

# Stabilization of Second-Order LTI Switched Systems

Xuping Xu<sup>1</sup> and Panos J. Antsaklis  
 Department of Electrical Engineering  
 University of Notre Dame  
 Notre Dame, IN 46556, USA  
 {xxu, antsaklis.1}@nd.edu

## Abstract

In this paper, the problem of asymptotically stabilizing switched systems consisting of second-order LTI subsystems is studied and solved. In a previous paper, the stabilization problem of switched systems consisting of subsystems with unstable foci was addressed. This paper extends the method therein to the stabilization of switched systems consisting of general second-order LTI subsystems. Necessary and sufficient conditions for stabilizability for such systems are obtained. Stabilizing switching control laws are also derived if the system is asymptotically stabilizable. Examples throughout the paper illustrate the approach and results.

## 1 Introduction

A switched system is a system that consists of several subsystems and a (switching) law that specifies which subsystem dynamics will be followed by the system trajectory at each instant of time. Switching is needed for the stabilization of a switched system if none of its subsystems is stable.

There are many papers on stability analysis and design of switched systems (see, e.g., [1, 3, 5, 6]). Common approaches include common Lyapunov function, multiple Lyapunov functions and LMI approaches which usually provide sufficient conditions for stability. In [10], the authors focused on the design of stabilizing switching control laws for switched systems consisting of second-order LTI subsystems with unstable foci. Necessary and sufficient conditions were established for the asymptotic stabilizability of such kind of switched systems. And conic stabilizing switching control laws were derived when the switched system was stabilizable.

In this paper, we extend the method in [10] to stabilize switched systems consisting of general second-order LTI subsystems. Specifically, we carefully study switched systems consisting of two second-order LTI subsystems with unstable nodes and saddle points (Sections 3 and 4). Necessary and sufficient conditions are also obtained for stabilizability of such systems. If a switched system is asymptotically stabilizable, then stabilizing switching control laws can be obtained based on the conic switching laws. Miscellaneous examples are presented to show the effectiveness of the method throughout the paper. Additional details for the methods and proofs in the paper can be found in [9].

## 2 Stabilization of Second-Order LTI Switched Systems with Foci

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First of all, we review some stabilization results from [10]. Note that these results are constructive and the stabilizability conditions derived are necessary and sufficient as opposed to other literature results.

A second-order system  $\dot{x} = Ax$  is said to be with focus, or node, or saddle point if the eigenvalues of  $A$  are complex conjugates or real numbers with the same sign, or real numbers with opposite signs, respectively (see [4] Chapter 1 for details). In the following, we say that the direction of a subsystem at  $x \neq 0$  is clockwise (resp. counterclockwise) if, starting from  $x$ , its trajectory has the potential to travel clockwise (resp. counterclockwise).

### 2.1 Two Subsystems of the Same Direction

Consider the following switched system,

$$\dot{x}(t) = A_1 x(t), \quad \dot{x}(t) = A_2 x(t), \quad (2.1)$$

whose subsystems are both with unstable foci and of clockwise directions. Note that the following method also applies for the case of counterclockwise directions.

Let  $x = (x_1, x_2)^T$  be a nonzero point on  $\mathbb{R}^2$  plane, and let  $f_1 = A_1 x = (a_1, a_2)^T$ ,  $f_2 = A_2 x = (a_3, a_4)^T$ . We can view  $x$ ,  $f_1$  and  $f_2$  as vectors in  $\mathbb{R}^2$  and denote  $\theta_i, i = 1, 2$  to be the angle from  $x$  to  $f_i$  measured counterclockwise with respect to  $x$  ( $\theta_i$  is confined to  $-\pi \leq \theta_i < \pi$ ). So in this case, when the  $i$ th subsystem is of clockwise direction,  $-\pi \leq \theta_i \leq 0$ .

As in [10], we define the regions

$$\begin{aligned} E_{is} &= \{x | x^T f_i(x) = x^T A_i x \leq 0\}, \quad i = 1, 2, \\ E_{iu} &= \{x | x^T f_i(x) = x^T A_i x \geq 0\}, \quad i = 1, 2. \end{aligned}$$

In other words, the interior of  $E_{is}$  ( $E_{iu}$ ) is the set of all points on  $\mathbb{R}^2$  plane where the trajectory of the  $i$ th subsystem would be driven closer to (farther from) the origin if the subsystem evolves for sufficiently small amount of time starting from the point.

