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Stabilization of second-order LTI switched systems

XUPING XU†‡ and PANOS J. ANTSAKLIS†

This paper studies and solves the problem of asymptotic stabilization of switched systems consisting of unstable second-order linear time-invariant (LTI) subsystems. Necessary and sufficient conditions for asymptotic stabilizability are first obtained. If a switched system is asymptotically stabilizable, then the conic switching laws proposed in the paper are used to construct a switching law that asymptotically stabilizes the system. Switched systems consisting of two subsystems with unstable foci are studied first and then the results are extended to switched systems with unstable nodes and saddle points. The results are applicable to switched systems that consist of more than two subsystems.

1. Introduction

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law specifying the active subsystem at each instant of time. The interest in switched systems stems from the fact that many real-world processes and systems in, for example, chemical, transportation and communication industries, can be modelled as switched systems (see, e.g. Morse 1997).

There have already been many results on the stability analysis of switched systems. Most of the results are based on Lyapunov's direct method, especially, on the common Lyapunov function (CLF) approach (see, e.g. Narendra and Balakrishnan 1994, Ooba and Funahashi 1997, Shorten and Narendra 1997, 1998, Skafidas *et al.* 1999) or the multiple Lyapunov functions (MLFs) approach (see, e.g. Peleties and DeCarlo 1991, Pettersson and Lennartson 1996, Branicky 1998, Johansson and Rantzer 1998, Pettersson 1999). Linear matrix inequalities (LMIs) are often formulated to help search for Lyapunov functions (see, e.g. Pettersson and Lennartson 1996, Johansson and Rantzer 1998, Pettersson 1999). Liberzon and Morse (1999), Michel (1999) and DeCarlo *et al.* (2000) provide good surveys of the related results. However, results using Lyapunov functions typically provide sufficient-only conditions for stability. The only necessary and sufficient conditions that the authors are aware of are the necessary and sufficient condition for the existence of a CLF for two second-order stable linear systems (Shorten and Narendra 1998) and the necessary and sufficient condition for the test of quadratic stability (Skafidas *et al.* 1999). Note that the first result is applicable only to stable subsystems and the second result is applicable only to a very special type of stability which demands strict decrease in some Lyapunov function.

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Another important related problem for switched systems is the control design for stability problem (stabilization). In many situations, one is given a collection of unstable subsystems and it is his or her task to design a stabilizing switching law. There have been very few such stabilization methods reported in the literature. Stability design methods based on CLF or MLFs and using LMIs have been proposed in Wicks *et al.* (1994), Malmborg *et al.* (1996), Wicks and DeCarlo (1997), Wicks *et al.* (1998), Pettersson (1999) and Skafidas *et al.* 1999). Pettersson (1999) proposed a min-projection strategy for the stability design. However, the above-mentioned methods may be complicated to work with and are sufficient only for stability or asymptotic stability, even in the case of second-order LTI switched systems. In other words, if a method fails to stabilize a switched system, one still cannot say much about the stabilizability of the system.

In this paper, we concentrate on the stability design of second-order LTI switched systems. The main characteristics of our results are the necessary and sufficient conditions for asymptotic stabilizability of such systems and the construction of stabilizing control laws when they exist. The idea behind our approach is to select an active subsystem so that the distance of the state to the origin ($\|x\|_2^2$) is minimized. To achieve this, we base our selection criterion on the angles of subsystem vector fields and the geometric properties of $\mathbb{R}^2$. Note that our selection criterion which is based on the angles is different from the above mentioned min-projection criterion in that, in the min-projection criterion, the contribution of the length of the vector fields may result to different selection of an active subsystem than our approach. Another important point worth noting is that, by fully utilizing the geometric properties of $\mathbb{R}^2$, our approach obtains necessary and sufficient conditions for the asymptotic stabilizability as opposed to the sufficient-only conditions obtainable by the min-projection and Lyapunov approaches. From the literature, it can be observed that the behaviours of switched systems are complicated and difficult to study in general. We point

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out that even for second-order LTI switched systems, the dynamic behaviours of the systems are very rich, and the study of such systems is therefore useful in also gaining insight into higher-order switched systems. It is then not surprising that second-order LTI switched systems have been attracting researchers for some time. For example, Shorten and Narendra (1998) studied the stability analysis problem of finding a CLF for two second-order linear systems. Another related interesting research is by Loparo (Loparo *et al.* 1987 a, b) which studies the local and global cycles, controllability and attainability for second-order LTI switched systems. Loparo *et al.* derived sufficient conditions for global controllability of such systems and indicated that attempts to determine necessary and sufficient conditions for global controllability had been unsuccessful. However, there was no result in the papers for the stabilization problem of such systems, which we investigate in the present paper.

Since the behaviour of a second-order LTI switched system can be quite complex, to derive necessary and sufficient conditions for asymptotic stabilizability, we need to study different cases exhaustively by categorizing subsystems. Although the approach in the paper is applicable to several second-order LTI subsystems, yet for clarity of motivation and presentation, we investigate in detail two subsystems with foci, nodes and saddle points. In § 3, we study two subsystems with foci by identifying different regions based on different properties of the angles of subsystem vector fields. The main results there are Theorem 1 and Theorem 2, which state necessary and sufficient conditions for asymptotic stabilizability for two subsystems with foci. The isocline properties of linear systems and the topology of $\mathbb{R}^2$ are central to the proofs of these theorems. Besides these theorems, we propose conic switching laws which asymptotically stabilize a system if the system is asymptotically stabilizable. In §§ 4 and 5, based on the methods for subsystems with foci, stability designs are carried out for two subsystems with nodes and two subsystems with saddle points. It can be seen that the behaviour of such systems is more complicated than two subsystems with foci; therefore we go through exhaustive case studies to completely explore the asymptotic stabilizability of such systems. Section 6 indicates that the approach is also applicable to several subsystems. Throughout the paper, examples are presented to illustrate our results.

