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Asymptotically Stabilizing Switching Scheme for Switched Systems Consisting of Second-Order Unstable LTI Autonomous Subsystems

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Abstract

In this paper, we look into the problem of finding asymptotically stabilizing switching schemes for switched systems consisting of several second-order LTI autonomous subsystems that have unstable foci at the origin. Firstly we study the method to decide the direction of the solution of second-order LTI autonomous system with focus at the origin. Then we study switched systems consisting of two subsystems, and we provide a conic switching rule based on studies of the vector fields. Moreover, we extend this conic switching rule to several subsystems. Sufficient and necessary conditions for the asymptotic stabilizability of a switched system are provided. If the switched system is stabilizable, an asymptotically stabilizing switching scheme can be obtained based on conic switching rules.

1 Introduction

A switched system is a system consisting of several subsystems. In this paper, we would study a special kind of switched systems, namely, switched systems whose subsystems are second-order unstable LTI autonomous subsystems $\dot{x}(t) = A_i x(t)$, $A_i \in \mathbb{R}^{2\times 2}$, $i = 1, 2, \cdots, N$.

In [1], Branicky pointed out that it is sometimes possible to make a switched system asymptotically stable by appropriate switching schemes even if none of the subsystems is stable individually. Yet the problem has not been solved as to how to decide whether a given switched system is asymptotically stabilizable and if yes, how to obtain a stabilizing switching scheme.

In [4], Linear Matrix Inequality (LMI) problems are formulated to search for the multiple Lyapunov functions for switched systems consisting of linear subsystems. If the multiple Lyapunov functions can be found, then the switched system is stabilizable and the switching rule can also be obtained. In [2], LMI problems are also formulated as to show the stabilizability of the switched systems. LMI provides a very good way to find sufficient conditions for the switched system. Yet it may not be necessary.

In this paper, we try to establish sufficient and necessary conditions for the asymptotic stabilizability for a special kind of switched systems, namely, switched systems consisting of second-order LTI subsystems. In [5], we classified second-order system by the type of its origin and we partly answer the question except for the case when both subsystems have unstable foci at the origin. In this paper, we would mainly look into such kind of systems. And such kind of subsystems $\dot{x}(t) = A_i x(t)$ is said to be with unstable focus hereafter. Throughout this paper, unless otherwise specified, the LTI autonomous subsystems we mention are all subsystems with unstable focus.

The outline of the paper is as follows. In Section 2, we talk about the direction of the solution of second-order systems with focus. In Section 3, we look into the asymptotic stabilizability problem for switched systems with two subsystems. In Section 4, the method in Section 3 is extended to several subsystems' cases. An algorithm and several examples are given in Section 5. Section 6 concludes the paper.

2 Direction of the Solution of $\dot{x} = Ax$, $A \in \mathbb{R}^{2\times 2}$ with Focus

The solution of the LTI system

$$\dot{x} = Ax, \; x(0) = x_0 \tag{2.1}$$

is well known to be

$$x(t) = e^{At}x_0 \text{ for } t \geq 0. \tag{2.2}$$

When $A \in \mathbb{R}^{2\times 2}$, the trajectory of the solution can be shown on the $\mathbb{R}^2$ plane. In this section, we are interested in the case when A has two eigenvalues $\alpha \pm \beta i$ with $\alpha \neq 0$ and $\beta \neq 0$. The origin $(0,0)^T$ are often called stable($\alpha < 0$) or unstable($\alpha > 0$) focus of system (2.1). And the solution (2.2) is a logarithmic spiral on the $X_2 - X_1$ plane. The question we would concern here is how to decide, without plotting the trajectory, whether the trajectory will be a spiral around the origin in clockwise or counterclockwise direction. In the following, coordinate transformation is considered so that the question can be answered.

Let us firstly consider the simplest case where $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$. We have the following Lemma for this case.

Lemma 2.1 *For the LTI autonomous system $\dot{x}(t) = Ax(t)$ with focus, where $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$. The solution with $x(0) = x_0 \neq 0$ has the following properties:*

- *If $\alpha < 0$ and $\beta > 0$, then the solution $x(t) = e^{At}x_0$ is a logarithmic spiral that converges to the origin clockwisely.*
- *If $\alpha < 0$ and $\beta < 0$, then the solution $x(t) = e^{At}x_0$ is a logarithmic spiral that converges to the origin counterclockwisely.*
- *If $\alpha > 0$ and $\beta > 0$, then the solution $x(t) = e^{At}x_0$ is a logarithmic spiral that diverges to ∞ clockwisely.*
- *If $\alpha > 0$ and $\beta < 0$, then the solution $x(t) = e^{At}x_0$is a logarithmic spiral that diverges to ∞ counterclockwisely.*

Proof: We just prove the $\alpha < 0$ and $\beta > 0$ case, the others follow similar arguments.

From linear system theory, we know that

$$e^{At} = e^{\alpha t}\begin{bmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{bmatrix}.$$

Notice that the initial state $x_0 = [x_{01},\; x_{02}]^T = [r_0\cos\theta_0,\; r_0\sin\theta_0]^T$, where $r_0 = \sqrt{x_{01}^2 + x_{02}^2}$ and $\cos\theta_0 = \frac{x_{01}}{r_0}$, $\sin\theta_0 = \frac{x_{02}}{r_0}$. In other words, the representation of x_0 in the polar coordinate system is (r_0, θ_0).

Now we get

$$\begin{aligned} x(t) &= e^{At}x_0 \\ &= r_0 e^{\alpha t}\begin{bmatrix} \cos\theta_0\cos\beta t + \sin\theta_0\sin\beta t \\ -\cos\theta_0\sin\beta t + \sin\theta_0\cos\beta t \end{bmatrix} \\ &= r_0 e^{\alpha t}\begin{bmatrix} \cos(\theta_0 - \beta t) \\ \sin(\theta_0 - \beta t) \end{bmatrix}. \end{aligned}$$

Observe that the representation of $x(t)$ in the polar coordinate system is $(r_0 e^{\alpha t}, \theta_0 - \beta t)$, so it is clear that if $\alpha < 0$ and $\beta > 0$, $x(t)$ is a logarithmic spiral that converges to the origin clockwisely, i.e., $\|x(t)\|_2 \to 0$ as $t \to \infty$. □

Before we proceed to the general case, we would look into linear transformation in $\mathbb{R}^2$.

Suppose $\mathbf{e}_{X_1}$ and $\mathbf{e}_{X_2}$ are two unit vectors on $\mathbb{R}^2$ for X_1 and X_2 axes, respectively. If a point x has coordinate representation $x = [x_1,\ x_2]^T$, the corresponding vector on $\mathbb{R}^2$ is

$$x = \left[\ \mathbf{e}_{X_1},\ \mathbf{e}_{X_2}\ \right] \left[\begin{array}{c} x_1 \\ x_2 \end{array} \right]. \tag{2.3}$$

If we choose another coordinate system $Y_2 - Y_1$ with unit vectors $\mathbf{e}_{Y_1}$ and $\mathbf{e}_{Y_2}$ for Y_1 and Y_2 axes, respectively. And

$$\left[\ \mathbf{e}_{Y_1},\ \mathbf{e}_{Y_2}\ \right] = \left[\ \mathbf{e}_{X_1},\ \mathbf{e}_{X_2}\ \right] T, \text{ where } T \in \mathbb{R}^{2\times 2} \text{ nonsingular.} \tag{2.4}$$

By this linear transformation, we know that a point x in the $X_2 - X_1$ coordinate system would have coordinate representation $y = T^{-1}x$ in $Y_2 - Y_1$ coordinate system.

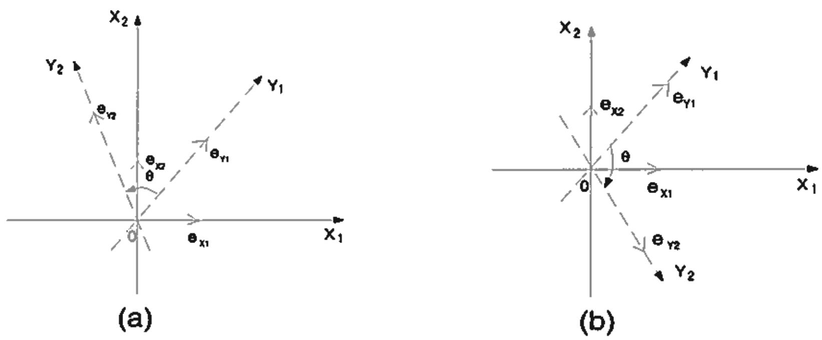

Figure 1: (a) $Y_2 - Y_1$ and $X_2 - X_1$ coordinate system agree in direction. (b) $Y_2 - Y_1$ and $X_2 - X_1$ coordinate system disagree in direction.