To design stabilizing switching control laws, we define the following conic regions.

$$\begin{aligned} \Omega_1 &= E_{1s} \cap E_{2u}, \quad \Omega_2 = E_{1u} \cap E_{2s}, \\ \Omega_3 &= E_{1s} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \leq 0\}, \\ \Omega_4 &= E_{1s} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \geq 0\}, \\ \Omega_5 &= E_{1u} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \leq 0\}, \\ \Omega_6 &= E_{1u} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \geq 0\}. \end{aligned}$$

The conic switching law is as follows: switch the switched system to subsystem 1 whenever the system trajectory enters  $\Omega_1, \Omega_3, \Omega_5$  and switch to subsystem 2 whenever the system trajectory enters  $\Omega_2, \Omega_4, \Omega_6$ .

Intuitively, the conic switching law just chooses the subsystem which has the potential to drive the trajectory closer to the origin at each point  $x$ . The following theorem is concerned with the stabilizability of the switched system (see [10]).

**Theorem 2.1** *Let  $l_1$  be a ray that goes through the origin. Let  $x_0 \neq 0$  be on  $l_1$ . Let  $x^*$  be the point on  $l_1$  where the trajectory intersects  $l_1$  for the first time after leaving  $x_0$ , when the switched system evolves according to the conic switching law. The switched system (2.1) consisting of two subsystems with unstable foci and of the same direction is asymptotically stabilizable if and only if  $\|x^*\|_2 < \|x_0\|_2$ .*

**Remark:** The approach mentioned above may look like the min-projection strategy proposed in [7], yet the difference between the two is that our approach chooses subsystem according to the angle from  $x$  to  $f$  while the min-projection depends on the projection of  $f$  on  $x$ .

## 2.2 Two Subsystems of Opposite Directions

Consider the switched system (2.1) whose subsystems are both with unstable foci, and with subsystem 1 of clockwise direction and subsystem 2 of counterclockwise direction.

We introduce the following conic regions.

$$\begin{aligned}\Omega_1 &= E_{1s} \cap E_{2s}, \quad \Omega_2 = E_{1u} \cap E_{2u}, \\ \Omega_3 &= E_{1s} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \geq 0\}, \\ \Omega_4 &= E_{1s} \cap E_{2u} \cap \{x | a_2 a_3 - a_1 a_4 \leq 0\}, \\ \Omega_5 &= E_{1u} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \geq 0\}, \\ \Omega_6 &= E_{1u} \cap E_{2s} \cap \{x | a_2 a_3 - a_1 a_4 \leq 0\}.\end{aligned}$$

The following theorem is concerned with the stabilizability of the switched system (see [10]).

**Theorem 2.2** *The switched system (2.1) consisting of two subsystems with unstable foci and of opposite directions is asymptotically stabilizable if and only if  $\text{Int}(\Omega_1) \cup \text{Int}(\Omega_3) \cup \text{Int}(\Omega_5) \neq \emptyset$ .*

If the switched system is asymptotically stabilizable, then a conic asymptotic stabilizing switching law can also be obtained. The **conic switching law** is as follows: first, by following subsystem 1, force the trajectory into the interior of one of the conic regions  $\Omega_1, \Omega_3, \Omega_5$  (there must be one available), and then switch to another subsystem upon intersecting the boundary of the region so as to keep the trajectory inside the region.

**Remark:** Let  $\theta_{f_1 f_2}$  denote the angle from vector  $f_1$  to  $f_2$  measured counterclockwise. Notice that a point  $x$  is in  $\Omega_1$  or  $\Omega_3$  or  $\Omega_5$ , if and only if the absolute value of the angle  $\theta_{f_1 f_2}$  satisfies  $|\theta_{f_1 f_2}| = |\theta_1| + |\theta_2| \geq \pi$ .

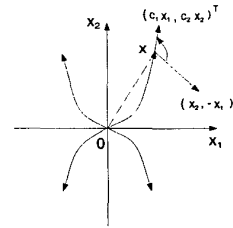
**Remark:** The above method may seem to be close to using  $V(x) = x^T x$  as a Lyapunov function. Yet  $V(x)$  is not monotonically decreasing in  $\Omega_3$  and  $\Omega_5$ , hence  $V(x)$  may not be readily used here.