2. Preliminaries

In this paper, we study second-order LTI switched systems of the form

$$\dot{x} = f_{i(t)}(x) = A_{i(t)}x, \qquad i \in I = \{1, 2, \ldots, M\} \tag{1}$$

where $A_i \in \mathbb{R}^{2\times 2}$ ($i \in I$) is the matrix for subsystem i and is unstable. $i(t)\colon [0, \infty) \to I$ is a piecewise constant function indicating the active subsystem at each instant and is determined by some switching law. Here we assume that we are given a collection of subsystems

$$\dot{x} = A_i x, \qquad i \in I \tag{2}$$

and are asked to design a switching law. In order to motivate and clearly explain our approach, we concentrate on the case of two subsystems ($I = \{1, 2\}$) in §§ 3, 4 and 5. In § 6, we mention extensions to the general M subsystem cases.

Remark 1: Note that a switching law is valid if and only if the switching function $i(t)$ it produces is piecewise constant. Therefore any law that will cause Zenoness (i.e. infinitely many switchings in a finite time interval) is regarded as unacceptable.

Remark 2: For the discussion of two subsystems, we assume that $A_1 \neq cA_2, \forall c \in \mathbb{R}$. For if $\exists c \leq 0, A_1 = cA_2$, then one subsystem will be stable and this contradicts our assumption. If $\exists c > 0, A_1 = cA_2$, then the trajectory of the two subsystems are similar except for the difference of speed; note that in this case, the system will behave as a single unstable system.

Focus, node, saddle point. For a second-order system $\dot{x} = Ax$, the origin of $\mathbb{R}^2$ is said to be a focus, or a node, or a saddle point if the eigenvalues of A are complex conjugates, or real numbers of the same sign, or real numbers of opposite signs, respectively (see Chapter 1 of Khalil (1996) for details). In this paper, a second-order LTI system will be said to be with *focus*, or *node*, or *saddle point* with regard to the corresponding type of the origin. In the following, we mainly discuss these three kinds of systems. Figure 1 shows some typical trajectories.

The direction of a subsystem at x. We say that the direction of a subsystem at $x \neq 0$ is *clockwise* (resp. *counterclockwise*) if, starting from x, its trajectory evolves in a clockwise (resp. counterclockwise) direction. To be more precise, let $x = (x_1, x_2)^{\mathrm{T}}$ be a nonzero point in $\mathbb{R}^2$, and denote

$$f_1 = A_1 x = (f_{11}, f_{12})^{\mathrm{T}} \tag{3}$$

$$f_2 = A_2 x = (f_{21}, f_{22})^{\mathrm{T}} \tag{4}$$

We view x, f_1 and f_2 as vectors in $\mathbb{R}^2$ and define $\theta_i, i = 1, 2$ to be the angle between x and f_i measured counterclockwise with respect to x (θ_i is confined to $-\pi \leq \theta_i < \pi$). So in this case, when $-\pi < \theta_i < 0$ (resp. $0 < \theta_i < \pi$), the ith subsystem is said to be *of clockwise* (resp. *of counterclockwise*) *direction at x*. Figure 2(a) shows an example in which subsystem 1 is of clockwise

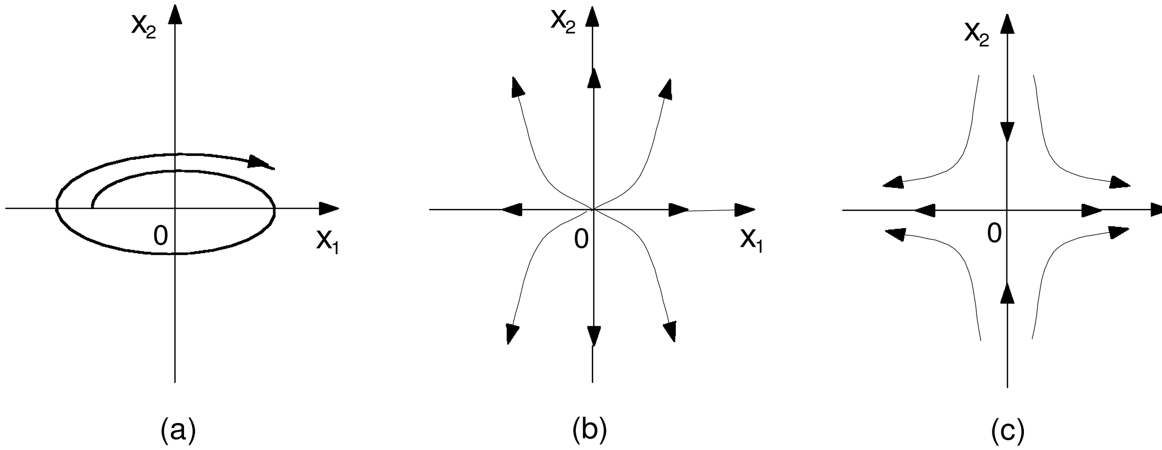

Figure 1. The typical trajectory of a second order LTI system with (*a*) unstable focus (*b*) unstable node (*c*) saddle point.