Definition 2.2 *The coordinate systems $Y_2 - Y_1$ and $X_2 - X_1$ are said to* **agree (disagree) in direction** *if the directional angle θ confined to $-\pi \le \theta < \pi$ from Y_1 axis to Y_2 axis satisfies $0 < \theta < \pi$ $(-\pi < \theta < 0)$ on $X_2 - X_1$ plane.*

Figure 1 shows $Y_2 - Y_1$ and $X_2 - X_1$ that agree and disagree in direction.

A simple criterion to decide the agreement or disagreement issue is to look at the linear transformation matrix T by the following Lemma.

Lemma 2.3 *The coordinate transforamtion in (2.4) results in a coordinate system $Y_2 - Y_1$ that agrees (disagrees) in direction with $X_2 - X_1$ system if $detT > 0$ $(detT < 0)$. In the following, we also say that T agrees (disagrees) in direction with $X_2 - X_1$ system.*

Proof: Suppose

$$T = \left[\begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array} \right].$$

By adding another axis X_3 which is perpendicular to the plane $X_2 - X_1$ as shown in Figure 2, we get a coordinate system $X_3 - X_2 - X_1$ in $\mathbb{R}^3$. By analytical geometry, the cross product

$$\begin{aligned} \mathbf{e}_{Y_1} \times \mathbf{e}_{Y_2} &= det \left[\begin{array}{ccc} \mathbf{e}_{X_1} & \mathbf{e}_{X_2} & \mathbf{e}_{X_3} \\ t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \end{array} \right] \\ &= det \left[\begin{array}{cc} t_{11} & t_{12} \\ t_{21} & t_{22} \end{array} \right] \mathbf{e}_{X_3} \\ &= (detT)\mathbf{e}_{X_3}. \end{aligned} \tag{2.5}$$

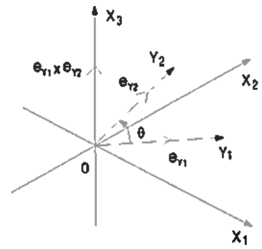

Figure 2: Coordinate system $X_3 - X_2 - X_1$ in $\mathbb{R}^3$.

Since the cross product, in this case, can also be written as

$$\mathbf{e}_{Y_1} \times \mathbf{e}_{Y_2} = (\|\mathbf{e}_{Y_1}\|_2 \|\mathbf{e}_{Y_2}\|_2 \sin\theta)\mathbf{e}_{X_3}, \tag{2.6}$$

so by (2.5), (2.6), we conclude that $\sin\theta > 0$ iff $detT > 0$. Therefore, if $detT > 0$ ($detT < 0$), we have $0 < \theta < \pi$ ($-\pi < \theta < 0$), that is, $Y_2 - Y_1$ system agrees (disagrees) in direction with $X_2 - X_1$ system. □

Now we come to the general case. Suppose $A \in \mathbb{R}^{2\times 2}$ has two eigenvalues $\alpha \pm \beta i$ with $\alpha \neq 0$ and $\beta \neq 0$. By linear algebra, we know that there exists nonsingular T such that:

$$A = T\Lambda T^{-1}, \text{ where } \Lambda = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}. \tag{2.7}$$

If $detT < 0$, we can let $T_1 = TE_{12}$, with $E_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to make $detT_1 > 0$, and

$$A = T_1\Lambda_1 T_1^{-1}, \text{ where } \Lambda_1 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}. \tag{2.8}$$

So for simplicity, we assume that $detT > 0$ in (2.7), i.e., T agrees with the direction of $X_2 - X_1$ system. By letting $y = T^{-1}x$, (2.1) can be reduced to

$$\dot{y}(t) = \Lambda y(t),\ y(0) = T^{-1}x(0) = T^{-1}x_0. \tag{2.9}$$

Upto this stage, we can use the following Theorem to decide the direction of the solution for (2.1).

Theorem 2.4 *If $A \in \mathbb{R}^{2\times 2}$ has two eigenvalues $\alpha \pm \beta i$ with $\alpha \neq 0$ and $\beta \neq 0$, and $A = T\Lambda T^{-1}$ as in (2.7), where T agrees with the direction of $X_2 - X_1$ system. The the direction of the solution (clockwise or counterclockwise) of (2.1) is the same as the direction of the solution of (2.9).*

Proof: Since T agrees with the direction of $X_2 - X_1$ system, so $Y_2 - Y_1$ system agrees with the direction of $X_2 - X_1$ system. Therefore a solution whose direction is clockwise (counterclockwise) in $Y_2 - Y_1$ system will also be clockwise (counterclockwise) in $X_2 - X_1$ system. □

By Theorem 2.4, we can find the direction of a solution in $X_2 - X_1$ system by reducing it to a problem in $Y_2 - Y_1$ system and then using Lemma 2.1 to obtain the conclusion. In the following, we would say a subsystem $\dot{x}(t) = A_i x(t)$ has clockwise (counterclockwise) direction if the direction of the solution has clockwise (counterclockwise) direction.

Example 2.5 *Consider $A = \begin{bmatrix} -3 & 13 \\ -2 & -1 \end{bmatrix}$, we have $A = T\Lambda T^{-1}$ with $T = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$ and $\Lambda = \begin{bmatrix} -2 & 5 \\ -5 & -2 \end{bmatrix}$. By Theorem 2.4 and Lemma 2.1, the solution of (2.1) associated with this A converges to the origin clockwisely.*

3 Two Subsystems

In this section, we will study the stabilizing switch schemes for switched systems consisting of two second-order unstable LTI autonomous subsystems. We will mainly study those switched systems whose subsystems are all with unstable foci.

Consider a switched system with two LTI autonomous subsystems with unstable focus:

$$\dot{x}(t) = A_1 x(t), \; \dot{x}(t) = A_2 x(t). \tag{3.1}$$

In this section, we will study the vector fields of both subsystems and give a very intuitive way of obtaining a stabilizing switching rule. In the following, we would firstly consider switched systems with subsystems in the same direction and then consider switched systems with subsystems in different directions.

In the following, when we mention the angle from one ray l_1 to another ray l_2, we take the counterclockwise angle to be positive and the clockwise angle to negative.

3.1 Two Subsystems in the Same Direction

We would consider switched systems, whose subsystems both have clockwise direction. For switched systems whose subsystems both have counterclockwise direction, all the following discussion can be applied similarly.

Let $x = (x_1, x_2)^T$ be a nonzero point on $\mathbb{R}^2$ plane, and let

$$f_1 = A_1 x = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \; f_2 = A_2 x = \begin{bmatrix} a_3 \\ a_4 \end{bmatrix}. \tag{3.2}$$

If we take x, f_1 and f_2 to be vectors in $\mathbb{R}^2$. We argue that the angle θ (θ is confined to $-\pi \leq \theta \leq \pi$) from x to f_1 (or f_2) satisfy $-\pi \leq \theta \leq 0$, in other words, f_1 must be on one side of vector x as shown in Figure 3(a). Since if otherwise, assume that f_1 is on the other side of x as shown in Figure 3(b), then in sufficiently small time elapse dt, the trajectory will travel counterclockwisely, which contradicts our assumption.

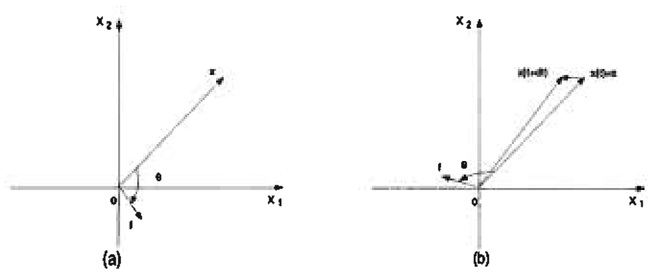

Figure 3: (a) The angle θ from x to f_1. (b) f_1 is on the other side of x.

Now we consider the switching scheme that asymptotically stabilize the switched system. In other words, by our switching schemes, we want to lead the trajectory closer and closer to the origin, i.e., $\|x(t)\|_2 \to 0$ as $t \to 0$. In the following, we will discuss the way by which we try to obtain such switching rules by choosing a sybsystem which has the potentiality to drive the trajectory closer to the origin.

