrix can be computed from (30) and (36) for the real and complex eigenvalues cases respectively by

$$\begin{aligned}K &= (\tilde{F} - \tilde{A}_1\tilde{V})\,(\tilde{C}\tilde{V})^{-1} & (39)\\ K &= (\hat{F} - \tilde{A}_1\hat{V})\,(\tilde{C}\hat{V})^{-1} & (40)\end{aligned}$$

In order to compute K from the relations above, it is imperative that $\tilde{C}\tilde{V} = CV$ has full rank. Therefore, we need to have desired eigenvectors such that the invertibility of the above matrices is guaranteed. On the other hand, as mentioned in [1], CV will be singular, when measurements taken (indicated by C) have little or no effect on the achievable eigenvectors

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

(indicated by V). Therefore, the possible singularity of CV gives a good indication of how unreasonable the desired eigenvectors may be with respect to the outputs measured and fed back.

Note that the output feedback matrix K computed above guarantees that q closed-loop eigenvalues of (19) will be located at $\{\lambda_i, i = 1, .., q\}$, as specified, and that their corresponding eigenvectors are the ones specified by the desired set $\{\tilde{v}_i, i = 1, .., q\}$. However, no control can be exercised upon the remaining $(n - q)$ eigenvalues which can be unstable. As mentioned before, for the computational procedure above it has been assumed that the system is fully controllable and that none of the desired closed-loop eigenvalues belongs to the set of open-loop eigenvalues. A procedure that relaxes both assumptions is presented in [11].

3 Reconfiguration and eigenstructure assignment

3.1 Problem formulation

For a linear multivariable system with the state-space description of (1)-(2) an output feedback control gain (5) has been selected such that the closed-loop eigenvalues are located at $\{\lambda_i, i = 1, .., n\}$ and the shape of the response is determined by the set of their associated eigenvectors $\{v_i, i = 1, .., n\}$. Note that the eigenstructure specified above characterizes the behavior of the closed-loop system since the eigenvalues determine the stability of the system and the eigenvectors the contribution of each system mode to the system (state or output) response.

Suppose that a system component (e.g. actuator or sensor) failure occurs in the system or that the operating conditions change. The state-space model of (1), (2) can no longer model the dymanics of the system, which is now described by

$$\begin{aligned} \dot{x}(t) &= A_f\, x(t) + B_f\, u(t) && (41) \\ y(t) &= C_f\, x(t) && (42) \end{aligned}$$

where the state-space matrices of the impaired system are of the same dimensions with the matrices of the nominal state-space model. Our objective is to design fast a new stabilizing output feedback control law

$$u(t) = K_f\, y(t) = K_f C_f\, x(t) \tag{43}$$

such that the new closed-loop system $A_f + B_f K_f C_f$ can capture as much of the eigenstructure information characterizing the nominal closed-loop system $A + BKC$ as possible. In other words, the new output feedback matrix has to be such that the shape of the response of the impaired system closely approximates the shape of the response of the nominal system. Therefore, in comparison to [7], [9], here we are really interested in the system performance, which is determined by the closed-loop eigenvalues/eigenvectors.

Without loss of generality, we assume that the nominal closed-loop eigenvalues are arranged in decreasing order with respect to their real parts, that is $\mathrm{real}(\lambda_1) \geq \mathrm{real}(\lambda_2) \geq \cdots \geq \mathrm{real}(\lambda_n)$. As discussed in the previous section, with output feedback we can only choose q closed-loop eigenvalues and partially assign the same number of closed-loop eigenvectors. Therefore, in order to maintain the performance of the nominal closed-loop system, we should determine the new control law (43) such that the set of the impaired closed-loop eigenvalues includes the q most dominant eigenvalues of the nominal closed-loop system, $\{\lambda_i, i = 1,..,q\}$. On the other hand, the eigenvectors of the impaired system that correspond to the above identical eigenvalues have to be as close to the corresponding eigenvalues of the nominal system, $\{v_i, i = 1,..,q\}$ as possible. Therefore, if we denote by $\{(\lambda_i^f, v_i^f), i = 1,..,n\}$ the closed-loop eigenvalues/eigenvectors for the impaired system, the above objectives are translated into

$$\lambda_i^f = \lambda(A_f + B_f K_f C_f) = \lambda_i = \lambda(A + BKC), \qquad i = 1,..,q \tag{44}$$

$$\min \left[\sum_{i=1}^{q} \|v_i^f - v_i\|^2 \right] \tag{45}$$

Similar to (13), we consider the state-transformation matrix $T_f = \begin{pmatrix} B_f & S_f \end{pmatrix}$, where S_f is selected such that $\mathrm{rank}(T_f) = n$. In the new state-coordinates, the impaired system is described by the matrices $(\tilde{A}_f, \tilde{B}_f, \tilde{C}_f)$, which are computed similar to (16)-(18). Note again the special structure of the input matrix $\tilde{B}_f$, as given in (17). The desired closed-loop eigenvectors, $\{v_i, i = 1,..,q\}$ together with the actual closed-loop eigenvectors of the impaired system, $\{v_i^f, i = 1,..,q\}$ need also to be transformed to the new state-coordinates. Define

$$\tilde{v}_i = T_f^{-1} v_i \tag{46}$$

$$\tilde{v}_i^f = T_f^{-1} v_i^f \tag{47}$$

as the desired and actual closed-loop eigenvectors for the transformed impaired system respectively. From now on, we continue our discussion considering the impaired system in the new state-coordinates specified above. Therefore, the objective of (45) for the transformed impaired system is given by

$$\min \left[\sum_{i=1}^{q} \|\tilde{v}_i^f - \tilde{v}_i\|^2 \right] \tag{48}$$

As it has already been discussed, all achievable eigenvectors $\tilde{v}_i^f$ that correspond to the closed-loop eigenvalue λ_i^f must lie in the subspace spanned by the columns of $(\lambda_i^f I_n - \tilde{A}_f)^{-1}\tilde{B}_f$. Define

$$\tilde{\Pi}_i = (\lambda_i^f I_n - \tilde{A}_f)^{-1}\tilde{B}_f \tag{49}$$

All achievable closed-loop eigenvectors of the impaired system that correspond to the eigenvalue λ_i^f should be of the form

$$\tilde{v}_i^f = \tilde{\Pi}_i \mu_i \tag{50}$$

where μ_i is an $(r \times 1)$ vector. Note that μ_i is a real vector if λ_i^f is a real eigenvalue or a complex vector if λ_i^f is a complex eigenvalue. In view of (50), the objective of (48) is rewritten as

$$\min \left[\sum_{i=1}^{q} ||\tilde{\Pi}_i \mu_i - \tilde{v}_i||^2\right] \tag{51}$$

and the minimizing quantity is defined as

$$J_1^{'} = \mathrm{Tr}\left[\sum_{i=1}^{q} (\tilde{\Pi}_i \mu_i - \tilde{v}_i)^H (\tilde{\Pi}_i \mu_i - \tilde{v}_i)\right] \tag{52}$$

where v^H denotes the complex conjugate transpose of a vector v. Each pair of closed-loop eigenvalues/eigenvectors should satisfy (27), which can equivalently be written as

$$\begin{aligned} (\tilde{A}_f^1 + K_f\tilde{C}_f - \lambda_i^f I_{r,n})\, \tilde{v}_i^f &= 0 \\ \Longrightarrow \quad (\tilde{A}_f^1 + K_f\tilde{C}_f - \lambda_i^f I_{r,n})\, \tilde{\Pi}_i \mu_i &= 0 \end{aligned} \tag{53}$$

where $\tilde{A}_f^1$ contains the first r rows of $\tilde{A}_f$ and

$$I_{r,n} = \begin{pmatrix} I_r & O_{r,n-r} \end{pmatrix} \tag{54}$$

We see that the vectors $\{\mu_i, i = 1, .., q\}$ that minimize (52) also need to satisfy the eigenstructure condition of (53). Therefore, we need to include this condition for the q eigenvectors of interest in the minimizing quantity, which becomes

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

$$J_1 = \mathrm{Tr}\left\{\sum_{i=1}^{q} (\tilde{\Pi}_i\mu_i - \tilde{v}_i)^H \, (\tilde{\Pi}_i\mu_i - \tilde{v}_i) + \sum_{i=1}^{q} M_i \, \left[(\tilde{A}_f^1 + K_f\tilde{C}_f - \lambda_i^f I_{r,n}) \, \tilde{\Pi}_i\mu_i\right]\right\} \tag{55}$$

where $\{M_i, i = 1,..,q\}$ are $(1 \times r)$ Lagrange multiplier vectors, which are real if they correspond to a real eigenvalue or complex if they correspond to a complex eigenvalue. So far, we have concentrated on the q closed-loop eigenvalues that we wish to preserve with the procedure outlined above. Although we have no control upon the remaining $(n-q)$ eigenvalues of the closed-loop system, we need to ascertain that they remain stable. Therefore, the output feedback gain needs to be such that the closed-loop system $\tilde{A}_f + \tilde{B}_f K_f \tilde{C}_f$ is stable. In other words, it suffices to satisfy the Lyapunov equation

$$(\hat{A}_f)^T P + P \, \hat{A}_f + Q = 0 \tag{56}$$

where

$$\hat{A}_f = \tilde{A}_f + \tilde{B}_f K_f \tilde{C}_f \tag{57}$$

As discussed in [7], [9], we also need to safeguard against possible uncertainties in the state-space matrices of the impaired system. It has been shown that this can be done by including the term $\mathrm{Tr}(P^2)$ in the minimizing quantity. Therefore, the overall minimizing quantity is finally given by

$$\begin{aligned} J \;=\; \mathrm{Tr}\ &\left\{\sum_{i=1}^{q}(\tilde{\Pi}_i\mu_i - \tilde{v}_i)^H \, (\tilde{\Pi}_i\mu_i - \tilde{v}_i) + L_1 \, \left[\hat{A}_f^T P + P\hat{A}_f + Q\right]\right. \\ &\left. + \sum_{i=1}^{q} M_i \, \left[(\tilde{A}_f^1 + K_f\tilde{C}_f - \lambda_i^f I_{r,n}) \, \tilde{\Pi}_i\mu_i\right] + P^2\right\} \end{aligned} \tag{58}$$

where $L_1 \in \Re^{n\times n}$ is another Lagrange multiplier matrix. To summarize the approach outlined above, we should state that with the minimization of the quantity in (58) above we seek an output feedback matrix K_f such that

- The q most dominant eigenvalues of the nominal closed-loop system belong to the set of the eigenvalues of the impaired closed-loop system $A_f + B_f K_f C_f$.

- The eigenvectors of the impaired system that correspond to the above set of closed-loop eigenvalues are as close to the corresponding eigenvalues of the nominal system as possible.

- The remaining $(n-q)$ closed-loop eigenvalues are stable.

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

- Possible uncertainties in the state-space matrices of the impaired system are taken care of by maximizing the stability margin allowed to the closed-loop system.

3.2 Algorithmic approach

Without loss of generality, we assume that the set of desired eigenvalues, that is the set of q most dominant eigenvalues of the nominal closed-loop system consists of a complex conjugate pair, that is $\lambda_1^f = (\lambda_2^f)^\star \in \mathcal{C}$, and $(q-2)$ real eigenvalues, that is $\{\lambda_i^f \in \Re, i = 3, .., q\}$. Then, $\tilde{v}_1^f = (\tilde{v}_2^f)^\star$. The generalization to the case of more complex conjugate pairs of eigenvalues is straightforward.