## 3 Stabilization of Second-Order LTI Switched Systems with Nodes

In this section, we study and design stabilizing switching laws for switched systems consisting of subsystems with unstable nodes. We mainly study switched systems consisting of two subsystems. The results can similarly be applied to several subsystems.

Let us begin our discussion by looking into the simplest second-order LTI system where  $A = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ ,

$c_1 > 0, c_2 > 0$ . Figure 1 shows the trajectory of the system when  $c_2 > c_1$  (note in this section, we generally do not consider the special case  $c_1 = c_2 > 0$ , though the case can also be analyzed using similar techniques).



**Figure 1:** Trajectory of the simplest second-order system with unstable node.

For the trajectory in quadrant I,  $(x_2, -x_1) \cdot (c_1 x_1, c_2 x_2)^T = (c_1 - c_2)x_1 x_2 < 0$ , so the trajectory travels counterclockwise in quadrant I. Similarly, the trajectories in quadrant II, III and IV travel clockwise, counterclockwise and clockwise, respectively.

For a general second-order system with unstable node, linear transformation can be used to find a  $Y_2 - Y_1$  coordinate system in which the system has the simplest form of system with node. So we can analyze the direction of the solution at each point in the  $Y_2 - Y_1$  coordinate system and then translate the result back to the original coordinate system.

Now we consider the switched system (2.1) with both subsystems having unstable nodes. In the following, we exclude the trivial case  $A_1 = cA_2$  for some fixed  $c$ . So whenever  $\text{Int}(\Omega_1) \cup \text{Int}(\Omega_3) \cup \text{Int}(\Omega_5) \neq \emptyset$ , asymptotic stabilizability can be assured (Here and in the following,  $\Omega_1, \Omega_3, \Omega_5$  are as defined in Section 2.2). Let

$$\begin{aligned}E_{ic} &= \{x = (x_1, x_2)^T | (x_2, -x_1)A_i(x_1, x_2)^T \geq 0\}, \\ E_{icc} &= \{x = (x_1, x_2)^T | (x_2, -x_1)A_i(x_1, x_2)^T \leq 0\}.\end{aligned}$$

In other words,  $E_{ic}$  (resp.  $E_{icc}$ ),  $i = 1, 2$  denotes the conic regions where the  $i$ th subsystem trajectory travels clockwise (resp. counterclockwise).

With these notations in mind, we let  $E_{c,c} = E_{1c} \cap E_{2c}$ , which denotes the conic region in which both the trajectories of subsystem 1 and 2 travel clockwise. In the same manner, we can define  $E_{c,cc} = E_{1c} \cap E_{2cc}$ ,  $E_{cc,c} = E_{1cc} \cap E_{2c}$  and  $E_{cc,cc} = E_{1cc} \cap E_{2cc}$ .

Figure 2 shows exhaustively the six possible arrangements of  $E_{c,c}, E_{c,cc}, E_{cc,c}$  and  $E_{cc,cc}$ . Without loss of generality, we illustrate these cases by fixing  $E_{1cc}$  to be in quadrants I, III and  $E_{1c}$  to be in quadrants II, IV, respectively. (We can always do so by some linear transformation, which will not affect the applicability of what we will discuss later).

**Case 1.** We only discuss Case 1(a) (Figure 2 Case 1(a)), since similar argument can be applied to Case 1(b).

Notice that in this case the two subsystems are of the same direction in  $E_{c,c}, E_{cc,cc}$ , which means that any trajectory starting in the conic region will eventually leave the conic region if the switched system stay at both subsystem 1 and subsystem 2 long enough. We call this kind of conic regions **Type A regions**. Of course, if the switched system stick to subsystem 1 in  $E_{c,c}$ , it will not leave  $E_{c,c}$ , yet the system will not be stable, so this kind of strategy is not of interest to us.