For a nonzero point $x \in \mathbb{R}^2$, let f_1 and f_2 as in (3.2) and let θ_1, θ_2 be angles from x to f_1, f_2, respectively.

Let l_2 be a line sufficiently close to line l_1 determined by x as shown in Figure 4(a). The trajectory will hit l_2 by point $x^{(1)}$ if it follows subsystem 1, it will hit l_2 by $x^{(2)}$ if it follows subsystem 2. Let x^* be the point where l_2 is hit by a line perpendicular to l_1. Let $\|x^{(1)}\|_2$, $\|x^{(2)}\|_2$ be the distance from $x^{(1)}$, $x^{(2)}$ to the origin, respectively. And let $ds = \|x^* - x\|_2$ (Figure 4). Also we denote the following conic regions

$$E_{is} = \{x | -\pi \leq \theta_i(f_i(x)) \leq -\frac{\pi}{2}\} = \{x | x^T f_i(x) = x^T A_i x \leq 0\}, \; i = 1, 2 \tag{3.3}$$

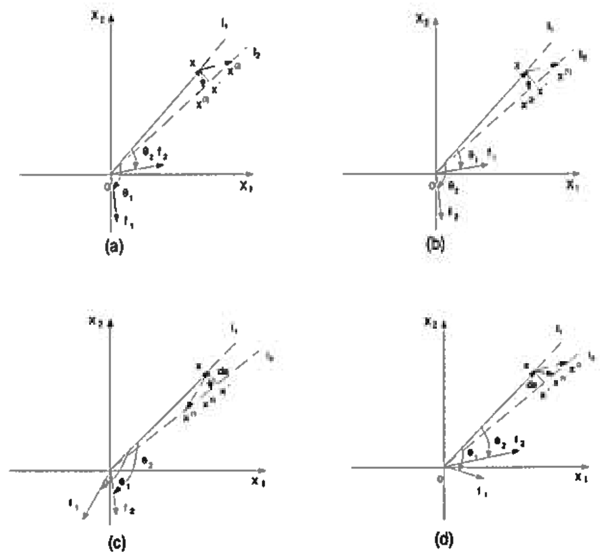

Figure 4: The four different cases.

$$E_{iu} = \{x| -\frac{\pi}{2} \leq \theta_i(f_i(x)) \leq 0\} = \{x|x^T f_i(x) = x^T A_i x \geq 0\},\ i = 1,2. \tag{3.4}$$

Now we try to associate a better subsystem for x according to the following different cases.

Case 1: Assume that

$$x \in E_{1s} \cap E_{2u}.$$

In this case, as shown in Figure 4(a), $\|x^{(1)}\|_2 \leq \|x^*\|_2 \leq \|x^{(2)}\|_2$ when ds is sufficiently small. So we can choose subsystem 1 in order to drive the trajectory closer to the origin. We denote the conic region $E_{1s} \cap E_{2u}$ as Ω_1.

Case 2: Assume that

$$x \in E_{1u} \cap E_{2s}.$$

In this case, as shown in Figure 4(b), we can choose subsystem 2 by the similar argument as in Case 1. We denote the conic region $E_{1u} \cap E_{2s}$ as Ω_2.

Case 3: Assume that

$$x \in E_{1s} \cap E_{2s}.$$

In this case, as shown in Figure 4(c), consider

$$\frac{\|x\|_2 - \|x^{(1)}\|_2}{\|x\|_2 - \|x^{(2)}\|_2} \text{ as } ds \to 0.$$

Let dt_1 and dt_2 be the time elapse the system would take to hit l_2 along the vector f_1 and f_2, respectively. When ds is sufficiently small, we approximately have

$$dt_1 = \frac{ds}{\frac{(a_1x_2 - a_2x_1)}{\sqrt{x_2^2+x_1^2}}}, \tag{3.5}$$

$$dt_2 = \frac{ds}{\frac{(a_3x_2 - a_4x_1)}{\sqrt{x_2^2+x_1^2}}}. \tag{3.6}$$

By (3.5) and (3.6) and using L'Hospital Rule, we obtain

$$\begin{aligned}
& \lim_{ds\to 0} \frac{\|x\|_2 - \|x^{(1)}\|_2}{\|x\|_2 - \|x^{(2)}\|_2} \\
= & \lim_{ds\to 0} \frac{\sqrt{x_1^2+x_2^2} - \sqrt{(x_1+a_1dt_1)^2+(x_2+a_2dt_1)^2}}{\sqrt{x_1^2+x_2^2} - \sqrt{(x_1+a_3dt_2)^2+(x_2+a_4dt_2)^2}} \\
= & \lim_{ds\to 0} \frac{\frac{(a_1(x_1+a_1dt_1)+a_2(x_2+a_2dt_1))\sqrt{x_1^2+x_2^2}}{(a_1x_2-a_2x_1)\sqrt{(x_1+a_1dt_1)^2+(x_2+a_2dt_1)^2}}}{\frac{(a_3(x_1+a_3dt_2)+a_4(x_2+a_4dt_2))\sqrt{x_1^2+x_2^2}}{(a_3x_2-a_4x_1)\sqrt{(x_1+a_3dt_2)^2+(x_2+a_4dt_2)^2}}} \\
= & \frac{(a_1x_1+a_2x_2)(a_3x_2-a_4x_1)}{(a_1x_2-a_2x_1)(a_3x_1+a_4x_2)} \qquad (3.7) \\
= & \frac{\cos\theta_1\cos(\theta_2+\frac{\pi}{2})}{\cos(\theta_1+\frac{\pi}{2})\cos\theta_2} \qquad (3.8)
\end{aligned}$$

By the above calculation, we learn that if

$$\frac{(a_1x_1+a_2x_2)(a_3x_2-a_4x_1)}{(a_1x_2-a_2x_1)(a_3x_1+a_4x_2)} \geq 1, \qquad (3.9)$$

in other words, by simplification,

$$a_2a_3 - a_1a_4 \leq 0, \qquad (3.10)$$

we will have $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$ for sufficiently small ds. In this case, we can choose subsystem 1. And we denote the conic region $E_{1s} \cap E_{2s} \cap \{x|a_2a_3 - a_1a_4 \leq 0\}$ as Ω_3.

This condition can also be translated into the following condition (since $\cos(\theta_1+\frac{\pi}{2})\cos\theta_2 \leq 0$ in this case)

$$\cos\theta_1\cos(\theta_2+\frac{\pi}{2}) - \cos(\theta_1+\frac{\pi}{2})\cos\theta_2 \leq 0,$$

that is

$$\sin(\theta_1-\theta_2) \leq 0,$$

that is

$$\theta_1 \leq \theta_2. \qquad (3.11)$$

So in this case, if $\theta_1 \leq \theta_2$, we choose subsystem 1. Similarly, if $a_2a_3 - a_1a_4 \geq 0$, i.e., $\theta_1 \geq \theta_2$, we choose subsystem 2 and we denote the conic region $E_{1s} \cap E_{2s} \cap \{x|a_2a_3 - a_1a_4 \geq 0\}$ as Ω_4.

Case 4: Assume that

$$x \in E_{1u} \cap E_{2u}.$$

In this case, as shown in Figure 4(d), consider

$$\frac{\|x^{(1)}\|_2 - \|x\|_2}{\|x^{(2)}\|_2 - \|x\|_2} \text{ as } ds \to 0.$$

By the similar calculation, we have

$$\begin{aligned}
& \lim_{ds\to 0} \frac{\|x^{(1)}\|_2 - \|x\|_2}{\|x^{(2)}\|_2 - \|x\|_2} \\
= & \frac{(a_1x_1+a_2x_2)(a_3x_2-a_4x_1)}{(a_1x_2-a_2x_1)(a_3x_1+a_4x_2)} \qquad (3.12) \\
= & \frac{\cos\theta_1\cos(\theta_2+\frac{\pi}{2})}{\cos(\theta_1+\frac{\pi}{2})\cos\theta_2} \qquad (3.13)
\end{aligned}$$

By the above calculation as in Case 3, we learn that if

$$\frac{(a_1x_1 + a_2x_2)(a_3x_2 - a_4x_1)}{(a_1x_2 - a_2x_1)(a_3x_1 + a_4x_2)} \leq 1, \tag{3.14}$$

in other words, by simplification,

$$a_2a_3 - a_1a_4 \leq 0, \tag{3.15}$$

we will have $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$ for sufficiently small ds, so we can choose subsystem 1. And we denote the conic region $E_{1u} \cap E_{2u} \cap \{x | a_2a_3 - a_1a_4 \leq 0\}$ as Ω_5.