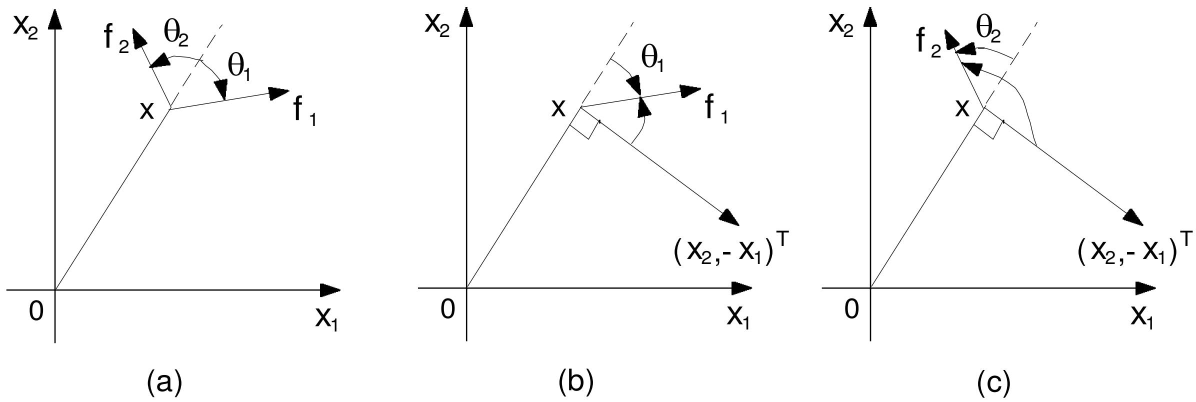

Figure 2. (*a*) Subsystem 1 is of clockwise direction and subsystem 2 is of counterclockwise direction at x. (*b*) The absolute value of the angle between $(x_2, -x_1)^{\mathrm{T}}$ and f_1 is less than $\pi/2$. (*c*) The absolute value of the angle between $(x_2, -x_1)^{\mathrm{T}}$ and f_2 is greater than $\pi/2$.

direction and subsystem 2 is of counterclockwise direction at x.

Notice that for any non-zero vector $x = (x_1, x_2)^{\mathrm{T}}$, $(x_2, -x_1)^{\mathrm{T}}$ is a vector obtained by rotating x clockwise by an angle $\pi/2$. It can be seen that subsystem i is of clockwise (resp. counterclockwise) direction at x if and only if the absolute value of the angle (confined to $[-\pi, \pi)$) between $(x_2, -x_1)^{\mathrm{T}}$ and f_i is less than $\pi/2$ (resp. greater than $\pi/2$), in other words, the inner product $\langle (x_2, -x_1)^{\mathrm{T}}, f_i \rangle > 0$ (resp. < 0). This actually provides us with an easy way to check the direction.

The regions E_{is} and E_{iu}. Define the following regions

$$\begin{aligned} E_{is} &= \{x \mid -\pi \le \theta_i \le -\pi/2 \text{ or } \pi/2 \le \theta_i < \pi\} \\ &= \{x \mid x^{\mathrm{T}} f_i(x) = x^{\mathrm{T}} A_i x \le 0\}, \qquad i = 1,2 \qquad (5) \\ E_{iu} &= \{x \mid -\pi/2 \le \theta_i \le \pi/2\} \\ &= \{x \mid x^{\mathrm{T}} f_i(x) = x^{\mathrm{T}} A_i x \ge 0\}, \qquad i = 1,2 \qquad (6) \end{aligned}$$

For subsystems with focus, node and saddle point, the interior of E_{is} (resp. E_{iu}) is the set of all points in $\mathbb{R}^2$ where the trajectory of the ith subsystem would be driven closer to (resp. farther from) the origin if the subsystem evolves for sufficiently small amount of time starting from the point x. It can be verified that E_{is} and E_{iu} both have two conic sections.

The regions E_{ic} and E_{icc}. Define the following regions

$$\begin{aligned} E_{ic} &= \{x = (x_1, x_2)^{\mathrm{T}} \mid \langle (x_2, -x_1)^{\mathrm{T}}, f_i \rangle \ge 0\} \\ &= \{x = (x_1, x_2)^{\mathrm{T}} \mid (x_2, -x_1) A_i (x_1, x_2)^{\mathrm{T}} \ge 0\}, \\ &\qquad i = 1,2 \qquad (7) \end{aligned}$$

$$\begin{aligned} E_{icc} &= \{x = (x_1, x_2)^{\mathrm{T}} \mid \langle (x_2, -x_1)^{\mathrm{T}}, f_i \rangle \le 0\} \\ &= \{x = (x_1, x_2)^{\mathrm{T}} \mid (x_2, -x_1) A_i (x_1, x_2)^{\mathrm{T}} \le 0\}, \\ &\qquad i = 1,2 \qquad (8) \end{aligned}$$

For subsystems with focus, node and saddle point, E_{ic} (resp. E_{icc}) denotes the regions in the interior of which the ith subsystem trajectory travels clockwise (resp. counterclockwise).