We need to compute the partial derivatives of the minimizing quantity of (58) with respect to all the matrix parameters entailed. These parameters are the Lagrange multiplier vectors $\{M_i, i = 1, .., q\}$, the Lagrange multiplier matrix L_1, the positive definite matrix P, the output feedback matrix K_f, and the vectors $\{\mu_i, i = 1, .., q\}$ that specify the closed-loop eigenvectors. Using the properties of [3] we have

$$\frac{\partial J}{\partial M_i} = \Delta_{M_i} = \left[(\tilde{A}_f^1 + K_f \tilde{C}_f - \lambda_i^f I_{r,n})\, \tilde{\Pi}_i\, \mu_i\right]^T, \qquad i = 1, .., q \tag{59}$$

$$\frac{\partial J}{\partial L_1} = \Delta_{L_1} = \hat{A}_f^T P + P \hat{A}_f + Q \tag{60}$$

$$\frac{\partial J}{\partial P} = \Delta_P = \hat{A}_f L_1^T + L_1^T \hat{A}_f^T + 2P \tag{61}$$

$$\frac{\partial J}{\partial K_f} = \Delta_{K_f} = \tilde{B}_f^T P L_1 \tilde{C}_f^T + \tilde{B}_f^T P L_1^T \tilde{C}_f^T + \sum_{i=1}^{q} M_i^T \mu_i^T \tilde{\Pi}_i^T \tilde{C}_f^T \tag{62}$$

$$\frac{\partial J}{\partial \mu_1} = \Delta_{\mu_1} = 2\, \tilde{\Pi}_2^H\, \tilde{\Pi}_2\, \mu_2 - 2\, \tilde{\Pi}_2^H\, \tilde{v}_2 \;+\; \tilde{\Pi}_1^T\, (\tilde{A}_f^1 + K_f \tilde{C}_f - \lambda_1^f I_{r,n})^T\, M_1^T \tag{63}$$

$$= \left(\frac{\partial J}{\partial \mu_2}\right)^\star \tag{64}$$

$$\frac{\partial J}{\partial \mu_i} = \Delta_{\mu_i} = 2\, \tilde{\Pi}_i^T\, \tilde{\Pi}_i\, \mu_i - 2\, \tilde{\Pi}_i^T\, \tilde{v}_i \;+\; \tilde{\Pi}_i^T\, (\tilde{A}_f^1 + K_f C_f - \lambda_i^f I_{r,n})^T\, M_i^T, \qquad i = 3, .., q \tag{65}$$

The derivation of (63) and the equivalence of (64) are shown in Remark 3.1. To minimize (58) we use a version of the BFGS optimization method of conjugate directions. Note that there are significant changes compared to similar algorithms used in [6], [7], [8], [9], [10]. This is due to the structure of the present problem, since now we update the vectors $\{\mu_i, i = 1, .., q\}$ instead of the output feedback matrix. On the other hand, the existence of complex eigenvalues/eigenvectors imposes certain modifications to the algorithmic scheme. The proposed algorithm is presented next.

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

Initialization Step Let $\epsilon > 0$ be the termination scalar. Choose an initial set of vectors $\{\mu_i, i = 1, .., q\}$ such that the output feedback matrix K_f^1 that results from the q vector equations of (53) stabilizes $(\tilde{A}_f, \tilde{B}_f, \tilde{C}_f)$, that is $\hat{A}_f$ stable. Let

$$\Theta_f^1 = \begin{pmatrix} \mu_1^1 & \mu_3^1 & \cdots & \mu_q^1 \end{pmatrix} \tag{66}$$

be a $[r \times (q-1)]$ matrix consisting of the above vectors. Note that $\mu_2^1 = (\mu_1^1)^\star$ is not included in this matrix. Define the $(2r \times 1)$ vector

$$\bar{\mu}_1^1 = \left(\left[\mu_1^1(1,1)\right]^R \ \left[\mu_1^1(1,1)\right]^I \ \left[\mu_1^1(2,1)\right]^R \ \left[\mu_1^1(2,1)\right]^I \ \cdots \ \left[\mu_1^1(r,1)\right]^R \ \left[\mu_1^1(r,1)\right]^I \right)^T \tag{67}$$

and form the $(rq \times 1)$ vector

$$\psi_1 = \chi_1 = \begin{pmatrix} \bar{\mu}_1^1 \\ \mu_3^1 \\ \vdots \\ \mu_q^1 \end{pmatrix} \tag{68}$$

consisting of the elements of the vectors chosen above. Also, choose an initial $[(rq) \times (rq)]$ symmetric positive definite matrix D_1, let $k_c = j = 1$, and go to the *Main Step*.

Main Step

S1. Substitute the vector columns of Θ_f^j in the gradients of (59)-(62), set them to zero, that is $\{\Delta_{M_i} = 0, i = 1, .., q\}$, $\Delta_{L_1} = 0$, $\Delta_P = 0$, $\Delta_{K_f^j} = 0$, and solve for K_f^j, P, L_1, $\{M_i, i = 1, .., q\}$ respectively, in that specific order.

S2. Substitute these matrix parameters in (63), (65) and compute

$$\Delta_{\mu_1}^j = \begin{pmatrix} \delta_{\mu_1}^j(1) & \delta_{\mu_1}^j(2) & \cdots & \delta_{\mu_1}^j(r) \end{pmatrix}^T \tag{69}$$

and $\{\Delta_{\mu_i}^j, i = 3, .., q\}$.

S3. Form the $(2r \times 1)$ vector

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

$$\bar{\Delta}^j_{\mu_1} = \left(\left[\delta^j_{\mu_1}(1)\right]^R \ \left[\delta^j_{\mu_1}(1)\right]^I \ \left[\delta^j_{\mu_1}(2)\right]^R \ \left[\delta^j_{\mu_2}(2)\right]^I \ \cdots \ \left[\delta^j_{\mu_1}(r)\right]^R \ \left[\delta^j_{\mu_1}(r)\right]^I \right)^T \tag{70}$$

and define

$$\nabla J(\psi_j) = \begin{pmatrix} \bar{\Delta}^j_{\mu_1} \\ \Delta^j_{\mu_3} \\ \vdots \\ \Delta^j_{\mu_q} \end{pmatrix} \tag{71}$$

If $\|\nabla J(\psi_j)\| < \epsilon$, *STOP*. The output feedback matrix that minimizes (58) is K^j_f. Otherwise, go to M4.

S4. If $j > 1$, update the positive definite matrix D_j as follows:

$$D_j = D_{j-1} + \frac{p_{j-1}p^T_{j-1}}{p^T_{j-1}q_{j-1}}\,[1 + \frac{q^T_{j-1}D_{j-1}q_{j-1}}{p^T_{j-1}q_{j-1}}] - \frac{[D_{j-1}q_{j-1}p^T_{j-1} + p_{j-1}q^T_{j-1}D_{j-1}]}{p^T_{j-1}q_{j-1}} \tag{72}$$

where

$$\begin{aligned} p_{j-1} &= \lambda_{j-1}d_{j-1} = \psi_j - \psi_{j-1} && (73) \\ q_{j-1} &= \nabla J(\psi_j) - \nabla J(\psi_{j-1}) && (74) \end{aligned}$$

S5. Define

$$d_j = -D_j \nabla J(\psi_j) \tag{75}$$

and let λ_j be an optimal solution to the problem of minimizing $J(\psi_j + \lambda d_j)$ subject to $\lambda \geq 0$. Let

$$\psi_{j+1} = \psi_j + \lambda_j d_j = \begin{pmatrix} \bar{\mu}^{j+1}_1 \\ \mu^{j+1}_3 \\ \vdots \\ \mu^{j+1}_q \end{pmatrix} \tag{76}$$

where $\bar{\mu}^{j+1}_1$ is an $(2r \times 1)$ vector and $\{\mu^{j+1}_i, i = 3, .., q\}$ are $(r \times 1)$ vectors. Defining

$$\mu_1^{j+1} = \begin{pmatrix} \mu_1^{j+1}(1,1) \\ \mu_1^{j+1}(2,1) \\ \vdots \\ \mu_1^{j+1}(r,1) \end{pmatrix} = \begin{pmatrix} \bar{\mu}_1^{j+1}(1,1) + j\bar{\mu}_1^{j+1}(2,1) \\ \bar{\mu}_1^{j+1}(3,1) + j\bar{\mu}_1^{j+1}(4,1) \\ \vdots \\ \bar{\mu}_1^{j+1}(2r-1,1) + j\bar{\mu}_1^{j+1}(2r,1) \end{pmatrix} \tag{77}$$

we obtain

$$\Theta_f^{j+1} = \begin{pmatrix} \mu_1^{j+1} & \mu_3^{j+1} & \cdots & \mu_q^{j+1} \end{pmatrix} \tag{78}$$

S6. If $j < (rq)$, replace j by $j+1$, and repeat the *Main Step.*

Otherwise, if $j = (rq)$, then let $\psi_1 = \chi_{k_c+1} = \psi_{(rq)+1}$, replace k_c by (k_c+1), let $j = 1$, and repeat the *Main Step.* □□

Several issues need to be discussed here.

Remark 3.1 The third term of (63), that is $\left\{\tilde{\Pi}_1^T \left(\tilde{A}_f^1 + K_f\tilde{C}_f - \lambda_1^f I_{r,n}\right) M_1^T\right\}$ can readily be derived from the term $\left\{M_1 \left[\left(\tilde{A}_f^1 + K_f\tilde{C}_f - \lambda_1^f I_{r,n}\right) \tilde{\Pi}_i\mu_i\right]\right\}$ of the minimizing quantity (58). Therefore, we only show how the remaining terms $\{2\,\tilde{\Pi}_2^H\,\tilde{\Pi}_2\,\mu_2 - 2\,\tilde{\Pi}_2^H\,\tilde{v}_2\}$ of (63) are computed. It is apparent that it suffices to consider the quantity

$$\begin{aligned} J_\mu^{12} &= \mathrm{Tr}\left[\sum_{i=1}^{2}(\tilde{\Pi}_i\mu_i - \tilde{v}_i)^H\,(\tilde{\Pi}_i\mu_i - \tilde{v}_i)\right] \\ &= \mathrm{Tr}\left[(\tilde{v}_1)^H\tilde{v}_1 + \mu_1^H\tilde{\Pi}_1^H\tilde{\Pi}_1\mu_1 - (\tilde{v}_1)^H\tilde{\Pi}_1\mu_1 - \mu_1^H\tilde{\Pi}_1^H\tilde{v}_1\right. \\ &\quad \left. + (\tilde{v}_2)^H\tilde{v}_2 + \mu_2^H\tilde{\Pi}_2^H\tilde{\Pi}_2\mu_2 - (\tilde{v}_2)^H\tilde{\Pi}_2\mu_2 - \mu_2^H\tilde{\Pi}_2^H\tilde{v}_2\right] \\ &= \mathrm{Tr}\left[(\tilde{v}_1)^H\tilde{v}_1 + \mu_2^T\tilde{\Pi}_1^H\tilde{\Pi}_1\mu_1 - (\tilde{v}_1)^H\tilde{\Pi}_1\mu_1 - \mu_2^T\tilde{\Pi}_1^H\tilde{v}_1\right. \\ &\quad \left. + (\tilde{v}_2)^H\tilde{v}_2 + \mu_1^T\tilde{\Pi}_2^H\tilde{\Pi}_2\mu_2 - (\tilde{v}_2)^H\tilde{\Pi}_2\mu_2 - \mu_1^T\tilde{\Pi}_2^H\tilde{v}_2\right] \end{aligned} \tag{79}$$

where the following equalities have been used

$$\mu_1^H = (\mu_1^\star)^T = \mu_2^T \tag{80}$$
$$\mu_2^H = (\mu_2^\star)^T = \mu_1^T \tag{81}$$

We compute

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

$$\begin{aligned}\frac{\partial\, J_\mu^{12}}{\partial \mu_1} &= \tilde{\Pi}_1^T \tilde{\Pi}_1^\star \mu_2 - \tilde{\Pi}_1^T \tilde{v}_1^\star + \tilde{\Pi}_2^H \tilde{\Pi}_2 \mu_2 - \tilde{\Pi}_2^H \tilde{v}_2 \\ &= 2\, \tilde{\Pi}_2^H \tilde{\Pi}_2 \mu_2 - 2\, \tilde{\Pi}_2^H \tilde{v}_2 \end{aligned} \tag{82}$$

since

$$\tilde{\Pi}_1^T \tilde{\Pi}_1^\star = (\tilde{\Pi}_2^\star)^T \tilde{\Pi}_2 = \tilde{\Pi}_2^H \tilde{\Pi}_2 \tag{83}$$

$$\tilde{\Pi}_1^T \tilde{v}_1^\star = (\tilde{\Pi}_2^\star)^T \tilde{v}_2 = \tilde{\Pi}_2^H \tilde{v}_2 \tag{84}$$

In view of (82) and the discussion above, (63) follows easily. Similar to (82), we have

$$\begin{aligned}\frac{\partial\, J_\mu^{12}}{\partial \mu_2} &= 2\, \tilde{\Pi}_1^H \tilde{\Pi}_1 \mu_1 - 2\, \tilde{\Pi}_1^H \tilde{v}_1 \\ &= \left(2\, \tilde{\Pi}_2^H \tilde{\Pi}_2 \mu_2 - 2\, \tilde{\Pi}_2^H \tilde{v}_2\right)^\star = \left(\frac{\partial\, J_\mu^{12}}{\partial \mu_1}\right)^\star \end{aligned} \tag{85}$$