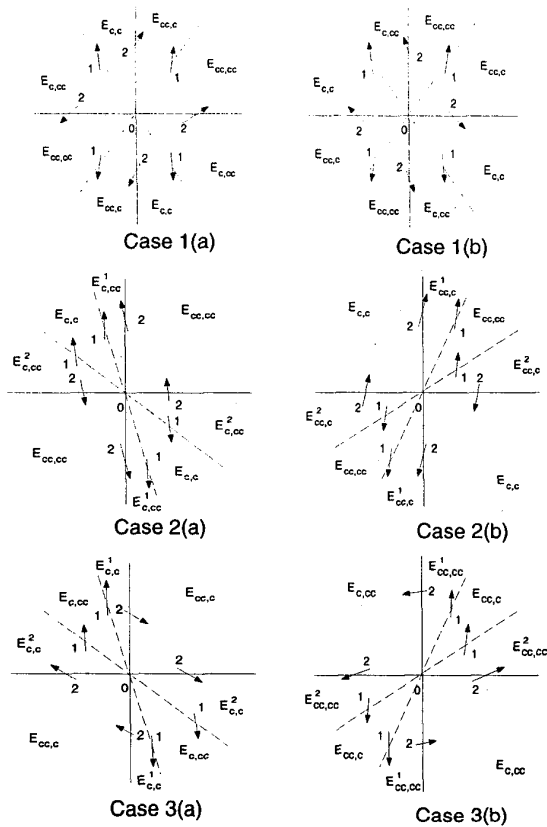


Figure 2: Cases for subsystems with nodes.

The conic region  $E_{cc,c}$  has the property that whenever a trajectory goes into it, the trajectory can never leave it again, we call it a **Type B region**.

For conic region  $E_{c,cc}$ , the trajectory of the systems can always leave the conic region from both boundaries under appropriate switchings. The region is called a **Type C region**.

Next we introduce a theorem.

**Theorem 3.1** In Case 1(a), for the Type B region  $E_{cc,c}$ , we have  $\text{Int}(E_{cc,c} \cap \Omega_i) = \emptyset$ ,  $i = 1, 3, 5$ .

**Proof:** See [9].  $\square$

In view of Theorem 3.1, once the trajectory goes into the Type B region  $E_{cc,c}$ , the trajectory cannot be stabilized for any switching sequences. So in general, the switched system is *not stabilizable* because stabilizability in general requires stabilizability from any initial point.

Also notice that in the Type C region  $E_{c,cc}$ , it can be shown that  $\text{Int}(E_{c,cc} \cap \Omega_i) = \emptyset$ ,  $i = 1, 3, 5$ .

*So in this case, the switched system cannot be stabilized regardless of the initial point.*  $\square$

**Case 2.** We only discuss Case 2(a) (Figure 2 Case 2(a)). Case 2(b) is analogous.  $E_{c,c}$  and  $E_{cc,cc}$  are Type A regions.  $E_{c,cc}^1$  is a Type B region and  $E_{c,cc}^2$  is a Type C region.

In this case, it can be shown that  $\text{Int}(E_{c,cc}^1 \cap \Omega_i) = \emptyset$ ,  $i = 1, 3, 5$ . So we conclude that the system is *not asymptotically stabilizable*. Yet, it is still possible for the tra-

jectory starting in the Type C regions to be stabilized by using some switching laws as long as the region satisfies the condition of the following theorem.

**Theorem 3.2** In Case 2(a), the trajectory starting inside the Type C region  $E_{c,cc}^2$  can be asymptotically stabilized if and only if  $\text{Int}(E_{c,cc}^2 \cap \Omega_1) \cup \text{Int}(E_{c,cc}^2 \cap \Omega_3) \cup \text{Int}(E_{c,cc}^2 \cap \Omega_5) \neq \emptyset$ .

**Proof:** See [9].  $\square$

In this case, it is possible that the conditions in Theorem 3.2 be satisfied, hence the trajectory starting from inside the Type C regions might be stabilized using the conic switching laws as in Section 2.2.

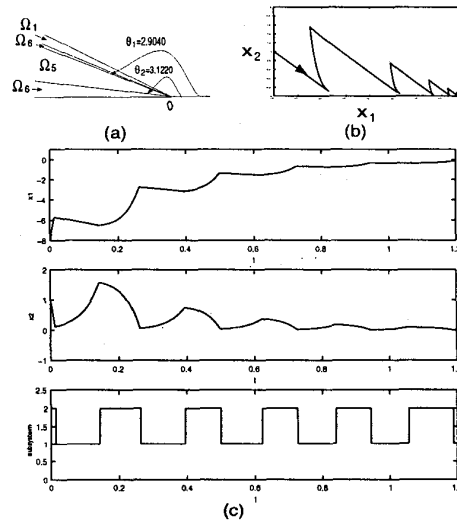


Figure 3: Example 3.3.