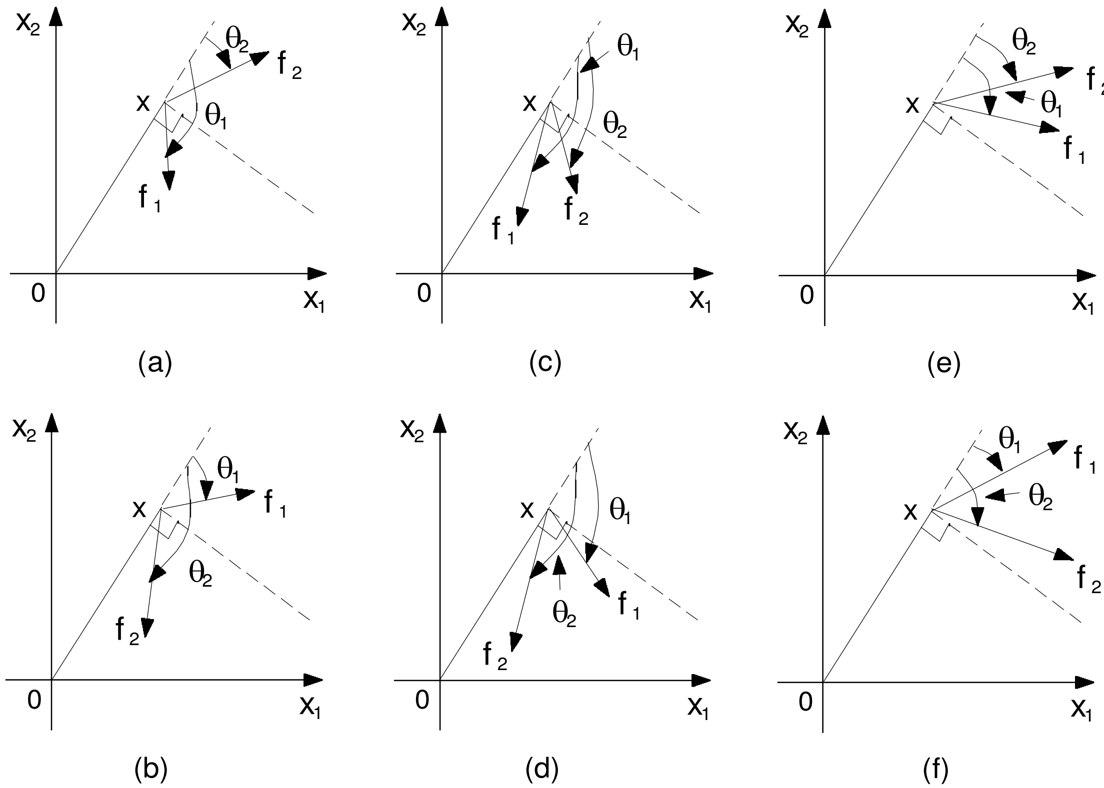

Figure 3. The appearance of vectors f_1 and f_2 for x in (a) Ω_1 (b) Ω_2 (c) Ω_3 (d) Ω_4 (e) Ω_5 (f) Ω_6 for two subsystems of the same direction.

3. Stabilization of second-order LTI switched systems with foci

In the present section, the switched system (2) consisting of two second-order LTI subsystems with unstable foci is studied in detail. Necessary and sufficient conditions for the asymptotic stabilizability of such systems are shown. If a switched system is asymptotically stabilizable, a switching law will also be obtained.

Note that for a second-order LTI system with focus, starting from any $x \neq 0$ the system trajectory is a spiral around the origin in the clockwise (or counterclockwise) direction. In view of the above discussion, the system is said to be of clockwise (or counterclockwise) direction. For a switched system consisting of two subsystems with unstable foci, we will study the vector fields of both subsystems and obtain stabilizing switching laws if they exist. The basic idea is to choose an appropriate subsystem at each instant so as to drive the trajectory closer to the origin, i.e. $\|x(t)\|_2 \to 0$ as $t \to \infty$. Hereafter we implement this idea to the case of two subsystems of the same direction and of opposite directions.

3.1. *Two subsystems of the same direction*

Without loss of generality, assume that both subsystems are of clockwise direction. We first define the following six regions

$$\Omega_1 = E_{1s} \cap E_{2u} \tag{9}$$

$$\Omega_2 = E_{1u} \cap E_{2s} \tag{10}$$

$$\Omega_3 = E_{1s} \cap E_{2s} \cap \{x | \, |\theta_1| \geq |\theta_2|\} \tag{11}$$

$$\Omega_4 = E_{1s} \cap E_{2s} \cap \{x | \, |\theta_1| \leq |\theta_2|\} \tag{12}$$

$$\Omega_5 = E_{1u} \cap E_{2u} \cap \{x | \, |\theta_1| \geq |\theta_2|\} \tag{13}$$

$$\Omega_6 = E_{1u} \cap E_{2u} \cap \{x | \, |\theta_1| \leq |\theta_2|\} \tag{14}$$

where $E_{is}, E_{iu}, \theta_1, \theta_2$ are as defined in §2. Figure 3 shows the appearance of the vector fields in Ω_1 to Ω_6. Some characteristics of these regions are now discussed.

The regions Ω_1, Ω_2. From (9), for any $x \in \Omega_1$, $-\pi < \theta_1 \leq -\pi/2$ and $-\pi/2 \leq \theta_2 < 0$ hold. Moreover, if $\mathrm{Int}(\Omega_1) \neq \emptyset$, for any $x \in \mathrm{Int}(\Omega_1)$, we have $-\pi < \theta_1 < -\pi/2$ and $-\pi/2 < \theta_2 < 0$ as shown in figure 3(a) since equality $\theta_i = -\pi/2$ only holds on the boundary of Ω_1.

Ω_2 can be similarly explained as Ω_1. If $\mathrm{Int}(\Omega_2) \neq \emptyset$, for any $x \in \mathrm{Int}(\Omega_2)$, $-\pi < \theta_1 < -\pi/2$ and $-\pi/2 < \theta_2 < 0$ hold (figure 3(b)).

The regions Ω_3, Ω_4, Ω_5 *and* Ω_6. From (11), for any $x \in \Omega_3$, $-\pi < \theta_1 \leq \theta_2 \leq -\pi/2$ must hold; note that strict inequalities hold for any $x \in \mathrm{Int}(\Omega_3)$ (see figure 3(c)). In this case, (11) is true if and only if, in addi-

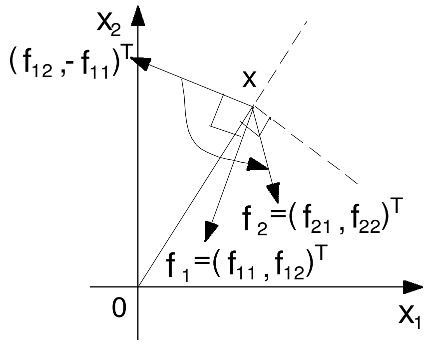

Figure 4. The absolute value of the angle between $(f_{12}, -f_{11})^{\mathrm{T}}$ and $f_2 = (f_{21}, f_{22})^{\mathrm{T}}$ is greater than or equal to $\pi/2$.