Similar to Case 3, this condition can also be translated into the following condition (since $\cos(\theta_1 + \frac{\pi}{2}) \cos\theta_2 \geq 0$ in this case)

$$\theta_1 \leq \theta_2. \tag{3.16}$$

So in this case, if $\theta_1 \leq \theta_2$, we choose subsystem 1. Similarly, if $a_2a_3 - a_1a_4 \geq 0$, i.e., $\theta_1 \geq \theta_2$, we choose subsystem 2 and we denote the conic region $E_{1u} \cap E_{2u} \cap \{x | a_2a_3 - a_1a_4 \geq 0\}$ as Ω_6.

Notice that Ω_i, $i = 1, 2, \cdots, 6$, are all conic regions that partition the $\mathbb{R}^2$ plane. We can associate with each region a subsystem according to the above discussion. In essence, we just associate with each point on $\mathbb{R}^2$ with a subsystem i whose θ_i is smaller (if $\theta_1 = \theta_2$, we may choose either of the subsystems). This partition of the $\mathbb{R}^2$ plane is of particular importance here. In the following, we call the switching rule by using the partition and adopting the associated subsystems the **conic switching rule**. We will show that by using the conic switching rule, we can decide whether the system (3.1) is asymptotically stabilizable or not. If (3.1) is asymptotically stabilizable, the partition and associated subsystems also provide a stabilizing switching scheme.

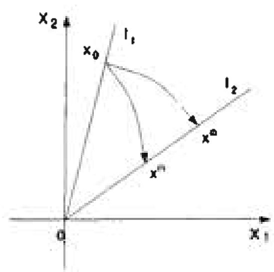

Figure 5: The two rays and related points.

Consider the system (3.1). Let l_1 and l_2 be two rays that go through the origin and are in the same conic region of one Ω_i. Suppose l_2 is to the clockwise side of l_1 in the conic region and the conic region with angle from l_1 to l_2 is inside Ω_i. Suppose x_0 is on l_1. Let $x^{(1)}$ be the point on l_2 where the trajectory of the system hits l_2 for the first time if the system evolves according to subsystem 1. Let $x^{(2)}$ be the point on l_2 where the trajectory of the system hits l_2 for the first time if the system evolves according to subsystem 2. Figure 5 shows the two rays and corresponding points. For the above-mentioned rays and points, we have the following lemma.

Lemma 3.1 *If l_1 and l_2 are in the conic region Ω_1, then $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$.*

Proof: Let $r_1(t) = \|x(t)\|_2 - \|x_0\|_2$ and $r_2(t) = \|x(t)\|_2 - \|x_0\|_2$ when the trajectory evolves according to subsystem 1 and subsystem 2, respectively. We have

$$dr_1 = d(\|x(t)\|_2) = \frac{x^T(t) f_1}{\|x(t)\|_2} dt = \frac{x^T(t) A_1 x(t)}{\|x(t)\|_2} dt,$$

$$dr_2 = d(\|x(t)\|_2) = \frac{x^T(t)f_2}{\|x(t)\|_2}dt = \frac{x^T(t)A_2x(t)}{\|x(t)\|_2}dt.$$

Assume t_1 and t_2 are the time the system take to go to $x^{(1)}$ and $x^{(2)}$, respectively. Since in Ω_1, $x^TA_1x \leq 0$ and $x^TA_2x \geq 0$, we have

$$r_1(t_1) = \|x^{(1)}\|_2 - \|x_0\|_2 = \int_0^{t_1} \frac{x^T(t)A_1x(t)}{\|x(t)\|_2}dt \leq \int_0^{t_1} 0dt = 0, \tag{3.17}$$

$$r_2(t_2) = \|x^{(2)}\|_2 - \|x_0\|_2 = \int_0^{t_2} \frac{x^T(t)A_2x(t)}{\|x(t)\|_2}dt \geq \int_0^{t_2} 0dt = 0. \tag{3.18}$$

Combining (3.17) and (3.18), we conclude that $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$. □

Lemma 3.2 *If l_1 and l_2 are in the conic region Ω_3, then $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$.*

Proof: Let α be the angle from l_1 to the line on which $x(t)$ lies and let α_1 be the angle from l_1 to l_2. Let $x^{(1)}(t)$ and $x^{(2)}(t)$ be the trajectory of the system if the system follows subsystem 1 and 2, respectively. If we use α as a variable, then we can also denote $x^{(1)}(t)$ and $x^{(2)}(t)$ as $x^{(1)}(\alpha)$ and $x^{(2)}(\alpha)$.

From the discussion of Case 3, we know that in a very small neighborhood of $x(0)$(i.e., $-\epsilon < \alpha < 0$ with $\epsilon > 0$ very small), $\|x^{(1)}(\alpha)\|_2 \leq \|x^{(2)}(\alpha)\|_2$.

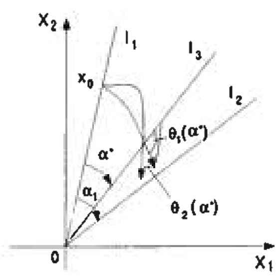

Figure 6: Figure for proof of Lemma 3.2.

Now choose α^*, $\alpha_1 < \alpha^* < 0$, such that $\|x^{(1)}(\alpha)\|_2 \leq \|x^{(2)}(\alpha)\|_2$ holds for any $\alpha \in (\alpha^*, \alpha^* + \epsilon)$ with some $\epsilon > 0$ very small.

If such α^* does not exist, then clearly $\|x^{(1)}(\alpha)\|_2 \leq \|x^{(2)}(\alpha)\|_2$ holds for any α, $\alpha_1 < \alpha < 0$.

Otherwise, assume that there exists an α^*, $\alpha_1 < \alpha^* < 0$, such that $\|x^{(1)}(\alpha^*)\|_2 = \|x^{(2)}(\alpha^*)\|_2$ on line l_3 which is inside the conic region formed by l_1 and l_2(Figure 6). Then we have

$$\theta_2(\alpha^*) \leq \theta_1(\alpha^*).$$

Yet combining (3.11), we obtain

$$\theta_2(\alpha^*) = \theta_1(\alpha^*),$$

that is

$$a_2a_3 - a_1a_4 = 0. \tag{3.19}$$

The solution of (3.19) are two lines on $\mathbb{R}^2$ plane. So l_3 is one of the lines.

Now observe (3.10)

$$a_2a_3 - a_1a_4 \leq 0, \tag{3.20}$$

the solution of which is a conic region formed by the two lines in (3.19). Therefore, there exists an α very close to α^* and $\alpha < \alpha^*$ such that $\theta_2(\alpha) < \theta_1(\alpha)$, yet this leads to a contradiction to (3.11).

Therefore $\|x^{(1)}(\alpha)\|_2 \leq \|x^{(2)}(\alpha)\|_2$ for any α, $0 < \alpha < \alpha_1$, by continuity, we have $\|x^{(1)}(\alpha_1)\|_2 \leq \|x^{(2)}(\alpha_1)\|_2$, that is $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$. □

Like the proofs of Lemma 3.1 and Lemma 3.2, we can prove the similar result for conic regions Ω_2 to Ω_6.

Now that we have the above lemmas, we can prove the following theorem.

Theorem 3.3 *Let l_1 be a ray that goes through the origin. Let x_0 be on l_1. If x^* on l_1 is the point where the trajectory hits l_1 for the first time after leaving x_0 when the switched system evolves according to the conic switching rule. And if x_1 on l_1 is the point where the trajectory hits l_1 for the first time after leaving x_0 when the switched system evolves according some arbitrary switching rule. Then we have $\|x^*\|_2 \leq \|x_1\|_2$.*

Proof: Let α be the angle from l_1 to the line on which $x(t)$ lies. Assume the s is a switching rule such that the system switches when α is equal to $0, \alpha_1, \alpha_1, \cdots, \alpha_n, \cdots$. Combining the switching α's above with the switching α's derived by conic switching rule and $\alpha - 2\pi$. For simplicity of notation, we still denote the combined switching αs as $0, \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots, \alpha - 2\pi, \cdots$. Let the trajectory for s be $x_s(\alpha)$ and the trajectory by conic switching be $x_c(\alpha)$.