Therefore, the equivalence of (64) is proven. □

Remark 3.2 The intermediate steps of *(S1)* need to be further exploited. First, the vector equations $\{\Delta_{M_i} = 0, i = 1,..,q\}$ of (59) are written in the compact form of (36), from which we compute the output feedback matrix K_f^j similar to (40). Then, $\Delta_{L_1} = 0$, $\Delta_P = 0$ are solved for P, $L_1 = L_1^T$ respectively. Note that the final matrix equation $\Delta_{K_f^j} = 0$ of (62) can be rewritten as

$$\begin{aligned} -2\, \tilde{B}_f^T P L_1 \tilde{C}_f^T &= \begin{pmatrix} M_1^T & (M_1^T)^\star & M_3^T & \cdots & M_q^T \end{pmatrix} \begin{pmatrix} \mu_1^T \tilde{\Pi}_1^T \\ (\mu_1^T \tilde{\Pi}_1^T)^\star \\ \mu_3^T \tilde{\Pi}_3^T \\ \vdots \\ \mu_q^T \tilde{\Pi}_q^T \end{pmatrix} \tilde{C}_f^T \\ &= \begin{pmatrix} (M_1^T)^R & (M_1^T)^I & M_3^T & \cdots & M_q^T \end{pmatrix} \begin{pmatrix} 2\, (\mu_1^T \tilde{\Pi}_1^T)^R \\ -2\, (\mu_1^T \tilde{\Pi}_1^T)^I \\ \mu_3^T \tilde{\Pi}_3^T \\ \vdots \\ \mu_q^T \tilde{\Pi}_q^T \end{pmatrix} \tilde{C}_f^T \\ &= \hat{M}\, \hat{\Pi} \end{aligned} \tag{86}$$

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

where obviously

$$\hat{M} \;=\; \left((M_1^T)^R \;\; (M_1^T)^I \;\; M_3^T \;\; \cdots \;\; M_q^T \right) \tag{87}$$

$$\hat{\Pi} \;=\; \begin{pmatrix} 2\,(\mu_1^T \tilde{\Pi}_1^T)^R \\ -2\,(\mu_1^T \tilde{\Pi}_1^T)^I \\ \mu_3^T \tilde{\Pi}_3^T \\ \vdots \\ \mu_q^T \tilde{\Pi}_q^T \end{pmatrix} \tilde{C}_f^T \tag{88}$$

are $(r \times q)$ and $(q \times q)$ real matrices respecticely. The above equation is solved for $\hat{M}$ to yield

$$\hat{M} = -2\; \tilde{B}_f^T P L_1 \tilde{C}_f^T \;\hat{\Pi}^{-1} \tag{89}$$

Remark 3.3 The line search in *(S5)* is restricted to λ's such that the resulting output feedback matrix from the q vector equations of (53) makes $\hat{A}_f$ stable.

Remark 3.4 We know that, in general, the desired eigenvectors $\{\tilde{v}_i, i = 1, .., q\}$ will not belong to the corresponding subspaces spanned by the columns of $\{\tilde{\Pi}_i, i = 1, .., q\}$. Furthermore, some of these desired eigenvectors may be such that we can not find any eigenvectors $\tilde{\Pi}_i \mu_i$ very close to them. In cases like that, we should use a more practical criterion compared to the stopping criterion of *(S3)*. Specifically, we need to monitor J and terminate the algorithm once we do not observe any significant changes in the value of J.

Remark 3.5 As it has already been discussed, the output feedback matrix does not change under state-transformation. Therefore, the optimal K_f that will be derived by the above algorithmic approach is the optimal gain for the impaired system in the original state-coordinates as well. However, the optimal vectors $\{\tilde{v}_i^f, i = 1, .., q\}$ determined by the algorithm need to be transformed back to the original state-coordinates using (47).

3.3 Interesting cases

The case $q < r$ In the analysis above, it has been assumed that the number of outputs q is greater than the number of inputs r. When $q < r$, we need to consider left eigenvectors instead of the right eigenvectors of (8). Then, all the results presented above for the $r < q$ case still hold but now for the dual of the system (A, B, C), that is for the system $(A_D = A^T, B_D = C^T, C_D = B^T)$. In that respect, similar to (11), it can been shown, [16], that all achievable left eigenvectors t_i that correspond to the desired closed-loop eigenvalue λ_i must lie to the subspace spanned by the columns of $(\lambda_i I_n - A^T)^{-1} C^T$. The controller that assigns the r closed-loop eigenvalues $\{\lambda_i, i = 1, .., r\}$ and their associated left eigenvectors $\{t_i, i = 1, .., r\}$ can be found from the r vector equations

$$(A^T + C^T K^T B^T)\, t_i = \lambda_i\, t_i \tag{90}$$

The duality mentioned above should now be straightforward.

<u>**The state-feedback case**</u> The procedure outlined above for the output feedback case can easily be extended to the case of state-feedback. In that case, we consider $C = I_n$. The number of outputs is identical to the number of system states $q = n$, which implies that all the eigenvalues of the nominal closed-loop system are preserved and all the nominal closed-loop eigenvectors can be approximated by the proposed optimization scheme. Therefore, the minimizing quantity of (58) for the control reconfiguration problem now can be written as

$$\begin{aligned} J \;=\; \mathrm{Tr} \Bigg\{ & \sum_{i=1}^{n} (\tilde{\Pi}_i \mu_i - \tilde{v}_i)^H\, (\tilde{\Pi}_i \mu_i - \tilde{v}_i) + L_1\, \left[\hat{A}_f^T P + P \hat{A}_f + Q\right] \\ & + \sum_{i=1}^{n} M_i\, \left[(\tilde{A}_f^1 + K_f \tilde{C}_f - \lambda_i^f I_{r,n})\, \tilde{\Pi}_i \mu_i\right] + P^2 \Bigg\} \end{aligned} \tag{91}$$

and the algorithmic approach of the previous subsection readily applies here as well. It is apparent that under state feedback the stability of the closed-loop system is guaranteed. However, the inclusion of the Lyapunov equation in (91) above is essential, since it is needed for the minimization of the robustness term $\mathrm{Tr}(P^2)$.

<u>**The dynamic compensator case**</u> The static output feedback case studied above can be extended to the dynamic output feedback case as well. Consider the dynamic output compensator described by

$$\dot{x}_c(t) = A_c\, x_c(t) + B_c\, y(t) \tag{92}$$
$$u(t) = C_c\, x_c(t) + D_c\, y(t) \tag{93}$$

where the compensator state vector $x_c(t)$ is of dimension l, with $0 \le l \le (n-q)$, and $A_c \in \Re^{l \times l}$, $B_c \in \Re^{l \times q}$, $C_c \in \Re^{r \times l}$, $D_c \in \Re^{r \times q}$. The closed-loop system is described by

$$\dot{x}_{cl}(t) = A_{cl}\, x_{cl}(t) + B_{cl}\, u_{cl}(t) \tag{94}$$
$$y_{cl}(t) = C_{cl}\, x_{cl}(t) \tag{95}$$
$$u_{cl}(t) = K_{cl}\, y_{cl}(t) \tag{96}$$

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

where

$$x_{cl}(t) = \begin{pmatrix} x(t) \\ x_c(t) \end{pmatrix}, \quad A_{cl} = \begin{pmatrix} A & O_{n,l} \\ O_{l,n} & O_{l,l} \end{pmatrix}, \quad B_{cl} = \begin{pmatrix} B & O_{n,l} \\ O_{l,r} & I_l \end{pmatrix} \tag{97}$$

$$C_{cl} = \begin{pmatrix} C & O_{q,l} \\ O_{l,n} & I_l \end{pmatrix}, \quad K_{cl} = \begin{pmatrix} D_c & C_c \\ B_c & A_c \end{pmatrix} \tag{98}$$

The closed-loop system has $(n+l)$ states, $(r+l)$ inputs and $(q+l)$ outputs, and closed-loop eigenvectors partitioned as

$$v_i = \begin{pmatrix} v_i(x) \\ v_i(x_c) \end{pmatrix} \tag{99}$$

where $v_i(x)$ is the ith subeigenvector that corresponds to the plant and $v_i(x_c)$ the ith subeigenvector that corresponds to the compensator. Therefore, we now can determine $(q+l)$ closed-loop eigenvalues and partially assign the same number of closed-loop eigenvectors. In other words, given a set of desired closed-loop eigenvalues $\{\lambda_i, i = 1, .., q+l\}$ and a set of desired closed-loop eigenvectors $\{v_i, i = 1, .., q+l\}$ such that they belong to the eigenspaces defined by $(\lambda_i I_{n+l} - A_{cl})^{-1} B_{cl}$, we can find a matrix K_{cl} that achieves the above given eigenstructure assignment. It should be mentioned that a separation property with regard to the eigenvalues of the plant and the eigenvalues of the dynamic compensator does not apply here.

In view of the definitions (97)-(98) for the closed-loop system, we see that the optimization procedure outlined above for the control reconfiguration problem applies to the dynamic output feedback case as well. Therefore, similar to (1), (2), (5), the nominal system is now (A_{cl}, B_{cl}, C_{cl}) and the output feedback matrix K_{cl}. For the impaired system of (41)-(43), similar to (92)-(93), we consider the dynamic output compensator

$$\dot{x}_c(t) = A_c^f\, x_c(t) + B_c^f\, y(t) \tag{100}$$

$$u(t) = C_c^f\, x_c(t) + D_c^f\, y(t) \tag{101}$$

such that the impaired closed-loop system is given by

$$\dot{x}_{cl}(t) = A_{cl}^f\, x_{cl}(t) + B_{cl}^f\, u_{cl}(t) \tag{102}$$

$$y_{cl}(t) = C_{cl}^f\, x_{cl}(t) \tag{103}$$

$$u_{cl}(t) = K_{cl}^f\, y_{cl}(t) \tag{104}$$

where

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

$$A^f_{cl} = \begin{pmatrix} A_f & O_{n,l} \\ O_{l,n} & O_{l,l} \end{pmatrix}, \quad B^f_{cl} = \begin{pmatrix} B_f & O_{n,l} \\ O_{l,r} & I_l \end{pmatrix} \tag{105}$$

$$C^f_{cl} = \begin{pmatrix} C_f & O_{q,l} \\ O_{l,n} & I_l \end{pmatrix}, \quad K^f_{cl} = \begin{pmatrix} D^f_c & C^f_c \\ B^f_c & A^f_c \end{pmatrix} \tag{106}$$

Hence, the impaired closed-loop system is $(A^f_{cl}, B^f_{cl}, C^f_{cl})$ with output feedback matrix K^f_{cl}. Therefore, the control reconfiguration problem for the dynamic output feedback case is defined as the problem of determining the matrix K^f_{cl} of the feedback law (104) such that

- The $(q+l)$ most dominant eigenvalues of the nominal closed-loop system $A_{cl}+B_{cl}K_{cl}C_{cl}$ belong to the set of the eigenvalues of the impaired closed-loop system $A^f_{cl} + B^f_{cl}K^f_{cl}C^f_{cl}$.

- The eigenvectors of the impaired closed-loop system $A^f_{cl} + B^f_{cl}K^f_{cl}C^f_{cl}$ that correspond to the above set of closed-loop eigenvalues are as close to the corresponding eigenvalues of the nominal closed-loop system as possible.

- The remaining $(n-q)$ closed-loop eigenvalues are stable.

- Possible uncertainties in the state-space matrices (A_f, B_f, C_f) of the impaired system are taken care of by maximizing the stability margin allowed to the impaired closed-loop system $A^f_{cl} + B^f_{cl}K^f_{cl}C^f_{cl}$.

Therefore, the algorithmic approach of the previous section readily applies here as well.