tion to $x \in E_{1s} \cap E_{2s}$, the absolute value of the angle (confined to $[-\pi, \pi)$) between $(f_{12}, -f_{11})^{\mathrm{T}}$ and $f_2 = (f_{21}, f_{22})^{\mathrm{T}}$ is greater than or equal to $\pi/2$ (see figure 4), in other words, the inner product $\langle (f_{12}, -f_{11})^{\mathrm{T}}, (f_{21}, f_{22})^{\mathrm{T}} \rangle = f_{12} f_{21} - f_{11} f_{22} \leq 0$. This provides us an alternative way to define Ω_3 in this case

$$\Omega_3 = E_{1s} \cap E_{2s} \cap \{x | f_{12} f_{21} - f_{11} f_{22} \leq 0\} \tag{15}$$

Similarly, for Ω_4, Ω_5 and Ω_6 we have

$$-\pi < \theta_2 \leq \theta_1 \leq -\pi/2, \quad \text{for } x \in \Omega_4$$
$$-\pi/2 \leq \theta_1 \leq \theta_2 < 0, \quad \text{for } x \in \Omega_5$$
$$-\pi/2 \leq \theta_2 \leq \theta_1 < 0, \quad \text{for } x \in \Omega_6$$

and the corresponding alternative definitions in this case for these regions are

$$\Omega_4 = E_{1s} \cap E_{2s} \cap \{x | f_{12} f_{21} - f_{11} f_{22} \geq 0\} \tag{16}$$
$$\Omega_5 = E_{1u} \cap E_{2u} \cap \{x | f_{12} f_{21} - f_{11} f_{22} \leq 0\} \tag{17}$$
$$\Omega_6 = E_{1u} \cap E_{2u} \cap \{x | f_{12} f_{21} - f_{11} f_{22} \geq 0\} \tag{18}$$

Remark 3: Note that because $A_1 \neq cA_2$, $\forall c \in \mathbb{R}$, it can be shown that $\theta_1 = \theta_2$ on only finitely many rays through the origin which are parts of the boundaries of some Ω_j.

Remark 4: The alternative definitions (15)–(18) provide us with an easier way to calculate Ω_3 to Ω_6 than (11)–(14). Note that if both subsystems are of counter-clockwise direction, equations (9)–(14) are still valid as definitions for Ω_1 to Ω_6, but the alternative definitions (15)–(18) need to be modified following the idea of the above discussion; for example, Ω_3 will then be $\Omega_3 = E_{1s} \cap E_{2s} \cap \{x | f_{12} f_{21} - f_{11} f_{22} \geq 0\}$.

Remark 5: Note that $\bigcup_{j=1}^{6} \Omega_j = \mathbb{R}^2$ and that the interiors of Ω_j's are disjoint.

An important observation can now be made. Refer to figure 5(a). Assume that $\mathrm{Int}(\Omega_1) \neq \varnothing$ and assume that the ray l_1 goes through the origin and the non-zero points on l_1 are in $\mathrm{Int}(\Omega_1)$. Let l_2 be the ray in the same conic section of Ω_1 as l_1 obtained by rotating l_1 around the origin for the angle $\alpha < 0$ ($\alpha < 0$ means rotating clockwise). Starting from a non-zero point $x \in l_1$, let $x^{(i)}, i = 1, 2$ be the point where the trajectory intersects l_2 for the first time by following subsystem i. Then from the definition of Ω_1 is not difficult to see that for $|\alpha|$ sufficiently small, we must have $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$.

Now refer to figure 5(b), let l_1 and l_2 be two different rays that go through the origin and are in the same conic section of Ω_1. Suppose that l_2 is to the clockwise side of l_1. Let $x \neq 0$ be on l_1 and $x^{(i)}, i = 1, 2$ be the point on l_2 where the trajectory of the system intersects l_2 for the first time if the system evolves solely by following subsystem i. Then it can be seen that $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$. Moreover, this observation can be extended to Ω_2 to Ω_6 as stated by the following lemma.

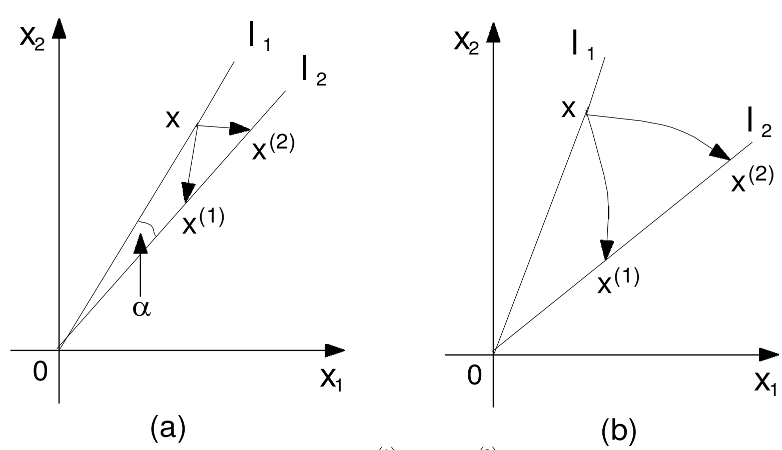

Figure 5. (a) For $x \in \mathrm{Int}(\Omega_1)$ and on l_1 and $|\alpha|$ small enough, $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$. ($b$) If l_1 and l_2 are in the same conic section of Ω_1, then $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$.