Now in the conic region $\alpha_1 \leq \alpha \leq 0$, using the above lemmas, it is better to follow the switching rule by the conic switching rule. So upon arriving α_1, we have $\|x_c(\alpha_1)\|_2 \leq \|x_s(\alpha_1)\|_2$.

In the conic region $\alpha_2 \leq \alpha \leq \alpha_1$, if $\|x_c(\alpha_1)\|_2 = \|x_s(\alpha_1)\|_2$, then by using the above lemmas, it is better to follow the conic switching rule. If $\|x_c(\alpha_1)\|_2 < \|x_s(\alpha_1)\|_2$, then it would be even clear that it is better to follow the conic switching rule, since every trajectory evolving according to any one subsystem starting from $x_c(\alpha_1)$ is closer to the origin than starting from $x_s(\alpha_1)$. Therefore upon arriving α_2, we have $\|x_c(\alpha_2)\|_2 \leq \|x_s(\alpha_2)\|_2$.

By induction, we can use the similar argument as above to prove that upon arriving α_2, we have $\|x_c(\alpha - 2\pi)\|_2 \leq \|x_s(\alpha - 2\pi)\|_2$, i.e., $\|x^*\|_2 \leq \|x_1\|_2$. □

Theorem 3.3 implies the following theorem for the sufficient and necessary condition of the asymptotic stabilizability of switched system (3.1) with subsystems in the same direction.

Theorem 3.4 *The switched system (3.1) with subsystems in the same direction is asymptotically stabilizable if and only if $\|x^*\|_2 < \|x_0\|_2$ by conic switching method, where x^* and x_0 are the same as in Theorem 3.3.*

3.2 Two Subsystems in Different Directions

Now we consider switched systems with two subsystems in different directions.

We would consider switched systems (3.1), where subsystem 1 has clockwise direction and subsystem 2 has counterclockwise direction.

Similar to the case of two subsystems in the same direction, we take x, f_1 and f_2 to be vectors in $\mathbb{R}^2$. We consider the angle θ_1 ($-\pi \leq \theta_1 \leq 0$) from x to f_1. And we consider the angle θ_2 $(0 \leq \theta_2 \leq \pi)$ from x to f_2 (Figure 7).

We consider switching schemes that asymptotically stabilize the switched system. In other words, by our switching schemes, we want to lead the trajectory closer and closer to the origin, i.e., $\|x(t)\|_2 \to 0$ as $t \to 0$. In the following, we will discuss the way by which we try to obtain such switching rules by choosing a sybsystem which has the potentiality to drive the trajectory closer to the origin. In the following, sets E_{is} and E_{iu} are as defined in (3.3) and (3.4). A lot of geomtric motivation is much like those in the previous

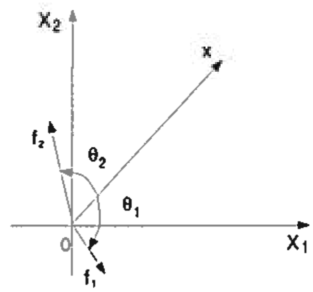

Figure 7: The angle θ_i from x to f_i, $i = 1, 2$.

subsection. Here we would firstly denote the following conic regions

$$\begin{aligned}
\Omega_1 &= E_{1s} \cap E_{2s}, & (3.21)\\
\Omega_2 &= E_{1u} \cap E_{2u}, & (3.22)\\
\Omega_3 &= E_{1s} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \geq 0\}, & (3.23)\\
\Omega_4 &= E_{1s} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \leq 0\}, & (3.24)\\
\Omega_5 &= E_{1u} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \geq 0\}, & (3.25)\\
\Omega_6 &= E_{1u} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \leq 0\}. & (3.26)
\end{aligned}$$

Notice, some Ω_i's we denote in this case are different from Ω_i's denoted in the previous subsection.

Case 1: If $Int(\Omega_1) \neq \emptyset$, it is clear that any trajectory of the switched system that is totally inside Ω_1 is stable (Figure 8(a)). Now if the initial point of the switched system is outside Ω_1, we can follow the trajectory of any one subsystem to go into Ω_1. Then we can use the conic switching rule, i.e., switch to another subsystem upon hitting the boundary of Ω_1, this will give us a trajectory that is totally inside Ω_1 and it is stable. If there exists $x \in \Omega_1$, such that $\theta_1(f_1(x)) \neq \theta_2(f_2(x))$, then the conic switching rule is asymptotically stable.

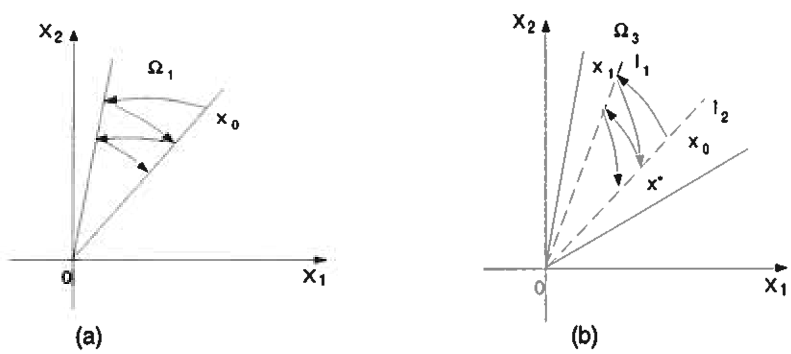

Figure 8: (a) Case 1:$Int(\Omega_1) \neq \emptyset$. (b) Case 2:$Int(\Omega_3) \neq \emptyset$.

Case 2: If $Int(\Omega_3) \neq \emptyset$. In this case, assume l_1 and l_2 be two rays that go through the origin and are in the same conic region Ω_3. And the conic region with angle from l_1 to l_2 is inside Ω_3 (Figure 8(b)). Suppose l_2 is to the clockwise side of l_1 in the conic region and suppose x_0 is on l_2, we have the following Lemma.

Lemma 3.5 *If the switched system follows subsystem 2 until the trajectory hit l_1 for the first time at x_1. Then the system switches to subsystem 1 and evolves according to subsystem 1. Suppose x^* is a point on l_2 where the trajectory hits l_2 for the first time after the switching, then we have $\|x^*\|_2 \leq \|x_0\|_2$.*

Proof: Figure 8(b) shows the case. It is quite clear that if we consider $\dot{x}(t) = A_3x(t)$ with $x(0) = x_1$, where $A_3 = -A_2$, then the trajectory is exactly the same as the trajectory from x_0 to x_1 but evolving in opposite direction. Also notice that $f_3 = (-a_3, -a_4)^T$.

Now consider $\dot{x}(t) = A_1x(t)$ and $\dot{x}(t) = A_3x(t)$, they are in the same direction. And since the conic region of angle from l_1 to l_2 are inside Ω_3 defined in (3.10), it is also inside $E_{1s} \cap E_{3s} \cap \{x|a_2(-a_3) - a_1(-a_4) \leq 0\}$. Therefore, by lemma 3.2, we obtain $\|x^*\|_2 \leq \|x_0\|_2$. □

According to Lemma 3.5, we can use the conic switching rule, i.e., switch to another subsystem upon hitting the boundary of Ω_3 so as to keep the trajectory inside Ω_3 (Figure 8(b)), this will give us a trajectory that is totally inside Ω_1 so it is stable.

Case 3: If $Int(\Omega_5) \neq \emptyset$. The the similar argument as in Case 2 can be applied and we find that the system can be stabilized by conic switching rule to keep the trajectory inside Ω_5.

The following theorem shows us that the system can only be stabilized for the above-mentioned three cases.

Theorem 3.6 *The switched system consisting of two subsystems in different direction is asymptotically stabilizable by switching rules if and only if* $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) \neq \emptyset$ *and there exists* $x \in Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5)$ *such that* $\theta_1(f_1(x)) \neq -\theta_2(f_2(x))$.

Proof: by using the conic switching rules associated with Ω_i, $i = 1, 3, 5$ if $Int(\Omega_i) \neq \emptyset$, the sufficiency is quite clear from the above discussion.

Necessity. Assume x_0 lies on ray l_1. Let α be the angle from l_1 to the line on which $x(t)$ lies. Assume s is an arbitrary switching rule such that the system switches when α is equal to $0, \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$ and starting from subsystem 2. And let the correponding points at switching moment be$x_0, x_1, x_2, \cdots, x_n, \cdots$. (The similar argument can be applied to an arbitrary switching rule starting from subsystem 1.)