**Example 3.3** Consider the switched system consisting of two subsystems with unstable nodes with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 20 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -37 & -152 \\ 14.25 & 58 \end{bmatrix}.$$

Using the conic switching law, the trajectory starting in  $E_{c,cc}^2$  is asymptotically stabilizable. Figure 3 shows (a) conic regions in  $E_{c,cc}^2$ , (b) the trajectory of the system in  $\Omega_5$ , (c) time domain responses, with  $x_0 = (-8, 1)^T$ .  $\square$

**Case 3.** We only discuss Case 3(a) (Figure 2 Case 3(a)). Case 3(b) is analogous. In this case,  $E_{c,c}^1$  and  $E_{c,c}^2$  are Type A regions. It can be shown that  $\text{Int}(E_{c,cc} \cap \Omega_i) = \emptyset$  and  $\text{Int}(E_{cc,c} \cap \Omega_i) = \emptyset$ ,  $i = 1, 3, 5$ . Hence, the trajectory must not always be inside  $E_{c,cc}$  ( $E_{cc,c}$ ) in order to possibly make the trajectory stable.

In fact, we can adopt similar (one round) test as in the case of two subsystems with unstable foci of the same direction in Section 2.1. In particular, the **conic switching law** can be chosen as follows. The corresponding subsystem can be chosen as subsystem 1 in  $E_{c,cc}$ , subsystem 2 in  $E_{cc,c}$ . While in  $E_{c,c}^1$  and  $E_{c,c}^2$ , the subsystem is chosen similar to the case of unstable foci of the same direction, the subsystem which has the potential to drive the system trajectory closer to the origin

is chosen at each point. The following theorem provides a necessary and sufficient condition for the asymptotical stabilizability of the switched system.

**Theorem 3.4** Let  $l_1$  be a ray that goes through the origin. Let  $x_0$  be on  $l_1$ . If  $x^*$  is the point on  $l_1$  where the trajectory intersects  $l_1$  for the first time after leaving  $x_0$ , when the switched system evolves according to the conic switching law stated above. Then the switched system as in Case 3(a) is asymptotically stabilizable if and only if  $\|x^*\|_2 < \|x_0\|_2$ .

**Proof:** See [9].  $\square$

The conditions in the above theorem is not easy to check. The following corollary provides a simplified sufficient condition to check whether a switched system is not stabilizable.

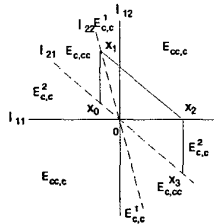


Figure 4: Figure for Corollary 3.5.

**Corollary 3.5 (The Parallelogram Sufficient Condition)** If in Case 3(a), the boundaries of the regions are denoted as  $l_{11}, l_{12}, l_{21}, l_{22}$  as shown in Figure 4. Let  $x_0$  be a nonzero point on  $l_{21}$ . If  $x_0x_1x_2x_3$  is a parallelogram on  $\mathbb{R}^2$ , where  $x_1 \in l_{11}, x_2 \in l_{12}, x_3 \in l_{21}$  and  $x_0x_1 \parallel x_2x_3 \parallel l_{12}, x_1x_2 \parallel x_0x_3$ . Then the switched system is not asymptotically stabilizable if  $\|x_3\|_2 \geq \|x_0\|_2$ .

**Proof:** See [9].  $\square$

**Example 3.6** Consider the switched system consisting of two subsystems with unstable nodes with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 58 & 152 \\ -14.25 & -37 \end{bmatrix}.$$

Using the conic switching law, the switched system is asymptotically stabilizable. Figure 5 shows (a) the regions for subsystems, (b) the trajectory of the system, (c) time domain responses, with  $x_0 = (-4, 0.5)^T$ .  $\square$

#### 4 Stabilization of Second-Order LTI Switched Systems with Saddle Points

In this section, we study and design stabilizing switching laws for switched systems consisting of subsystems with saddle points. We mainly study switched systems consisting of two subsystems. The results can similarly be applied to several subsystems.

Let us begin our discussion by looking into the simplest second-order system where  $A = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}$ . Here we assume  $c_1 > 0 > c_2$ . Figure 6 shows the trajectory of the system.