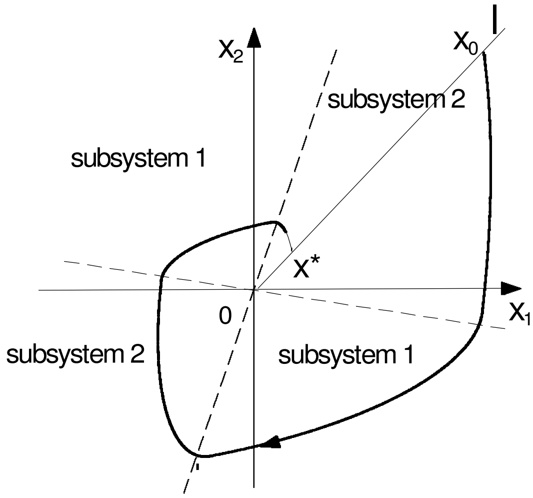

Figure 6. Figure for Theorem 1.

Lemma 1: *Consider a switched system* (2) *consisting of two subsystems with unstable foci and of the same direction. Let l_1 and l_2 be different rays that go through the origin and be in the same conic section of the region Ω_j and l_2 be to the clockwise side of l_1. If $x_0 \neq 0$ is on l_1 and $x^{(i)}, i = 1,2$ is the point on l_2 where the trajectory of the system, evolving solely by following subsystem i, intersects l_2 for the first time, then*

(a) *for* $j = 1,3,5$, $\|x^{(1)}\|_2 \leq \|x^{(2)}\|_2$,
(b) *for* $j = 2,4,6$, $\|x^{(2)}\|_2 \leq \|x^{(1)}\|_2$.

The proof of this result is straightforward (see Xu and Antsaklis 1999 c). From Lemma 1, it can be seen that for asymptotic stability it is preferable to choose subsystem 1 in Ω_1, Ω_3 and Ω_5 and subsystem 2 in Ω_2, Ω_4 and Ω_6. So we propose the following switching law.

Conic Switching Law I: *Switch to subsystem* 1 *whenever the system trajectory enters the conic sections of the regions Ω_1, Ω_3, Ω_5 and switch to subsystem* 2 *whenever the system trajectory enters the conic sections of the regions Ω_2, Ω_4, Ω_6.*

It is clear that this conic switching law selects the subsystem which drives the trajectory closer to the origin at each point $x \neq 0$.

Remark 6: If at some instant t, the trajectory of a switched system is at a point x where $\theta_1 = \theta_2$, then the point x is on the boundary of two Ω_j regions. In this case, we associate with x the subsystem corresponding to the region which is to be entered. This would make the switching function right continuous.

By using Conic Switching Law I, we have the following theorem that provides necessary and sufficient conditions for the asymptotic stabilizability of the switched system.

Theorem 1: *Consider a switched system* (2) *consisting of two subsystems with unstable foci and of the same direction. Let l be a ray that goes through the origin and let $x_0 \neq 0$ be on l. Let x^* be the point on l where the trajectory intersects l for the first time after leaving x_0, when the switched system evolves according to Conic Switching Law I (see figure 6). The switched system is asymptotically stabilizable if and only if $\|x^*\|_2 < \|x_0\|_2$.*

Proof: If $\|x^*\|_2 < \|x_0\|_2$, then, by the isocline property of linear systems, the trajectory of the switched system under Conic Switching Law I will evolve in the similar fashion as in the first round (i.e. the round from x_0 to x^*). Hence the trajectory will go to the origin as $t \to \infty$. Therefore Conic Switching Law I is an asymptotically stabilizing switching law.

Next we prove that the 'only if' part is also true. We claim that if s is an arbitrary switching law and if $\tilde{x}$ is the point on l where the trajectory of the system intersects l for the first time after leaving x_0, when the system evolves using s, then we must have $\|x^*\|_2 \leq \|\tilde{x}\|_2$. This can be proved as follows. Let α be the angle from ray l to the ray on which $x(t)$ lies, measured counterclockwise. Note that s generates a finite number of switchings in any finite time interval by Remark 1 in § 2; we assume that using s the system switches when α is equal to $0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$. Now combine the above switching α's with the switching α's obtained when using Conic Switching Law I and -2π. For simplicity of notation, we still denote the combined switching α's as $0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots, -2\pi, \ldots$. Let the trajectory for s be $x_s(\alpha)$ and the trajectory using Conic Switching Law I be $x_c(\alpha)$ indexed by α.

Now in the conic section $\alpha_1 \leq \alpha \leq 0$, it is better to follow the subsystem which is specified by Conic Switching Law I in view of Lemma 1. So upon arriving at α_1, we have $\|x_c(\alpha_1)\|_2 \leq \|x_s(\alpha_1)\|_2$.

In the conic section $\alpha_2 \leq \alpha \leq \alpha_1$, if $\|x_c(\alpha_1)\|_2 = \|x_s(\alpha_1)\|_2$, then from Lemma 1, it is still better to follow Conic Switching Law I. If $\|x_c(\alpha_1)\|_2 < \|x_s(\alpha_1)\|_2$, then it would be even clearer that it is better to follow Conic Switching Law I, since every trajectory evolving according to any one subsystem starting from $x_c(\alpha_1)$ is closer to the origin than starting from $x_s(\alpha_1)$. Therefore upon arriving at α_2, we have $\|x_c(\alpha_2)\|_2 \leq \|x_s(\alpha_2)\|_2$. By induction, we can use similar argument as above to prove that upon arriving at -2π, we have $\|x_c(-2\pi)\|_2 \leq \|x_s(-2\pi)\|_2$, i.e., $\|x^*\|_2 \leq \|\tilde{x}\|_2$.