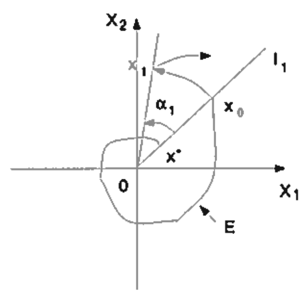

Figure 9: Figure for the proof of Theorem 3.6.

Now we consider the trajectory of subsystem 2 and go backward in time, i.e., $x(-t)$. Assume x^* is the point where the trajectory hits l_1 for the first time (at time $-t^*$) (Figure 9). Let

$$E = \{x(t)| - t^* \leq t \leq 0\}.$$

If we consider the trajectory from x_0 to x_1, it is clear that for any points x on the trajectory between x_0 to x_1, $\|x\|_2$ would be greater than or equal to the norm of the corresponding points on E. (x' on E corresponds to x_1 when they are on the same ray through the origin).

If $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) \neq \emptyset$ or $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) \neq \emptyset$ but $\forall x \in Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5)$ we have $\theta_1(f_1(x)) \neq -\theta_2(f_2(x))$. By induction, we can show that any x on the trajectory between x_k and x_{k+1} would have a norm greater than or equal to the norm of the corresponding points on E. Therefore, any x on the trajectory of the switched system would have a norm than the smallest norm of the points on E. Therefore, the switched system is not asymptotically stablized by any s, which is a contradiction to our assumption. □

4 Several Subsystems

Now we study the stabilizing switch scheme for switched systems with several LTI autonomous subsystems with unstable focus

$$\dot{x}(t) = A_i x(t), \; i = 1, 2, \cdots, N. \tag{4.1}$$

We would consider the following case.

4.1 All Subsystems in the Same Direction

Assume that all the N subsystems are in the clockwise direction (the similar argument can also be applied to several subsystems in the counterclockwise direction). Notice from the discussion of two subsystems in the same direction, we find the conic switching rule associate with each point on $\mathbb{R}^2$ with a subsystem i where θ_i is smaller. In fact, we can apply the similar method to several subsystems in clockwise direction. We let the conic switching rule associated with each point on $\mathbb{R}^2$ where θ_i is the smallest (if some $\theta_j = \theta_i$, we may choose subsystem i or j).

Except for the number of subsystems, we can follow the similar argument as in subsection 3.1, and we have the following Theorem.

Theorem 4.1 *Let l_1 be a ray that goes through the origin. Let x_0 be on l_1 and consider the conic switching rule. Let x^* on l_1 be the point where the trajectory hits l_1 for the first time after leaving x_0. The switched system (4.1) is asymptotically stabilizable if and only if $\|x^*\|_2 < \|x_0\|_2$.*

4.2 Not All Subsystems in the Same Direction

Assume that $K(K > 0)$ subsystems are in clockwise direction and $M(M > 0)$ subsystems are in counterclockwise direction $(K + M = N)$. by some observation and combination of the previous results, the following Theorem can be obtained.

Theorem 4.2 *Thw switched system with $K(K > 0)$ subsystems $S_1^-, \cdots, S_K^-$ in clockwise direction and $M(M > 0)$ subsystems $S_1^+, \cdots, S_M^+$ in counterclockwise direction is asymptotically stabilizable if and only if one of the following three conditions holds:*

1. *The switched system consisting of $S_1^-, \cdots, S_K^-$ is asymptotically stabilizable.*
2. *The switched system consisting of $S_1^+, \cdots, S_M^+$ is asymptotically stabilizable.*
3. *There exists $1 \le i \le K$ and $1 \le j \le M$ such that the switched system consisting of two subsystems S_i and S_j is asymptotically stabilizable.*

5 A Practical Algorithm and Exmaples

The conic switching rule in Section 3 and Section 4 depends on the computation of many conic regions. For each conic region, we are going to solve second-order equations of x_2 in terms of x_1 which gives us two lines as boundaries of the conic region. Yet in practice, along with the increase of the number of subsystems, the compuation can be cumbersome. So in practice, we would not compute the accurate boundaries of the conic regions but use the following algorithm to obtain the conic regions and the switching rule associated with each region. We just explain the algorithm for two subsystems, teh similar argument can be applied to several subsystems.

Assume two subsystems are in the clockwise direction. Consider a circle with center at the origin and radius r (r not too small) on $\mathbb{R}^2$. Pick n points $x_k = (r\cos\alpha_k, r\sin\alpha_k)^T$ with $\alpha_k = \frac{2\pi}{n}(k-1)$, $k = 1, 2, \cdots, n$, on the circle. Compute the $\theta_i(f_i(x_k))$ $(-\pi \leq \theta_i \leq 0)$ for subsystem 1,2 and associate a subsytem to x_k with smaller θ_i. If we use "*" to denote those points associated with subsytem 1 and "o" to denote points associated with subsytem 2. Now draw lines l_i's through the boundary of "*" and "o" regions as in Figure 10, then l_1 and l_2 will give us a rough boundary of conic regions associated with individually subsystems. In this case, some of the Ω_i's are in the same conic region by l_1 and l_2 and we do not have to identify them differently.

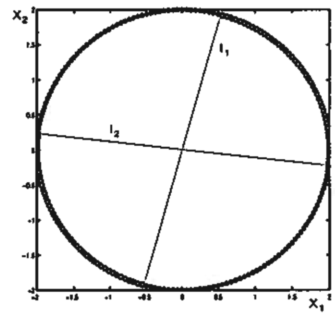

Figure 10: The boundaries of conic regions.

The similar method can be carried out to check whether $Int(\Omega_1) \cup Int(\Omega_3) \cup Int(\Omega_5) \neq \emptyset$ in the case of two subsystems in different directions to decide whether the system is stabilizable and if yes, to obtain the conic switching rule.

In the following we show several examples.

Example 5.1 *The switched system (3.1) with two subsystems in the same direction*

$$A_1 = \begin{bmatrix} 1 & 13 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}^{-1},$$

$$A_2 = \begin{bmatrix} -1 & 2 \\ -10 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}^{-1},$$

is asymptotically stabilizable using the conic switching rule, though each subsystem is unstable. Figure 11 shows the trajectories of the individual subsystem. Figure 14 shows the conic regions, the trajectory and the time domain response of the switched system.

Example 5.2 *The switched system (3.1) with two subsystems in the same direction*

$$A_1 = \begin{bmatrix} 5 & 0.2 \\ -40 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 20 \end{bmatrix}^{-1},$$

$$A_2 = \begin{bmatrix} 1 & 0.2 \\ -40 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 20 \end{bmatrix}^{-1},$$

is not asymptotically stabilizable. Figure 12 shows the trajectories of the individual subsystem. Figure 15 shows the conic regions and the trajectory of the switched system using the conic switching rule.

Example 5.3 *The switched system (3.1) with two subsystems in different directions*

$$A_1 = \begin{bmatrix} -2 & 52 \\ -8 & 6 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 20 \\ -20 & 2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}^{-1},$$

$$A_2 = \begin{bmatrix} 11 & -10 \\ 50 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 20 \\ -20 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}^{-1},$$

is asymptotically stabilizable. Figure 13 shows the trajectories of the individual subsystem. Figure 16 shows the conic regions, the trajectory and the time domain response of the switched system.

Example 5.4 *The switched system with three subsystems in the same direction*

$$A_1 = \begin{bmatrix} 1 & 0.5 \\ -50 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}^{-1},$$

$$A_2 = \begin{bmatrix} 25.75 & 25.75 \\ -25.75 & -23.75 \end{bmatrix} = \begin{bmatrix} 0.7071 & -7.071 \\ 0.7071 & 7.071 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 0.7071 & -7.071 \\ 0.7071 & 7.071 \end{bmatrix}^{-1},$$

$$A_3 = \begin{bmatrix} -23.75 & 25.75 \\ -25.75 & 25.75 \end{bmatrix} = \begin{bmatrix} 0.7071 & 7.071 \\ -0.7071 & 7.071 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 0.7071 & 7.071 \\ -0.7071 & 7.071 \end{bmatrix}^{-1},$$

is asymptotically stabilizable. Figure 17 shows the conic regions, the trajectory and the time domain response of the switched system.