For the trajectory in quadrant I, we have  $(x_2, -x_1) \cdot (c_1x_1, c_2x_2)^T = (c_1 - c_2)x_1x_2 > 0$ , so the trajectory travels clockwise in quadrant I. Similarly, the trajectories in quadrant II, III and IV travel counterclockwise,

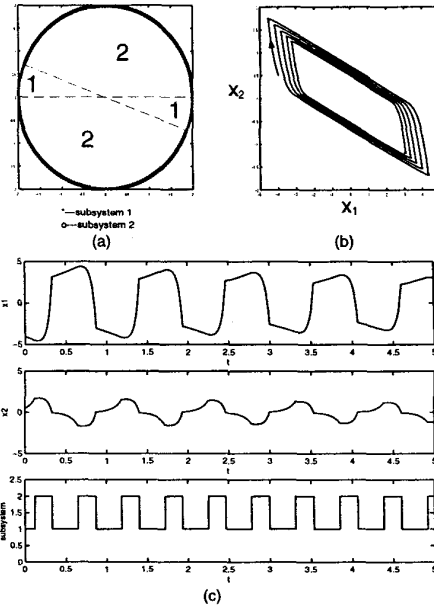


Figure 5: Example 3.6.

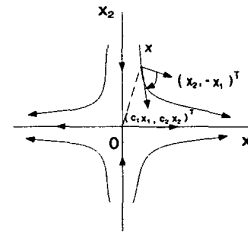


Figure 6: Trajectory of the simplest second-order system with saddle point.

clockwise and counterclockwise, respectively. While on  $X_2$  axis, the trajectory tends toward the origin. On  $X_1$  axis, the trajectory tends toward  $\infty$ .

Now consider the switched system (2.1) whose subsystems are both with saddle points. Similarly to the previous section, we can define  $E_{ic}, E_{icc}, i = 1, 2$  to be the conic regions where the  $i$ th subsystem trajectory travels clockwise (counterclockwise).

Figure 7 shows exhaustively the six possible arrangements of  $E_{c,c}, E_{c,cc}, E_{cc,c}$  and  $E_{cc,cc}$ . As in Section 3, we illustrate these cases by fixing  $E_{1c}$  to be in the I, III quadrants and  $E_{1cc}$  to be in the II, IV quadrants.

**Case 1.** We only discuss Case 1(a) (Figure 7 Case 1(a)). Case 1(b) is analogous. In this case,  $E_{c,c}, E_{cc,cc}$  are Type A regions.  $E_{cc,c}$  is a Type B region and  $E_{c,cc}$  is a Type C region.

In this case, we have the following theorem.

**Theorem 4.1** In Case 1(a), for the Type C region  $E_{c,cc}$ ,  $Int(E_{c,cc} \cap \Omega_1) \cup Int(E_{c,cc} \cap \Omega_3) \cup Int(E_{c,cc} \cap \Omega_5) \neq \emptyset$ .

**Proof:** See [9].  $\square$

For the Type B region  $E_{cc,c}$ , we have the following

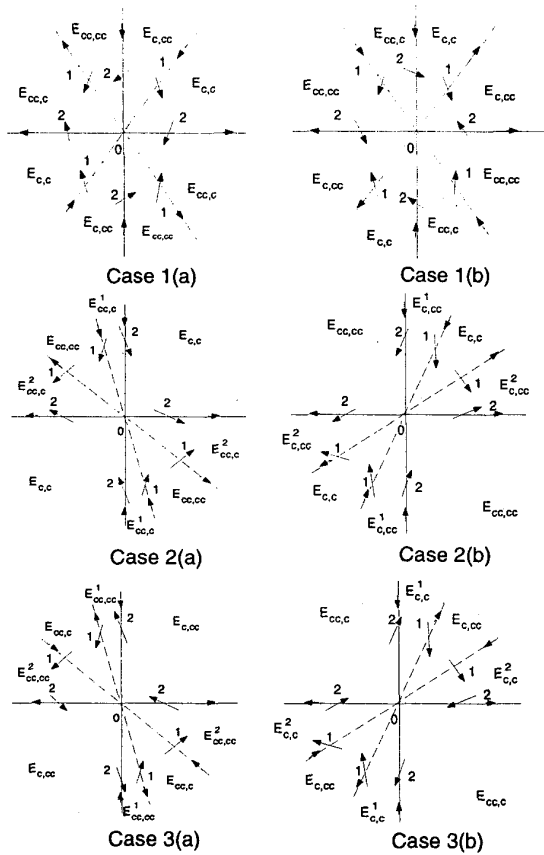


Figure 7: Cases for subsystems with saddle points.

theorem.