Next we assume that the system is asymptotically stabilizable but $\|x^*\|_2 \geq \|x_0\|_2$, therefore by the above arguments, any switching law will have $\|x_0\|_2 \leq \|x^*\|_2 \leq \|\tilde{x}\|_2$. So when the system evolves around the origin for -2π, the trajectory will not be closer to the origin than x_0 for any switching law. By

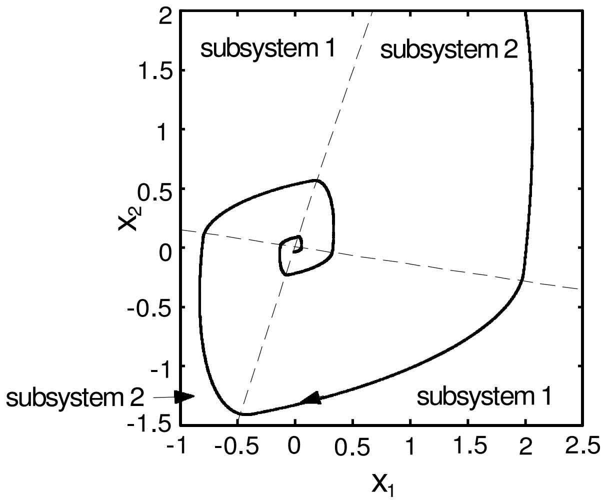

Figure 7. The trajectory for Example 1.

induction, we can prove that for any $-2k\pi$ $(k>0)$ around the origin, the trajectory still cannot be closer to the origin than x_0 for any switching law. Therefore the system is not asymptotically stabilizable, which is a contradiction to our assumption. Consequently it must be true that $\|x^*\|_2 < \|x_0\|_2$. □

Example 1: Consider a switched system (2) consisting of two subsystems with unstable foci and of the same direction where

$$A_1 = \begin{bmatrix} 1 & 13 \\ -2 & 3 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -1 & 2 \\ -10 & 3 \end{bmatrix}$$

It is asymptotically stabilizable. Figure 7 shows the trajectory of the system with $x_0 = (2,2)^{\mathrm{T}}$. Here the system starts with subsystem 2 and when the trajectory first intersects a dashed line, it switches to subsystem 1 and then to subsystem 2, 1, etc. upon intersecting the dashed lines according to Conic Switching Law I. Note that here the active region for subsystem 1 (resp. 2) is actually the union of Ω_1, Ω_3, Ω_5 (resp. Ω_2, Ω_4, Ω_6) and Ω_j's are not shown individually.

3.2. *Two subsystems of opposite directions*

Assume that subsystem 1 is of clockwise direction and subsystem 2 is of counterclockwise direction. Similar to § 3.1, define the following regions

$$\hat{\Omega}_1 = E_{1s} \cap E_{2s} \tag{19}$$

$$\hat{\Omega}_2 = E_{1u} \cap E_{2u} \tag{20}$$

$$\hat{\Omega}_3 = E_{1s} \cap E_{2u} \cap \{x|\ |\theta_1| + |\theta_2| \geq \pi\} \tag{21}$$

$$\hat{\Omega}_4 = E_{1s} \cap E_{2u} \cap \{x|\ |\theta_1| + |\theta_2| \leq \pi\} \tag{22}$$

$$\hat{\Omega}_5 = E_{1u} \cap E_{2s} \cap \{x|\ |\theta_1| + |\theta_2| \geq \pi\} \tag{23}$$

$$\hat{\Omega}_6 = E_{1u} \cap E_{2s} \cap \{x|\ |\theta_1| + |\theta_2| \leq \pi\} \tag{24}$$

Figure 8 shows the appearance of the vector fields in $\hat{\Omega}_1$ to $\hat{\Omega}_6$. Notice, the $\hat{\Omega}_j$'s in this subsection are different from Ω_j's in the previous subsection. Some characteristics of these regions are now discussed.

The regions $\hat{\Omega}_1$, $\hat{\Omega}_2$. From (19), for any $x \in \hat{\Omega}_1$, $-\pi < \theta_1 \leq -\pi/2$ and $\pi/2 \leq \theta_2 < \pi$ hold. Moreover, if $\mathrm{Int}(\hat{\Omega}_1) \neq \varnothing$, for any $x \in \mathrm{Int}(\hat{\Omega}_1)$, we must have $-\pi < \theta_1 < -\pi/2$ and $\pi/2 < \theta_2 < \pi$ as shown in figure 8(*a*) since equality $\theta_1 = -\pi/2$ or $\theta_2 = \pi/2$ only holds on the boundary of $\hat{\Omega}_1$.

$\hat{\Omega}_2$ can be similarly explained as $\hat{\Omega}_1$. In $\mathrm{Int}(\hat{\Omega}_2)$, we have $-\pi/2 < \theta_1 < 0 < \theta_2 < \pi/2$ (figure 8(*b*)).

The regions $\hat{\Omega}_3$, $\hat{\Omega}_4$, $\hat{\Omega}_5$ *and* $\hat{\Omega}_6$. From (21), for any $x \in \hat{\Omega}_3$, $-\pi < \theta_1 \leq -\pi/2$, $0 < \theta_2 \leq \pi/2$ and $|\theta_1| + |\theta_2| \geq \pi$ must hold; note that strict inequalities hold for any $x \in \mathrm{Int}(\hat{\Omega}_3)$. In this case, (21) is true if and only if, in addition to $x \in E_{1s} \cap E_{2u}$, the absolute value of the angle (confined to $[-\pi,\pi)$) between $(f_{12},-f_{11})^{\mathrm{T}}$ and $f_2 = (f_{21},f_{22})^{\mathrm{T}}$ is smaller than or equal to $\pi/2$ (see figure 9), in other words, the inner product $\langle (f_{12},-f_{11})^{\mathrm{T}}, (f_{21},f_{22})^{\mathrm{T}} \rangle = f_{12}f_{21} - f_{11}f_{22} \geq 0$. This provides an alternative way to define $\hat{\Omega}_3$ in this case

$$\hat{\Omega}_3 = E_{1s} \cap E_{2u} \cap \{x | f_{12}f_{21} - f_{11}f_{22} \geq 0\} \tag{25}$$