6 Conclusion

In this paper, we look into the problem of finding asymptotically stabilizing switching schemes for switched systems consisting of several second-order unstable LTI autonomous subsystems. Sufficient and necessary conditions are given for the asymptotic stabilizability of the switched system.

The method developed in this paper uses the vector fields of second-order LTI autonomous subsystems. The method does not directly use Lyapunov function. And the computation of the conic regions are easy to obtain and implement. The relationship between the method in this paper and the LMI formulation problem in [4] are to be explored more clearly.

The method used in this paper is mainly depending on the geometric nature of vector fields on $\mathbb{R}^2$ plane. Since the topological structure of $\mathbb{R}^2$ and $\mathbb{R}^n, n \geq 3$ is quite different, the method may not be extended to $\mathbb{R}^n$ space directly. In fact, there has no sufficient and necessary conditions for the asymptotic stabilizability of switched systems consisting of n-th order LTI autonomous subsystems up to now. Though many LMI method has been proposed, yet they are not necessary conditions.

By look into the vector fields of second-order LTI autonomous subsystems, the reachability problem may also be solved, yet several sliding mode may appear in the process. We will not discuss it in detail here.

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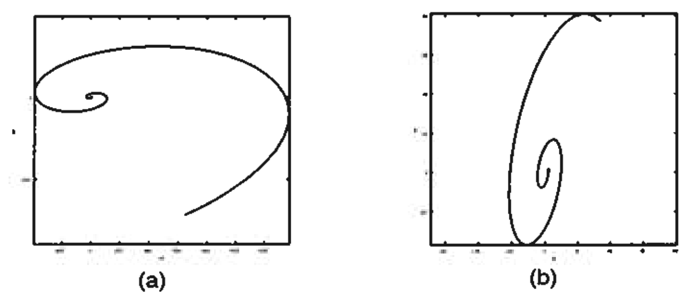

Figure 11: Example 5.1: (a) The trajectory of the subsystem 1. (b) The trajectory of the subsystem 2.($x_0 = (2,2)^T$)

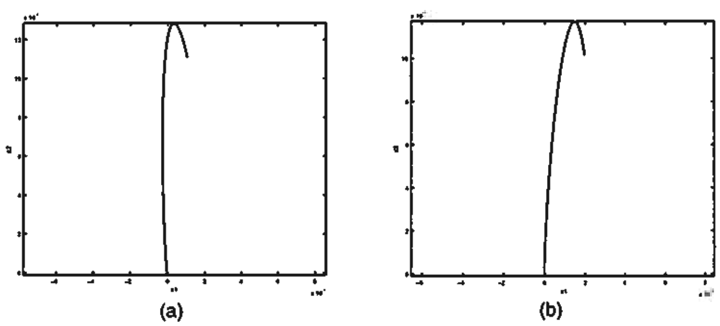

Figure 12: Example 5.2: (a) The trajectory of the subsystem 1. (b) The trajectory of the subsystem 2.($x_0 = (2,2)^T$)

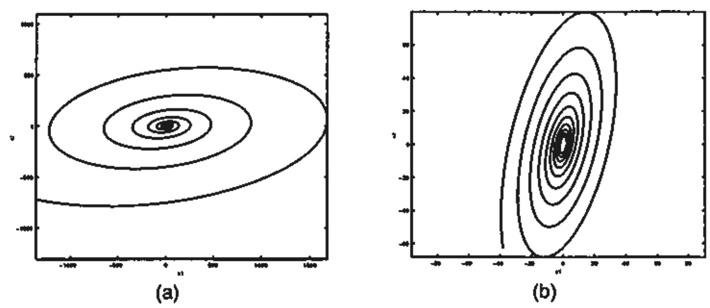

Figure 13: Example 5.3: (a) The trajectory of the subsystem 1. (b) The trajectory of the subsystem 2.($x_0 = (2,2)^T$)

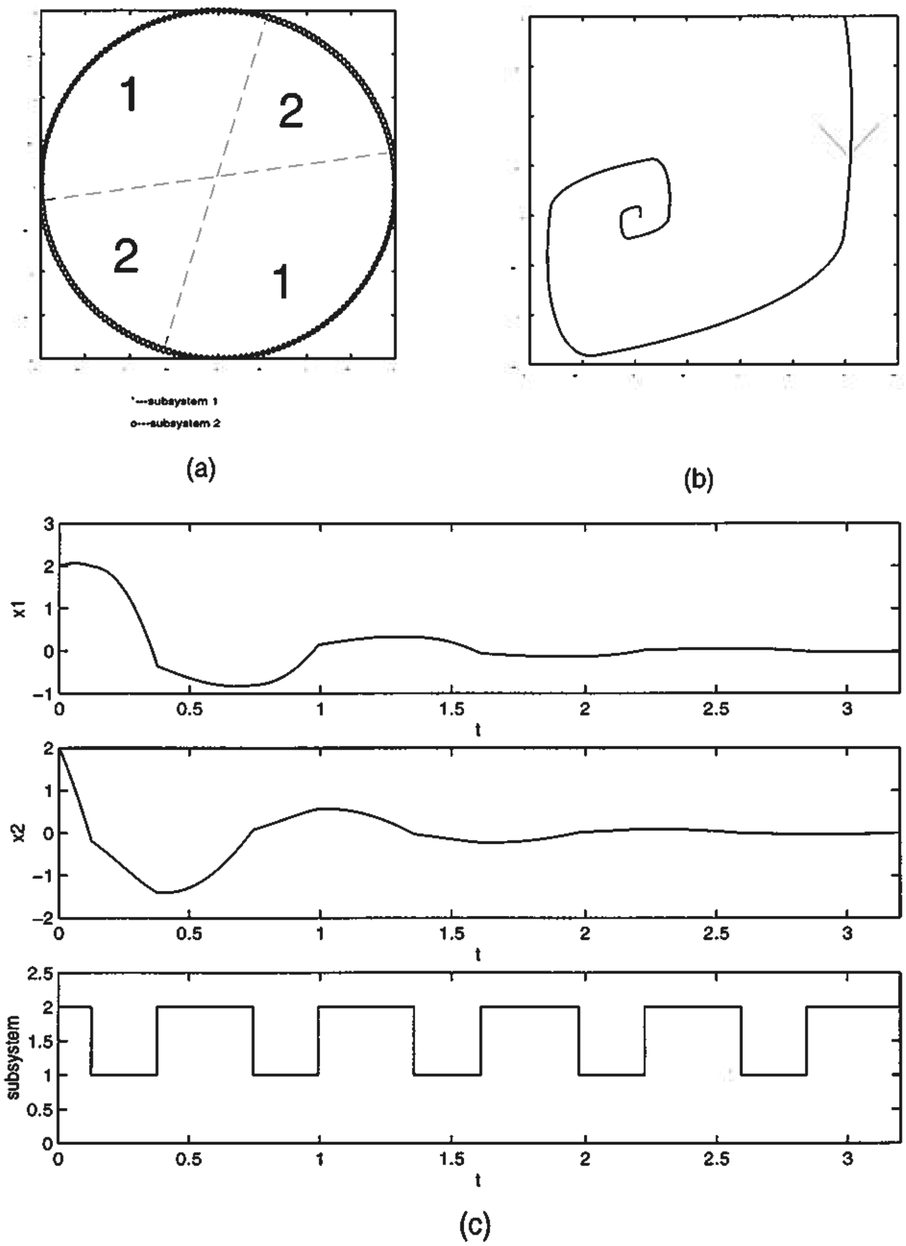

Figure 14: Example 5.1: (a) The switching region for conic rule. (b) The trajectory of the system using conic switching rule. (c) Time domain response. $(x_0 = (2, 2)^T)$