**Theorem 4.2** If  $Int(E_{cc,c} \cap \Omega_1) \cup Int(E_{cc,c} \cap \Omega_3) \cup Int(E_{cc,c} \cap \Omega_5) \neq \emptyset$ , then any trajectory starting in  $E_{cc,cc}$ ,  $E_{c,c}$  and  $E_{cc,c}$  can be asymptotically stabilized.

By Theorem 4.1 and 4.2, the system is asymptotically stabilizable from any initial point on  $\mathbb{R}^2$  if and only if the condition of Theorem 4.2 holds. And conic switching law can also be obtained similarly to Section 2.2.

**Example 4.3** Consider the switched system consisting of two subsystems with saddle points with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -10 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4.5 & -5.5 \\ -5.5 & -4.5 \end{bmatrix}.$$

The trajectory starting in  $E_{cc,c}$  is asymptotically stabilizable. Figure 8 shows (a) conic regions in  $E_{cc,c}$ , (b) the trajectory of the system in a conic region in  $\Omega_1$ , (c) time domain responses, with  $x_0 = (-2, 0.8)^T$ .  $\square$

**Case 2.** We only discuss Case 2(a) (Figure 7 Case 2(a)). Case 2(b) is analogous. In this case,  $E_{c,c}$  and  $E_{cc,cc}$  are Type A regions,  $E_{cc,c}^2$  is a Type B region and  $E_{cc,c}^1$  is a Type C region.

In this case, it can be shown that  $Int(E_{cc,c}^1 \cap \Omega_1) \cup Int(E_{cc,c}^1 \cap \Omega_3) \cup Int(E_{cc,c}^1 \cap \Omega_5) \neq \emptyset$ . Furthermore,

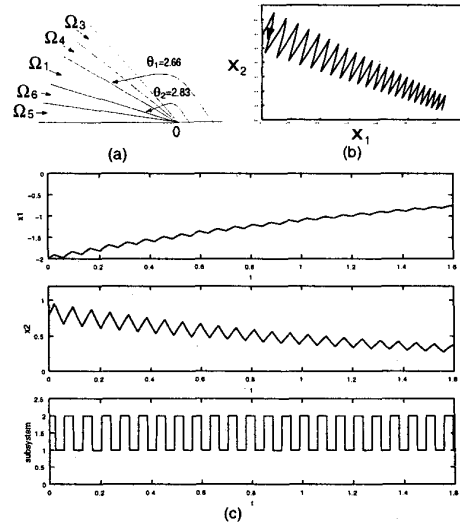


Figure 8: Example 4.3.

we can also prove that  $Int(E_{cc,c}^2 \cap \Omega_i) = \emptyset$ ,  $i = 1, 3, 5$ . Therefore, in this case, the system is *not asymptotically stabilizable*. Yet for initial points in  $E_{cc,c}^1$ , we can use the conic switching law as in Section 2.2 to keep the trajectory in  $E_{cc,c}^1$  so that the trajectory can be stabilized.

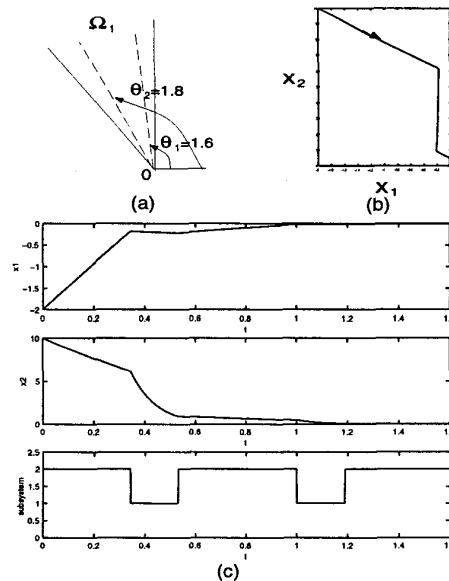


Figure 9: Example 4.4.