4 Illustrative examples

Example 4.1 Consider the aircraft longitudinal control system of [5], whose linearized dynamic model is given by

$$\begin{pmatrix} \dot{\alpha}(t) \\ \dot{\beta}(t) \\ \dot{\psi}(t) \\ \dot{\theta}(t) \end{pmatrix} = \begin{pmatrix} -0.0582 & 0.0651 & 0 & -0.171 \\ -0.303 & -0.685 & 1.109 & 0 \\ -0.0715 & -0.658 & -0.947 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha(t) \\ \beta(t) \\ \psi(t) \\ \theta(t) \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -0.0541 & 0 \\ -1.11 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \eta(t) \\ \tau(t) \end{pmatrix} \tag{107}$$

$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha(t) \\ \beta(t) \\ \psi(t) \\ \theta(t) \end{pmatrix} \tag{108}$$

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

where $\alpha(t)$ and $\beta(t)$ are the forward and vertical speeds, $\psi(t)$ is the pitch rate and $\theta(t)$ is the pitch angle. The control inputs $\eta(t)$ and $\tau(t)$ are the elevator angle and throttle position respectively. When we consider the static output feedback law of (5), the controller that assigns the closed-loop eigenvalues at $\{-0.5973, -1.5 \pm j2, -2\}$ and their corresponding eigenvectors at

$$\begin{aligned} V &= (v_1 \;\; v_2 \;\; v_3 \;\; v_4) \\ &= \begin{pmatrix} -0.1887 & 0.1465 + j0.0958 & 0.1465 - j0.0958 & 0.9680 \\ -0.9634 & 0.2257 - j0.2492 & 0.2257 + j0.2492 & 0.1441 \\ -0.0977 & 0.3790 + j0.6047 & 0.3790 - j0.6047 & 0.0905 \\ 0.1636 & 0.1025 - j0.2664 & 0.1025 + j0.2664 & -0.0453 \end{pmatrix} \end{aligned} \tag{109}$$

is given in [5] by

$$K = \begin{pmatrix} -0.00031 & 4.77004 & 1.70457 \\ -2.01505 & -1.13002 & 0.02904 \end{pmatrix} \tag{110}$$

Next, we suppose that the system dynamics change due to operating condition variations. The state-space matrices of the impaired model are given below.

$$\begin{aligned} A_f &= \begin{pmatrix} -0.0582 & 0.10 & 0.0 & -0.171 \\ -0.103 & -0.685 & 1.109 & 0 \\ -0.0715 & -0.658 & 1.98 & 0 \\ 0 & 0 & 1.5 & 0 \end{pmatrix}, \quad B_f = \begin{pmatrix} 0 & 0.9 \\ -0.09 & 0.0 \\ -1.11 & 0.0 \\ 0 & 0.0 \end{pmatrix} \\ C_f &= \begin{pmatrix} 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{aligned} \tag{111}$$

The algorithmic approach outlined above is used to find the optimal output feedback matrix K_f, that is the controller gain that minimizes J of (58), for the impaired system. Our objective is to preserve the first 3 most dominant eigenvalues of the nominal closed-loop system, that is $\{-0.5973, -1.5 \pm j2\}$, and achieve closed-loop eigenvectors as close to their corresponding eigenvectors of (109) as possible. First we need to transform the impaired system (A_f, B_f, C_f) to new state-coordinates. Select

$$T = \begin{pmatrix} 0 & 0.9 & 0 & 0 \\ -0.09 & 0 & 1 & 0 \\ -1.11 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{112}$$

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The best results, with regard to closeness of the closed-loop eigenvectors of the impaired system to the desired eigenvectors specified in (109) are obtained when we assign a weight factor of 0.1 to the term $\left\{(\tilde{\Pi}_3\mu_3 - \tilde{v}_3)^T\left(\tilde{\Pi}_3\mu_i - \tilde{v}_3\right)\right\}$ of the minimizing quantity of (58). This is the term that corresponds to the real eigenvalue -0.5973. By assigning this weight, we are able to emphasize the task of achieving optimal eigenvectors for the complex conjugate pair of eigenvalues, $(-1.5 \pm \jmath 2)$. Note that this task is the most difficult to achieve due to the complex nature of the corresponding eigenvectors. The introduction of this weight factor only affects (65), whose first 2 terms $\left\{2\,\tilde{\Pi}_3^T\,\tilde{\Pi}_3\,\mu_i - 2\,\tilde{\Pi}_3^T\,\tilde{v}_3\right\}$ need to be multiplied by this weight factor.

The algorithm gives the following results with regard to the closeness of the derived eigenvectors to the ideal eigenvectors of (109)

$$\begin{aligned} \|\tilde{v}_1^f - \tilde{v}_1\|^2 &= 0.0242 && (113) \\ \|\tilde{v}_2^f - \tilde{v}_2\|^2 &= \|\tilde{v}_3^f - \tilde{v}_3\|^2 = 0.0230 && (114) \end{aligned}$$

whereas the robustness term and the corresponding robustness bound are

$$\begin{aligned} Tr(P^2) &= 0.0788 && (115) \\ \sigma_{max}(\Delta\hat{A}) &= 0.4037 && (116) \end{aligned}$$

and the output feedback gain that achieves these results is

$$K_f = \begin{pmatrix} -4.42776 & 5.95419 & 5.59306 \\ -4.15014 & -0.71481 & 0.49365 \end{pmatrix} \tag{117}$$

With the above controller, the fourth closed-loop eigenvalue is placed at -4.7358. Note that the above results concern the impaired system in the new state-coordinates specified by (112). However, the controller is the same, as discussed before. The obtained eigenvectors transformed back to the original coordinates of the impaired system are given by

$$\begin{aligned} V_f &= \begin{pmatrix} v_1^f & v_2^f & v_3^f \end{pmatrix} \\ &= \begin{pmatrix} -0.0674 & 0.1424 + \jmath 0.0945 & 0.1424 - \jmath 0.0945 \\ -0.9950 & 0.1834 - \jmath 0.1722 & 0.1834 + \jmath 0.1722 \\ -0.0453 & 0.3519 + \jmath 0.5667 & 0.3519 - \jmath 0.5667 \\ 0.1136 & 0.1453 - \jmath 0.3729 & 0.1453 + \jmath 0.3929 \end{pmatrix} && (118) \end{aligned}$$

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where obviously the first column is the eigenvector that corresponds to the real eigenvalue -0.5973, and the last two columns are the eigenvectors that correspond to the complex conjugate pair of eigenvalues $(-1.5 \pm j2)$. As we see, the above eigenvectors are indeed very close to the desired eigenvectors of (109), as suggested by (113)-(114) above. This can also be shown by computing

$$\begin{aligned} \|v_1^f - v_1\|^2 &= 0.0231 && (119) \\ \|v_2^f - v_2\|^2 &= \|v_3^f - v_3\|^2 = 0.0210 && (120) \end{aligned}$$

In Figures 1-2, we compare the state respone of the nominal system (A, B, C) of (107)-(108) with the output feedback matrix K of (110) and the state response of the impaired system of (111) with the output feedback matrix K_f of (117). The initial conditon vector is chosen as

$$V_{in}^1 = (\, 0.75 \quad 0.5 \quad 0.3 \quad 1 \,)^T \tag{121}$$

The same is done in Figures 3-4 for the initial condition vector

$$V_{in}^2 = (\, 0.75 \quad 0.65 \quad -0.5 \quad -0.6 \,)^T \tag{122}$$

As we see, the algorithm is capable of recovering the performance of the nominal system. This should be expected, since the eigenvectors of the impaired closed-loop system are assigned very close to the eigenvectors of the nominal closed-loop system, as shown in (119)-(120) above.

The external input case For the system of (1)-(2) we assume the output feedback law

$$u(t) = K\, y(t) + G\, w(t) = KC\, x(t) + G\, w(t) \tag{123}$$

where $w \in \Re^r$ is the external input vector and G an $\Re^{r \times r}$ matrix. With this feedback law, the closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= (A + BKC)\, x(t) + BG\, w(t) && (124) \\ y(t) &= C\, x(t) && (125) \end{aligned}$$

For the impaired system of (41)-(42), we consider the output feedback law

$$u(t) = K_f\, y(t) + G_f\, w(t) = K_f C_f\, x(t) + G_f\, w(t) \tag{126}$$

so that the closed-loop system becomes

$$\dot{x}(t) = (A_f + B_f K_f C_f)\, x(t) + B_f G_f\, w(t) \tag{127}$$
$$y(t) = C_f\, x(t) \tag{128}$$

The steady-state outputs of the above closed-loop systems in response to a unit step input either in the first or the second external input are given by

$$\begin{aligned} \bar{h}_i(\infty) &= \lim_{s\to 0}\left\{sC(sI_n - A - BKC)^{-1}BGJ_i\frac{1}{s}\right\} \\ &= -C(A - BKC)^{-1}BGJ_i = \Phi\ J_i \end{aligned} \tag{129}$$
$$\begin{aligned} \bar{h}_i^f(\infty) &= \lim_{s\to 0}\left\{sC_f(sI_n - A_f - B_fK_fC_f)^{-1}B_fG_fJ_i\frac{1}{s}\right\} \\ &= -C_f(A_f - B_fK_fC_f)^{-1}B_fG_fJ_i = \Psi\ G_fJ_i \end{aligned} \tag{130}$$

where

$$J_1 = (\,1 \quad 0\,), \qquad\qquad J_2 = (\,0 \quad 1\,) \tag{131}$$

correspond to a unit step input in w_1 and w_2 respectively, and the following definitions have been used

$$\Phi = -C(A + BKC)^{-1}BG \tag{132}$$
$$\Psi = -C_f(A_f + B_fK_fC_f)^{-1}B_f \tag{133}$$

In view of the definitions above, the problem of maintaining the steady-state output performance in response to unit step changes in the external inputs is reduced in [5] to the problem of determining the matrix G_f that minimizes the quantity

$$J_{ss} = \|\Phi - \Psi G_f\|_F \tag{134}$$

The optimal solution, that is the solution that minimizes the Frobenius norm of the above quantity, is given by

$$G_f = \Psi^{\dagger}\Phi = (\Psi^T\Psi)^{-1}\Psi^T\Phi \tag{135}$$

where $A^\dagger$ denotes the pseudo-inverse of a matrix A.

Assume that the output feedback law of (123) is applied to the nominal system of (107)-(108) with $G = I_2$. When we apply the feedback law of (126) to the impaired system of (111), the optimal gain K_f that minimizes the quantity of (58) has already been determined in (117). In view of (135), we find that the optimal external input matrix that maintains the closed-loop steady-state performance is given by

$$G_f = \begin{pmatrix} 1.7784 & 1.9438 \\ 0.3342 & 1.9835 \end{pmatrix} \tag{136}$$

In Figures 5-6, we compare the output responses of the nominal closed-loop system of (124)-(125) and the impaired closed-loop system of (127)-(128), when a unit step input is applied in the external input w_1. The same is done in Figures 7-8 for a unit step input in the external input w_2. As we can see, with the feedback gain K_f of (117) and the external input matrix G_f of (136) the steady-state response of the nominal closed-loop system of (124)-(125) has successfully been recovered.

Finally, the closed-loop systems discussed above are subjected to impulse changes in the external inputs. In Figures 9-10, we have the respones to an impulse change in the external input w_1 and in Figures 11-12, the responses to an impulse change in the external input w_2. As we see, the impaired system of (127)-(128) designed to maintain the steady-state state response of the nominal system (124)-(125), via the selection of G_f, is capable of recovering the impulse response of the nominal system as well.

Example 4.2 Consider the lateral flight control system from [4]

$$\begin{aligned}
\begin{pmatrix} \dot{p}_s(t) \\ \dot{r}_s(t) \\ \dot{\beta}(t) \\ \dot{\phi}(t) \end{pmatrix} &= \begin{pmatrix} -0.746 & 0.387 & -12.9 & 0 \\ 0.024 & -0.174 & 4.31 & 0 \\ 0.006 & -0.999 & 0.0578 & 0.0369 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_s(t) \\ r_s(t) \\ \beta(t) \\ \phi(t) \end{pmatrix} \\
&+ \begin{pmatrix} 0.952 & 6.05 \\ -1.76 & -0.416 \\ 0.0092 & -0.0012 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \delta_r(t) \\ \delta_a(t) \end{pmatrix} \\
y(t) &= \begin{pmatrix} p_s(t) & r_s(t) & \beta(t) & \phi(t) \end{pmatrix}^T
\end{aligned} \tag{137}$$

where $p_s(t)$, $r_s(t)$ are the stability axis roll and yaw rates, $\beta(t)$ the angle of sideslip, and $\phi(t)$ the bank angle. The control inputs $\delta_r(t)$, δ_a are the rudder and aileron deflections respectively. As we see, all states are available for measurement. In [4], it is assumed that the desired closed-loop eigenvalues and their associated closed-loop eigenvectors are as follows

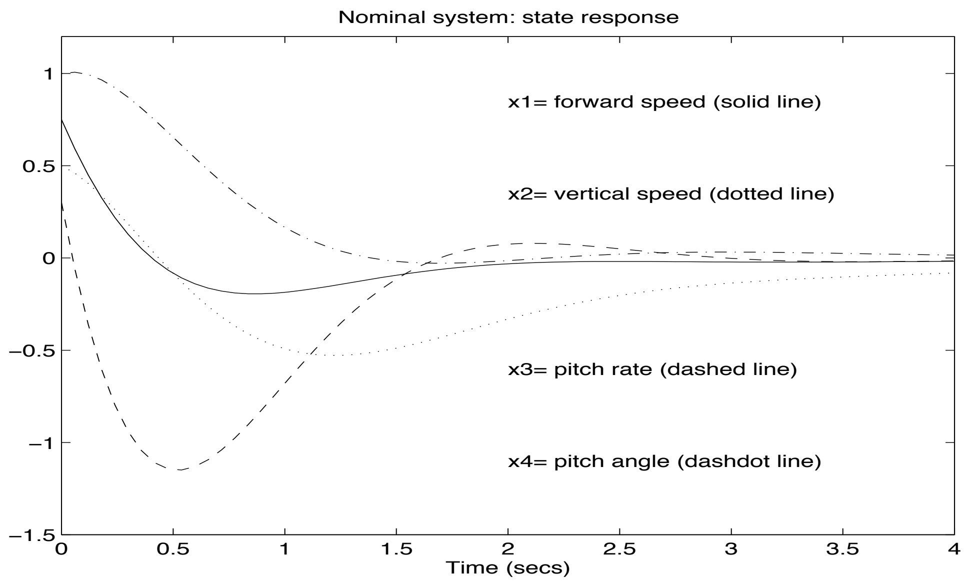

Figure 1: Example 4.1: Nominal system. Closed-loop state response (no external inputs) for the initial condition vector V_{in}^1.