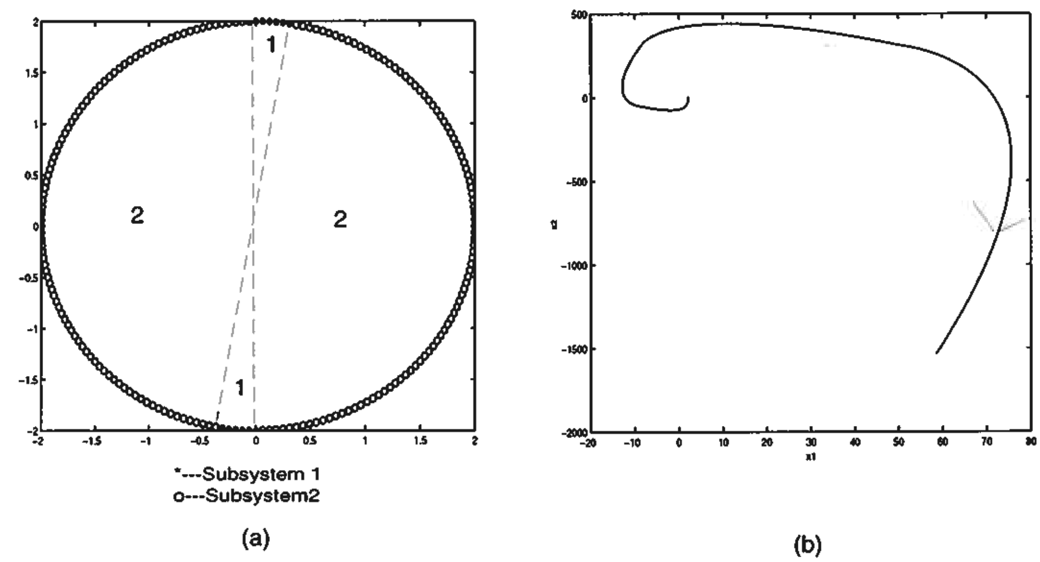

Figure 15: Example 5.2: (a) The switching region for conic rule. (b) The trajectory of the system using conic switching rule. $(x_0 = (2,2)^T)$

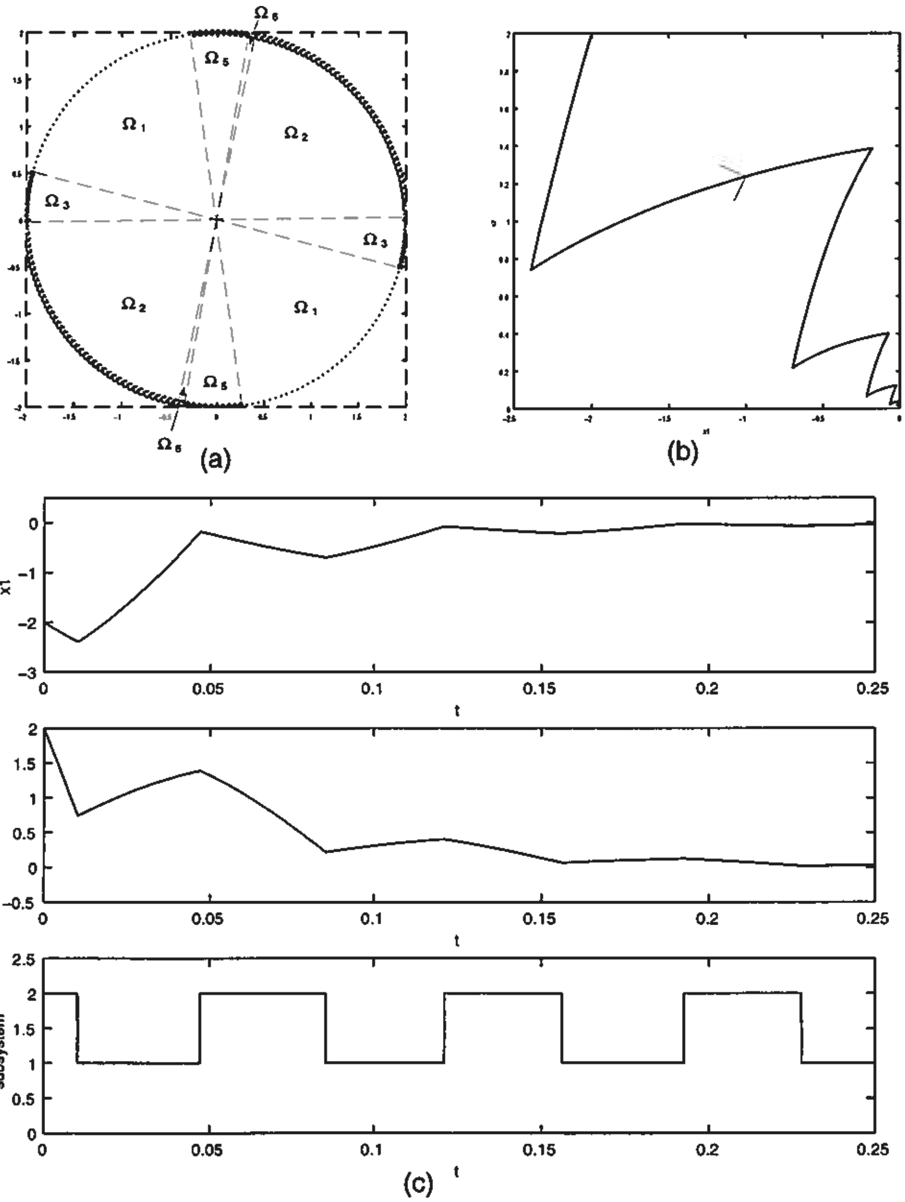

Figure 16: Example 5.3: (a) The conic regions. (b) The trajectory of the system. (c) Time domain response. $(x_0 = (-2, 2)^T)$

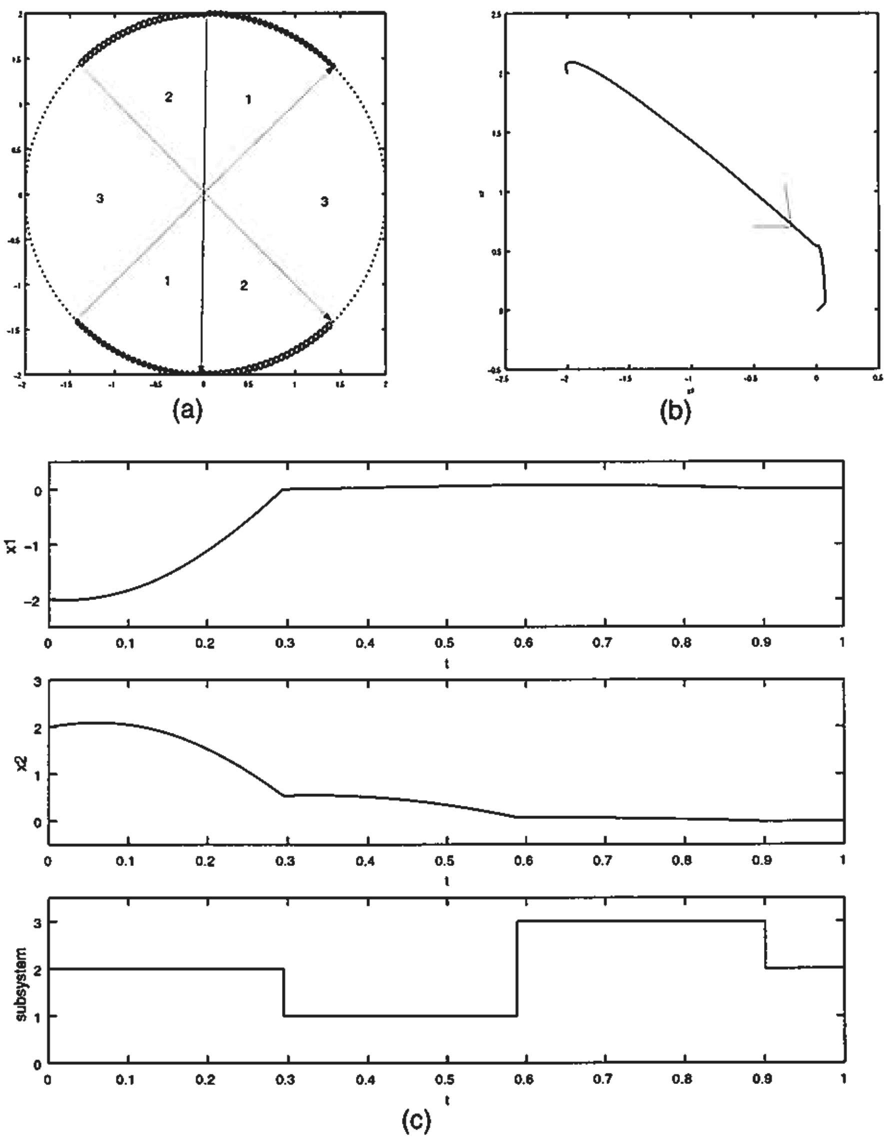

Figure 17: Example 5.4: (a) The switching region for conic rule. (b) The trajectory of the switched system. (c) Time domain response. ($x_0 = (-2, 2)^T$)