**Example 4.4** Consider the switched system consisting of two subsystems with saddle points with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -10 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.6667 & 0.8889 \\ -2.0000 & -1.6667 \end{bmatrix}.$$

The trajectory starting in  $E_{cc,c}^1$  is asymptotically stabilizable.

lizable since  $\text{Int}(E_{cc,c}^1 \cap \Omega_1) \neq \emptyset$ . We keep the trajectory between the rays  $l_1$  with angle 1.6 and  $l_2$  with angle 1.8. Figure 9 shows (a)  $l_1$  and  $l_2$  in  $E_{cc,c}^1$ , (b) the trajectory of the system in a conic region between  $l_1$  and  $l_2$ , (c) time domain responses, with  $x_0 = (-2, 10)^T$ .  $\square$

**Case 3.** We only discuss Case 3(a) (Figure 7 Case 3(a)). Case 3(b) is analogous. In this case,  $E_{cc,cc}^1$  and  $E_{cc,cc}^2$  are Type A regions.

In this case, we can prove that  $\text{Int}(E_{c,cc} \cap \Omega_1) \cup \text{Int}(E_{c,cc} \cap \Omega_3) \cup \text{Int}(E_{c,cc} \cap \Omega_5) \neq \emptyset$ . This is because for a point  $x \in E_{c,cc}$  which is very close to the  $X_2$  axis, we will have  $|\theta_{f_1, f_2}| > \pi$ .

Furthermore, we claim that the system is stabilizable in this case. This is because for any initial point  $x \in \mathbb{R}^2$ , we can always first choose appropriate switchings such that the system trajectory is driven into  $E_{c,cc}$  and then adopt the conic switching law as in Section 2.2 so as to keep the system trajectory in  $\text{Int}(E_{c,cc} \cap \Omega_1)$  or  $\text{Int}(E_{c,cc} \cap \Omega_3)$  or  $\text{Int}(E_{c,cc} \cap \Omega_5)$ . In this way, the system can be asymptotically stabilized.

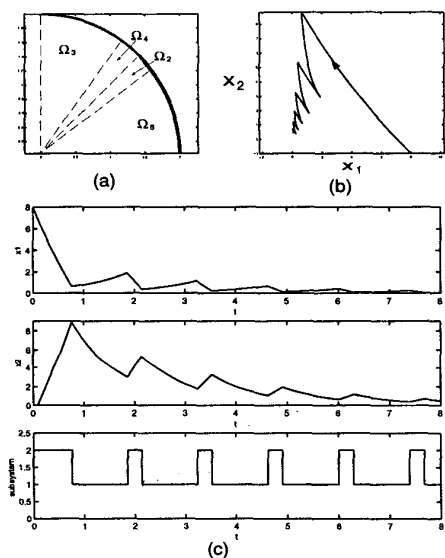


Figure 10: Example 4.5.

**Example 4.5** Consider the switched system consisting of two subsystems with saddle points with

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.5714 & -0.8571 \\ 1.7143 & 1.5714 \end{bmatrix}.$$

The trajectory is asymptotically stabilizable since  $\text{Int}(E_{c,cc} \cap \Omega_3) \neq \emptyset$  and trajectories from any initial point can be driven into  $E_{c,cc} \cap \Omega_3$ . Here we keep the trajectory between the rays  $l_1$  with angle 1 and  $l_2$  with angle 1.5. Figure 10 shows (a) conic regions in  $E_{c,cc}$ , (b) the trajectory of the system, (c) time domain responses, with  $x_0 = (8, -1)^T$ .  $\square$

## 5 Conclusions

In this paper, we study the stabilization problem for switched systems consisting of general second-order

LTI subsystems. Necessary and sufficient conditions were obtained for the asymptotic stabilizability of such switched systems. If the system is asymptotically stabilizable, stabilizing laws were also derived. The method developed in this paper uses the geometric properties of vector fields of second-order LTI systems. The stabilization problem is studied according to the type of the origin (focus, node or saddle point) and the direction of the vector field at a point (clockwise, or counterclockwise). For switched systems consisting of subsystems with mixed types of the origin (e.g., one subsystem with focus and the other one with node), similar techniques can be adopted to find the modified conic switching laws. The method does not use Lyapunov functions directly. Note that the computation to obtain the conic regions are easy to carry out and the method can readily be used to study several subsystems. It should also be noted that the conic switching laws have been shown in [2] to be robust.

Since the topological structures of  $\mathbb{R}^2$  and  $\mathbb{R}^n, n \geq 3$  are quite different, it may be difficult to extend the method to  $\mathbb{R}^n$  space directly. Note that up to now, there has been no necessary and sufficient conditions for the asymptotic stabilizability of switched systems consisting of  $n$ -th ( $n \geq 3$ ) order LTI subsystems.

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