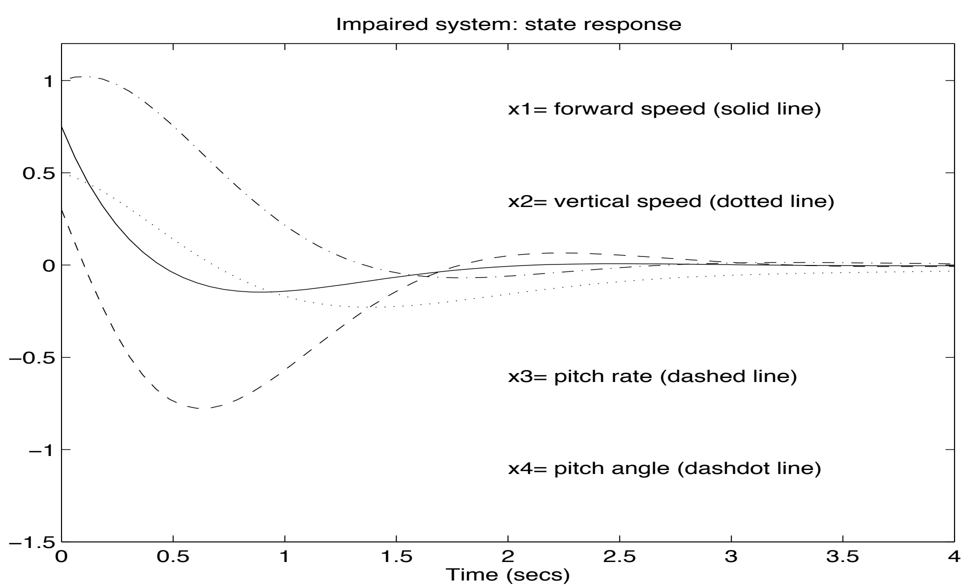

Figure 2: Example 4.1: Impaired system. Closed-loop state response (no external inputs) for the initial condition vector V_{in}^1.

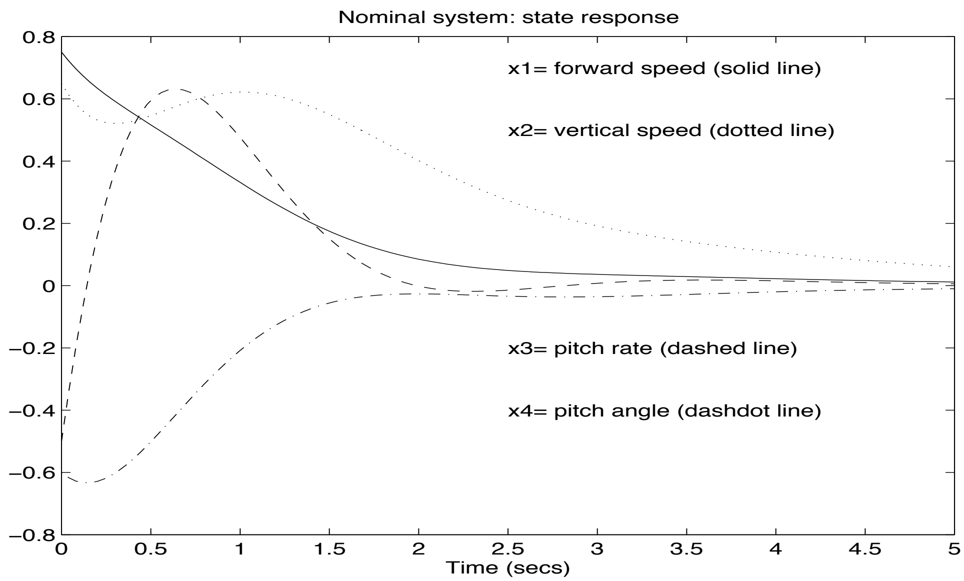

Figure 3: Example 4.1: Nominal system. Closed-loop state response (no external inputs) for the initial condition vector V_{in}^2.

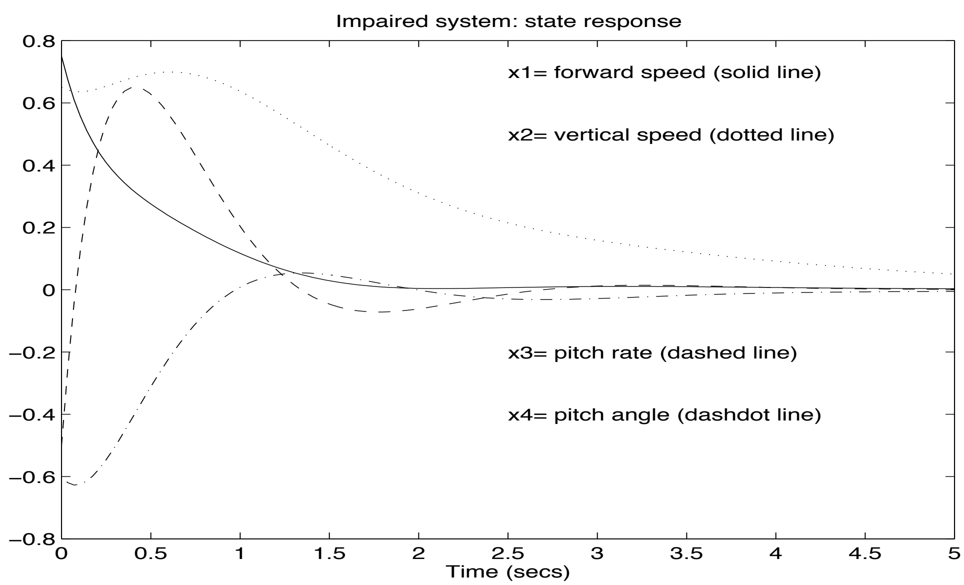

Figure 4: Example 4.1: Impaired system. Closed-loop state response (no external inputs) for the initial condition vector V_{in}^2.

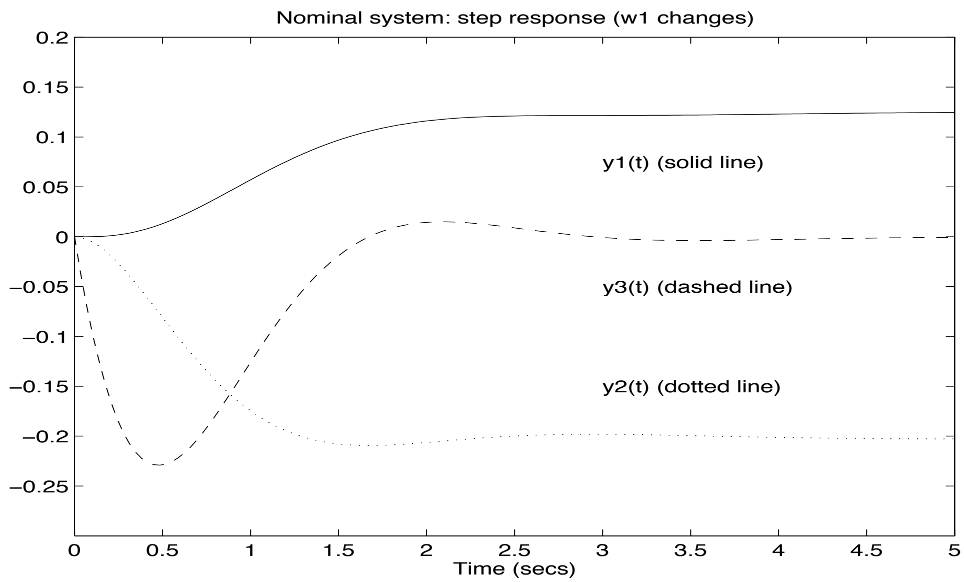

Figure 5: Example 4.1: Nominal system. Output response to a unit step change in the external input w_1.

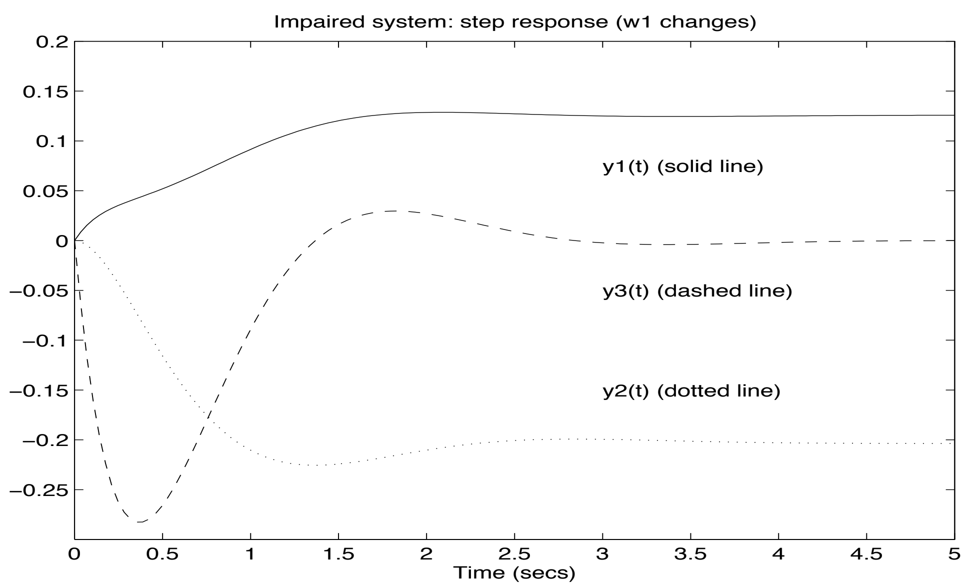

Figure 6: Example 4.1: Impaired system. Output response to a unit step change in the external input w_1.

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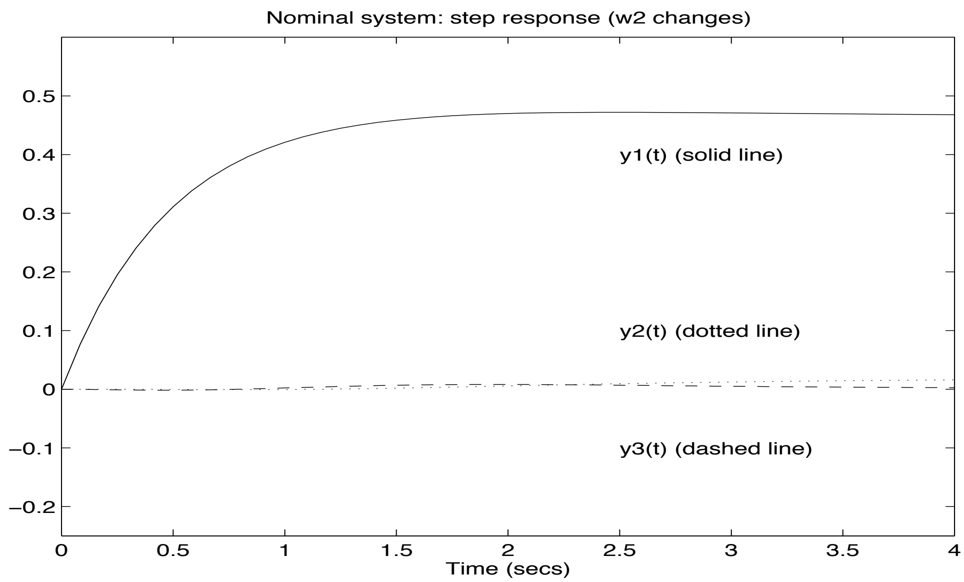

Figure 7: Example 4.1: Nominal system. Output response to a unit step change in the external input w_2.

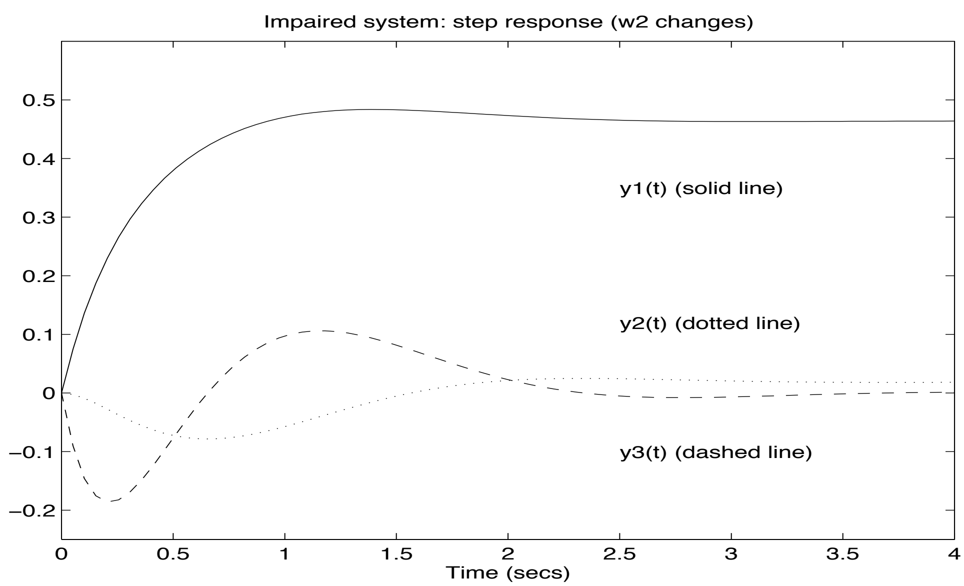

Figure 8: Example 4.1: Impaired system. Output response to a unit step change in the external input w_2.

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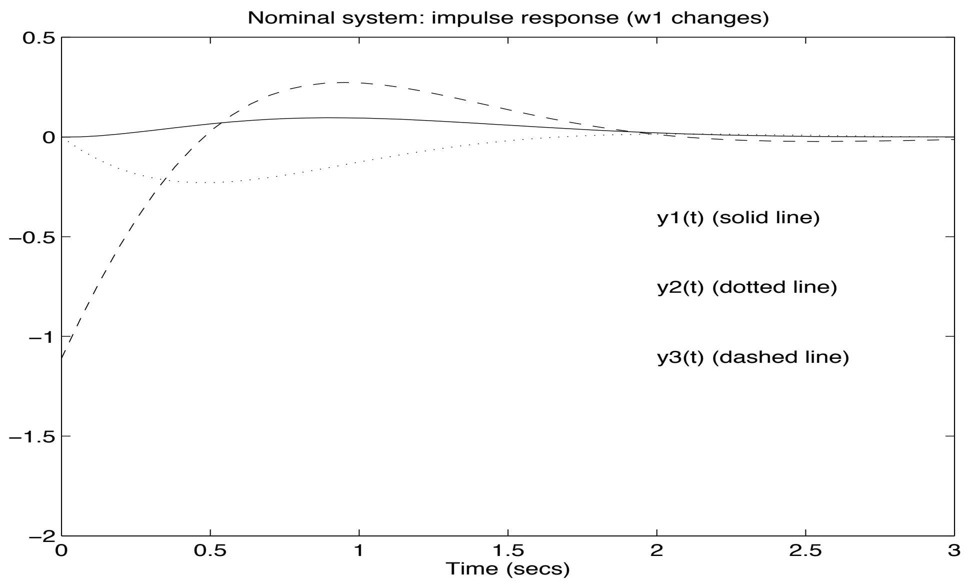

Figure 9: Example 4.1: Nominal system. Output response to an impulse change in the external input w_1.

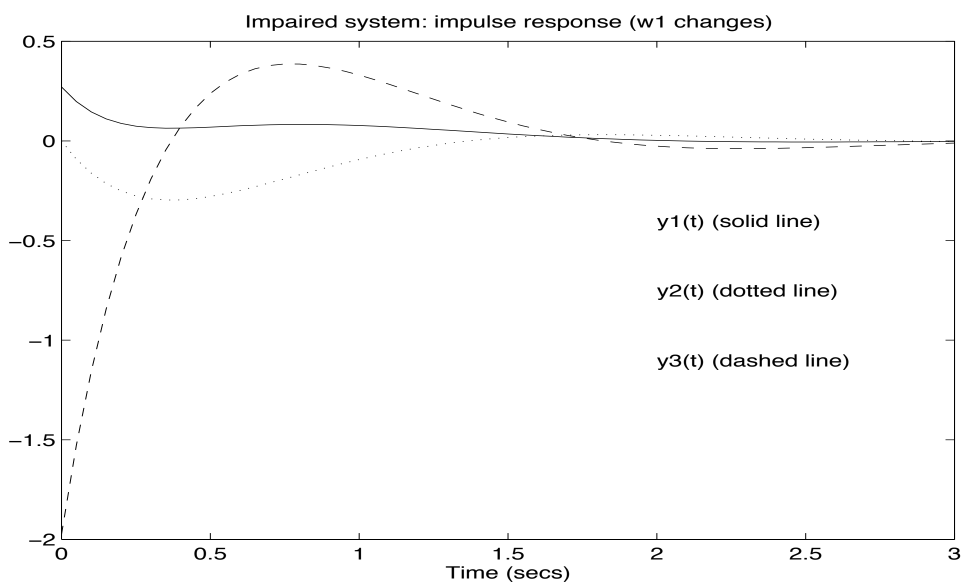

Figure 10: example 4.1: Impaired system. Output response to an impulse change in the external input w_1.

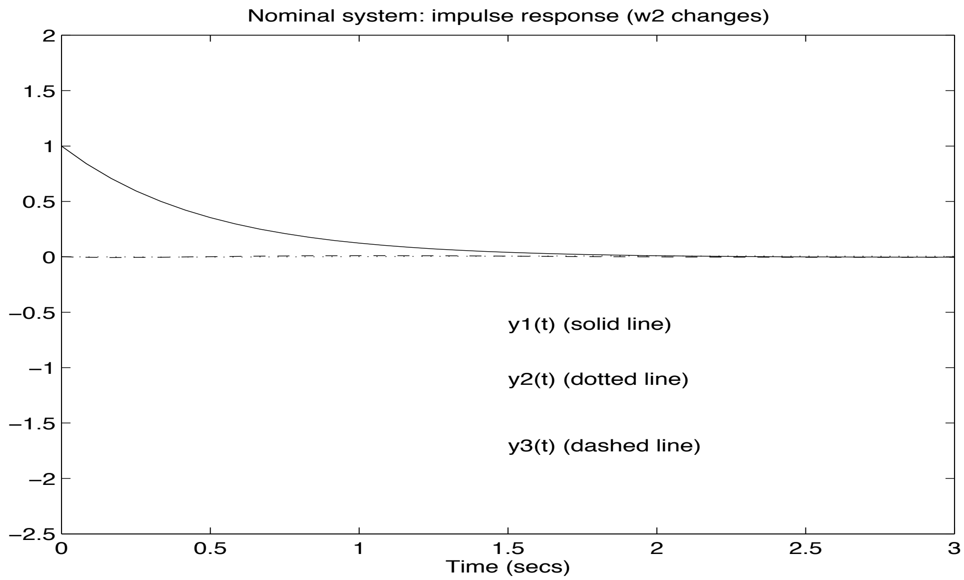

Figure 11: Example 4.1: Nominal system. Output response to an impulse change in the external input w_2.

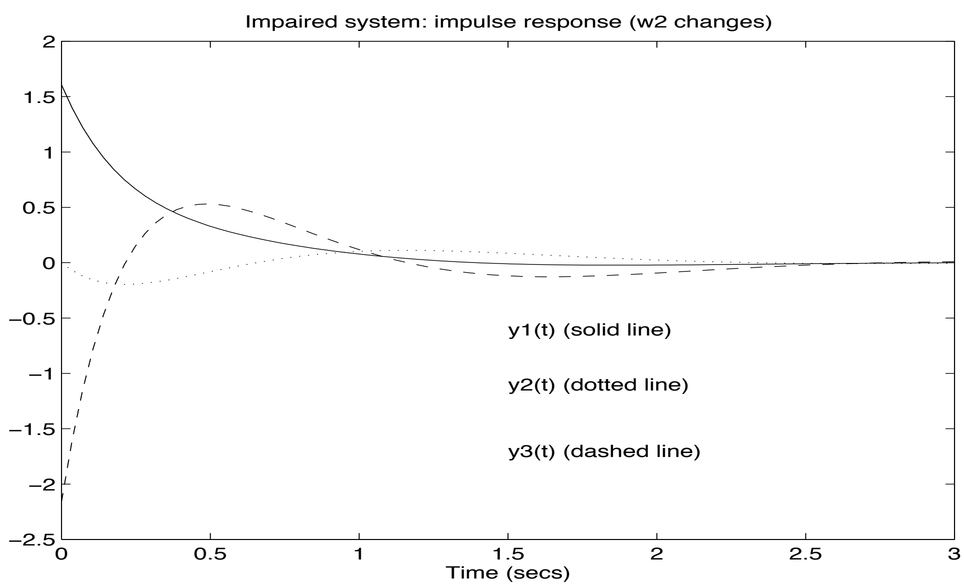

Figure 12: Example 4.1: Impaired system. Output response to an impulse change in the external input w_2.

$$\Lambda_D = \{-1, -1.25 \pm j1.75, -3\} \tag{138}$$

$$V_D = \begin{pmatrix} x & 0+jx & 0-jx & 1 \\ x & 1+j & 1-j & 0 \\ 0 & x+jx & x-jx & x \\ 1 & x+j0 & x-j0 & x \end{pmatrix} \tag{139}$$

where x represents an unspecified component. When we apply the state-feedback control law $u(t) = K\ x(t)$, the gain matrix that achieves the above design specifications is

$$K = \begin{pmatrix} 0.138879 & 1.416315 & -0.821448 & 0.086284 \\ -0.559704 & -0.286832 & 2.261491 & -0.509444 \end{pmatrix} \tag{140}$$

and the actual closed-loop eigenvectors are given by

$$\begin{aligned} V &= \begin{pmatrix} v_1 & v_2 & v_3 & v_4 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0.03080 & 1+j & 1-j & 0 \\ 0 & -0.0940+j0.6329 & -0.0940-j0.6329 & 0.00158 \\ 1 & 0 & 0 & -0.33333 \end{pmatrix} \end{aligned} \tag{141}$$

It is apparent that the above specifications for the closed-loop eigenvectors are completely satisfied. Next, we suppose that a failure occurs in the actuator. Although the state-transition matrix remains the same, the input matrix changes. Therefore

$$A_f = A, \qquad B_f = \begin{pmatrix} 0.952 & 4.50 \\ -1.5 & -0.416 \\ 0.0092 & -0.0100 \\ 0 & 0 \end{pmatrix} \tag{142}$$

The algorithmic approach presented above is used again to find the optimal state-feedback gain. Note that here all closed-loop eigenvalues will be recovered. Therefore, our objective is to determine a new state feedback law $u(t) = K_f\ x(t)$ such that the impaired closed-loop eigenvectors are as close to the eigenvectors of (141) as possible. First, we transform the impaired system to new state-coordinates by selecting the state-transformation matrix

$$T = \begin{pmatrix} 0.952 & 4.50 & 0 & 0 \\ -1.5 & -0.416 & 0 & 0.2 \\ 0.0092 & -0.0100 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \end{pmatrix} \tag{143}$$

In order to emphasize performance, which is determined by the closed-loop eigenvectors, we assign a weight factor of 0.01 to the robustness term $\mathrm{Tr}(P^2)$. Note that the introduction of this weight only affects (61), where the term $2P$ needs to be replaced by the term $2w_P P$, where w_P is the weight factor discussed above.

The algorithm gives the following state feedback gain

$$K_f = \begin{pmatrix} 0.220764 & 1.699474 & -1.208819 & 0.154278 \\ -0.769002 & -0.493509 & 3.242570 & -0.697044 \end{pmatrix} \tag{144}$$

with which the achievable eigenvectors transformed back to the original state-coordinates are

$$\begin{aligned} V_f &= \begin{pmatrix} v_1^f & v_2^f & v_3^f & v_4^f \end{pmatrix} \\ &= \begin{pmatrix} -1.00279 & -0.10677 + \jmath 0.10062 & -0.10677 - \jmath 0.10062 & 1.12200 \\ 0.03033 & 1.01196 + \jmath 0.98766 & 1.01196 - \jmath 0.98766 & 0.00428 \\ 0.00002 & -0.09381 + \jmath 0.63279 & -0.09381 - \jmath 0.63279 & 0.00116 \\ 1.00279 & 0.06693 + \jmath 0.01321 & 0.06693 - \jmath 0.01321 & -0.37400 \end{pmatrix} \end{aligned} \tag{145}$$

As we see, the obtained closed-loop eigenvectors are indeed close to the corresponding eigenvectors of (141). This can also be seen by computing

$$\|v_1^f - v_1\|^2 = 0.000015 \tag{146}$$

$$\|v_2^f - v_2\|^2 = \|v_3^f - v_3\|^2 = 0.0265 \tag{147}$$

$$\|v_4^f - v_4\|^2 = 0.0166 \tag{148}$$

The state response of the nominal system (A, B, C) of (137) with the state feedback matrix K of (140) and the state response of the impaired system of (142) with the state feedback feedback matrix K_f of (144) are compared in Figures 13-14. The initial conditon vector is $\begin{pmatrix} -0.50 & 0.4 & -0.75 & 0.9 \end{pmatrix}^T$. As we see, the algorithm is capable of completely recovering the performance of the nominal system. This is not surprising, since with state feedback we maintain all the nominal closed-loop eigenvalues, whereas with the proposed optimization scheme we achieve closed-loop eigenvectors very close to the eigenvectors of the nominal closed-loop system, as shown in (146)-(148) above.

The external input case For the system of (1)-(2) we assume the state feedback law

$$u(t) = K\,x(t) + G\,w(t) \tag{149}$$

where again $w \in \Re^r$ is the external input vector and G is an $\Re^{r\times r}$ matrix. With this feedback law, the closed-loop system is given by

$$\begin{aligned} \dot{x}(t) &= (A + BK)\,x(t) + BG\,w(t) && (150) \\ y(t) &= x(t) && (151) \end{aligned}$$

For the impaired system of (41)-(42), we consider the output feedback law

$$u(t) = K_f\,x(t) + G_f\,w(t) \tag{152}$$

so that the closed-loop system becomes

$$\begin{aligned} \dot{x}(t) &= (A_f + B_f K_f)\,x(t) + B_f G_f\,w(t) && (153) \\ y(t) &= x(t) && (154) \end{aligned}$$

As shown in the previous example, in order to recover the steady-state performance of the nominal closed-loop system of (150)-(151) in response to a unit step input in either of the external inputs, we need to determine the external input matrix G_f that minimizes the quantity J_{ss} of (134). Assuming again $G = I_2$, in view of (135) we easily obtain

$$G_f = \begin{pmatrix} 1.2019 & -0.0970 \\ -0.0877 & 1.3517 \end{pmatrix} \tag{155}$$

In Figures 15-16, we see that the output response of the nominal closed-loop system of (150)-(151) and the output response of the impaired closed-loop system of (153)-(154) for a unit step change in the external input w_1 are almost identical. The same is true for the output responses to a unit step change in the external input w_2, which are illustrated in Figures 17-18. Therefore, with the state feedback gain K_f of (144) and the external input matrix G_f of (155) the steady-state performance of the nominal closed-loop system of (150)-(151) is recovered as well.

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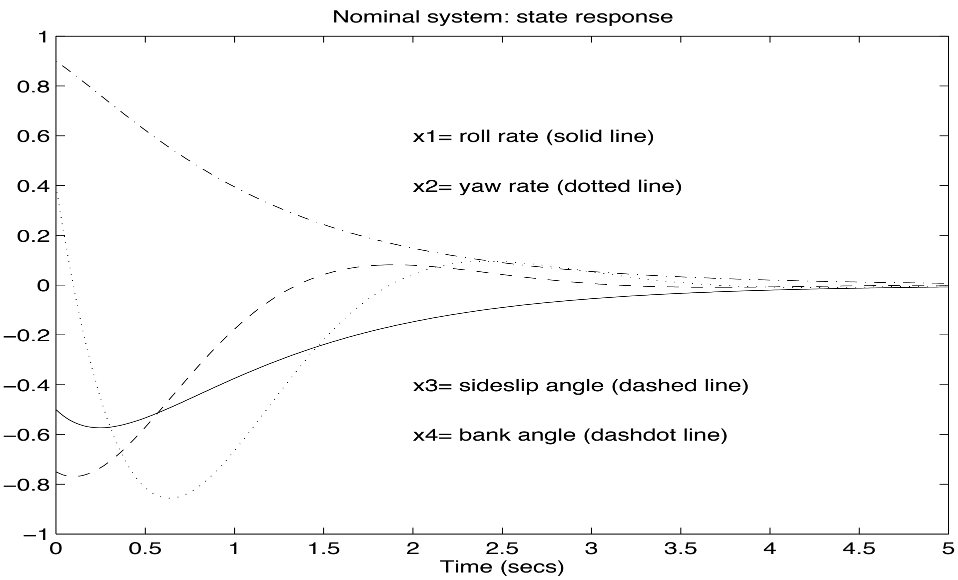

Figure 13: Example 4.2: Nominal system. Closed-loop state response (no external inputs).

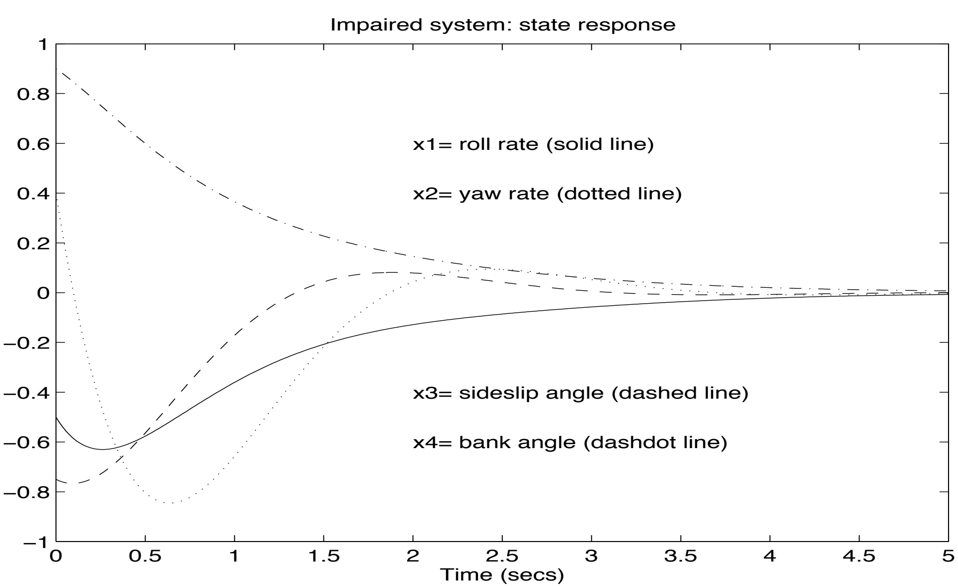

Figure 14: Example 4.2: Impaired system. Closed-loop state response (no external inputs).

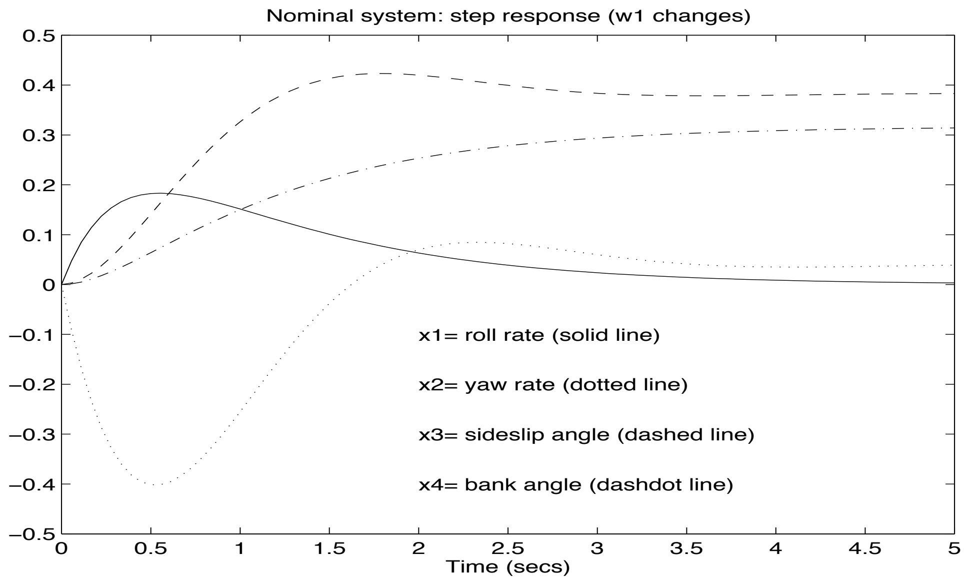

Figure 15: Example 4.2: Nominal system. Output response to a unit step change in the external input w_1.

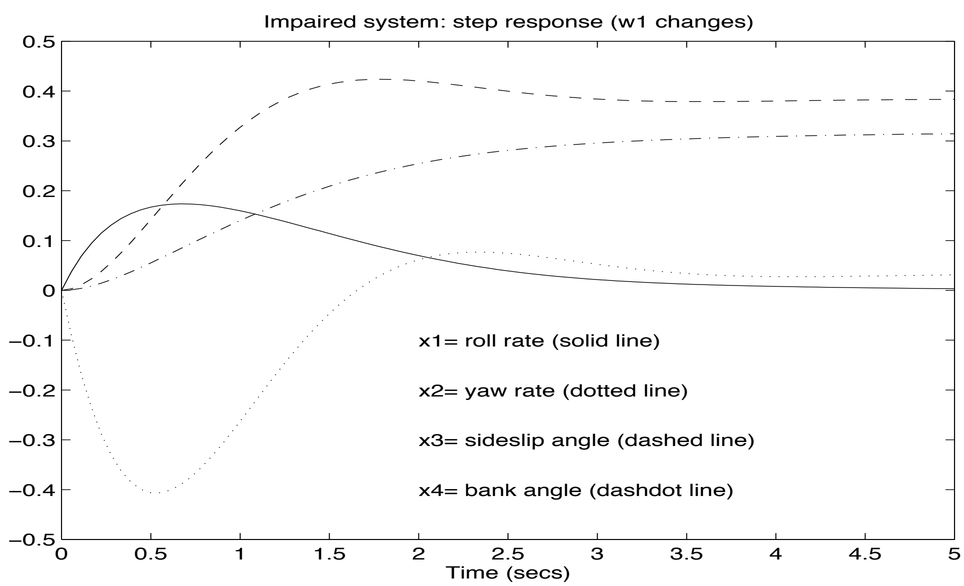

Figure 16: Example 4.2: Impaired system. Output response to a unit step change in the external input w_1.

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

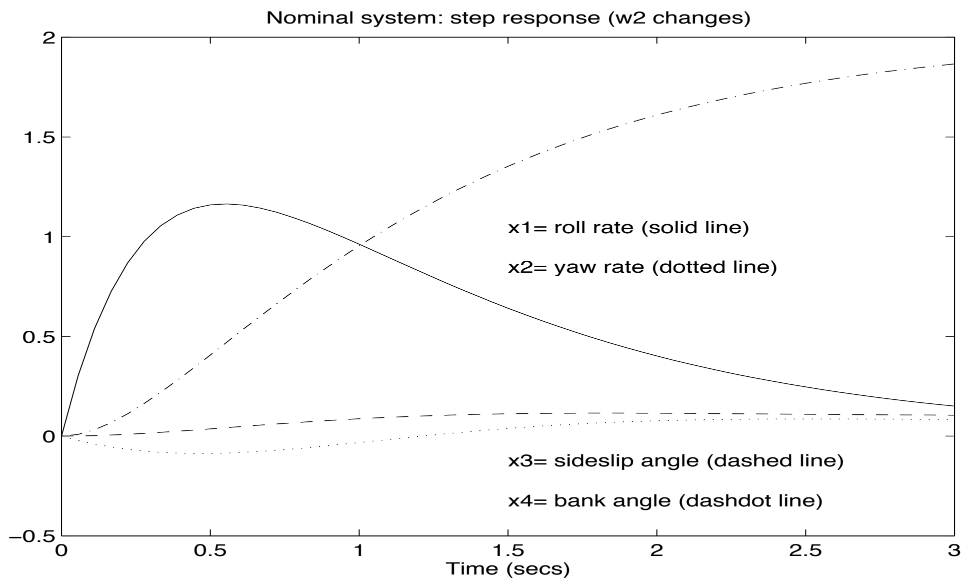

Figure 17: Example 4.2: Nominal system. Output response to a unit step change in the external input w_2.

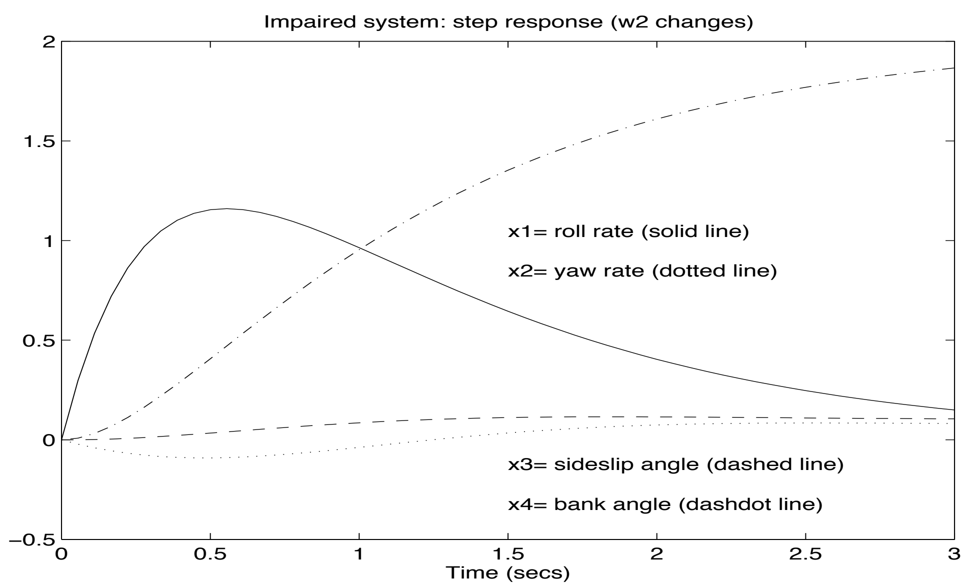

Figure 18: Example 4.2: Impaired system. Output response to a unit step change in the external input w_2.

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

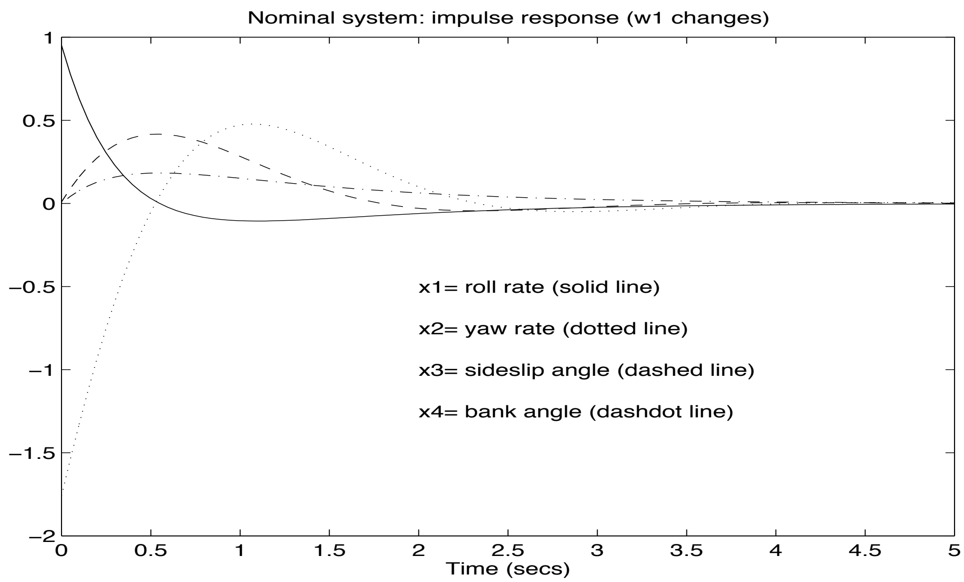

Figure 19: Example 4.2: Nominal system. Output response to an impulse change in the external input w_1.

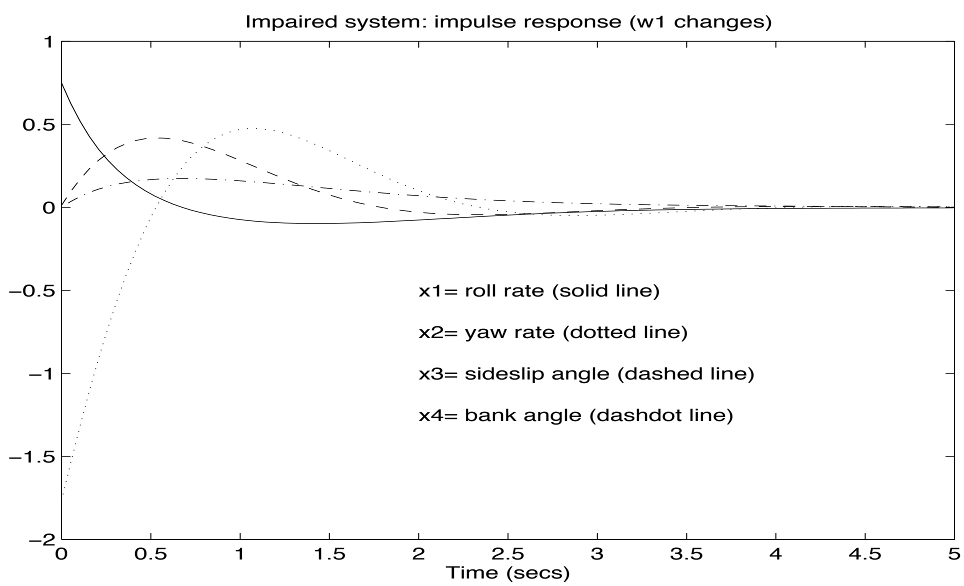

Figure 20: example 4.2: Impaired system. Output response to an impulse change in the external input w_1.

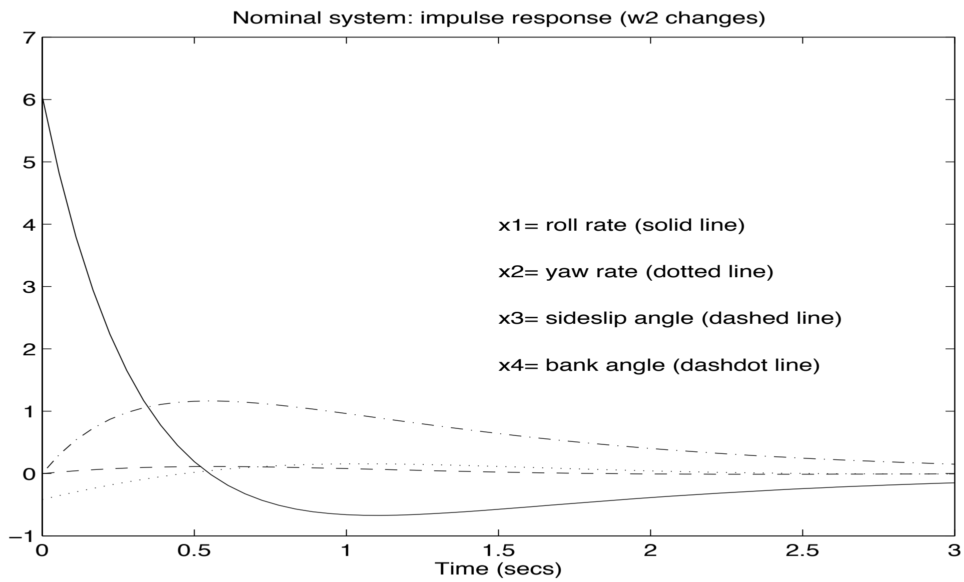

Figure 21: Example 4.2: Nominal system. Output response to an impulse change in the external input w_2.

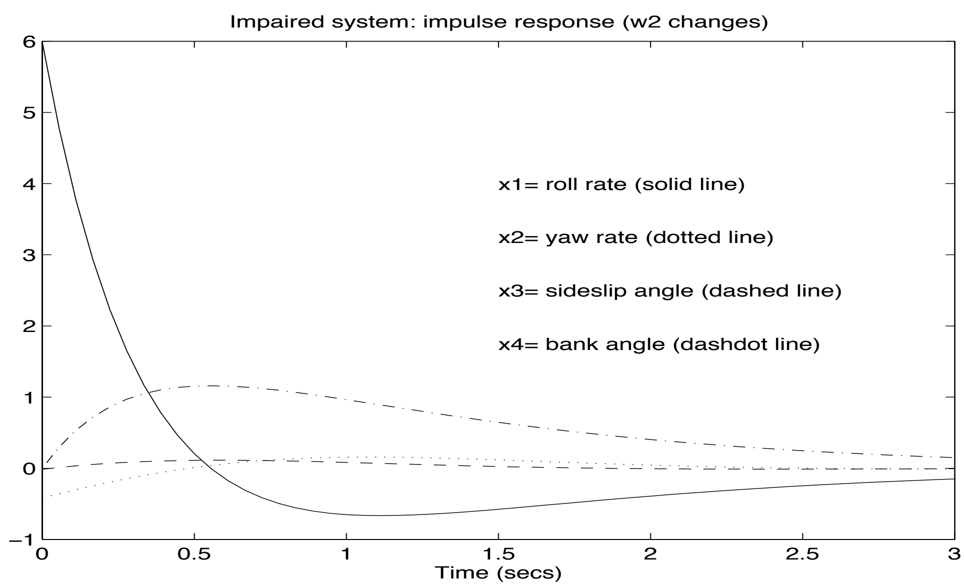

Figure 22: Example 4.2: Impaired system. Output response to an impulse change in the external input w_2.

I.K. Konstantopoulos and P.J. Antsaklis, "Eigenstructure Assignment in Reconfigurable Control Systems," Technical Report of the ISIS Group, ISIS-96-001, Dept. of Electrical Engr, Univ of Notre Dame, January 1996.

Finally, we compare the responses of the nominal closed-loop system and the impaired closed-loop system, with K_f and G_f determined above, to impulse changes in the external inputs. The results for an impulse change in w_1 are given in Figures 19-20, and for an impulse change in w_2, in Figures 21-22. As expected, the impulse response of the nominal system also is recovered.

We see that for the state feedback case the results obtained here are even better compared to the results obtained in the output feedback example studied before. This is due to the fact that all closed-loop poles are recovered. In addition, the optimization scheme proposed here determines all n impaired closed-loop eigenvectors close to the corresponding eigenvectors of the nominal system, since we are capable of exercising control upon all the modes of the system. This results in a performance (state response, unit step response, impulse response) which is very close to the performance of the nominal system.

5 Conclusions

An optimization approach to control reconfiguration, based on eigenstructure assignment, for control systems with output feedback has been presented. The emphasis has been on the recovery of the nominal closed-loop performance, which is determined by the closed-loop eigenvalues and eigenvectors. The proposed scheme preserves the $\max(r, q)$ most dominant eigenvalues of the nominal closed-loop system and determines their associated closed-loop eigenvectors as close to the corresponding eigevectors of the nominal closed-loop system as possible. Additionally, the stability of the remaining closed-loop eigenvalues is guaranteed by the satisfaction of an appropriate Lyapunov equation. The overall design is also robust with respect to uncertainties in the state-space matrices of the impaired/reconfigured system. Although the emphasis here was on static output feedback, the cases of state feedback and dymanic output feedback have also been studied. The approach has been applied to an aircraft control examples, where it was shown to not only preserve the shape of the transient response but recover much of the characteristics of the steady-state response as well.

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