Similarly for $\hat{\Omega}_4$, $\hat{\Omega}_5$ and $\hat{\Omega}_6$ we have

$$-\pi < \theta_1 \leq -\pi/2, \quad 0 < \theta_2 \leq \pi/2, \quad |\theta_1| + |\theta_2| \leq \pi \qquad \text{for } x \in \hat{\Omega}_4$$

$$-\pi/2 \leq \theta_1 < 0, \quad \pi/2 \leq \theta_2 < \pi, \quad |\theta_1| + |\theta_2| \geq \pi \qquad \text{for } x \in \hat{\Omega}_5$$

$$-\pi/2 \leq \theta_1 < 0, \quad \pi/2 \leq \theta_2 < \pi, \quad |\theta_1| + |\theta_2| \leq \pi \qquad \text{for } x \in \hat{\Omega}_6$$

and the corresponding alternative definitions in this case for these regions are

$$\hat{\Omega}_4 = E_{1s} \cap E_{2u} \cap \{x | f_{12}f_{21} - f_{11}f_{22} \leq 0\} \tag{26}$$

$$\hat{\Omega}_5 = E_{1u} \cap E_{2s} \cap \{x | f_{12}f_{21} - f_{11}f_{22} \geq 0\} \tag{27}$$

$$\hat{\Omega}_6 = E_{1u} \cap E_{2s} \cap \{x | f_{12}f_{21} - f_{11}f_{22} \leq 0\} \tag{28}$$

Remark 7: Note that in $\hat{\Omega}_1$, $\hat{\Omega}_3$ and $\hat{\Omega}_5$, $|\theta_1| + |\theta_2| \geq \pi$, but in $\hat{\Omega}_2$, $\hat{\Omega}_4$ and $\hat{\Omega}_6$, $|\theta_1| + |\theta_2| \leq \pi$. Moreover, $|\theta_1| + |\theta_2| = \pi$ can be true only on a finite number of rays through the origin which are parts of the boundaries of some $\hat{\Omega}_j$.

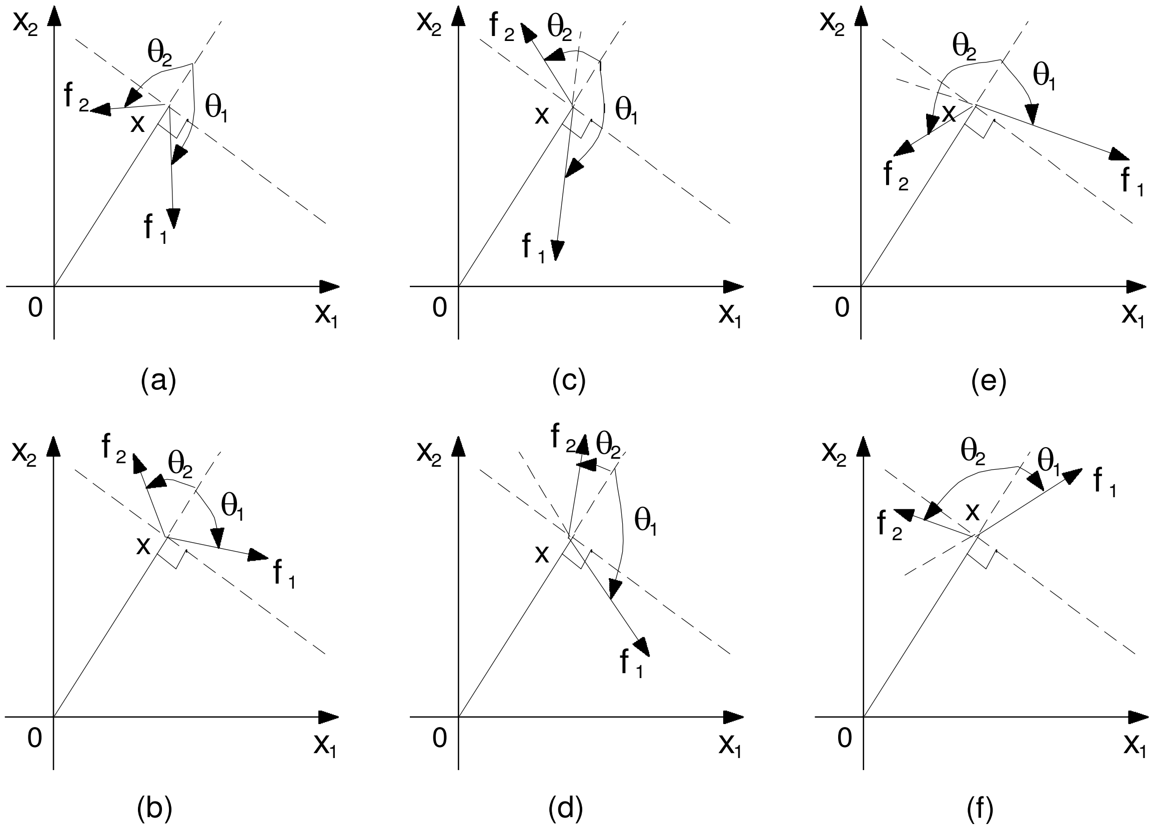

Figure 8. The appearance of vectors f_1 and f_2 for x in (a) $\hat{\Omega}_1$ (b) $\hat{\Omega}_2$ (c) $\hat{\Omega}_3$ (d) $\hat{\Omega}_4$ (e) $\hat{\Omega}_5$ (f) $\hat{\Omega}_6$ for two subsystems of opposite directions.

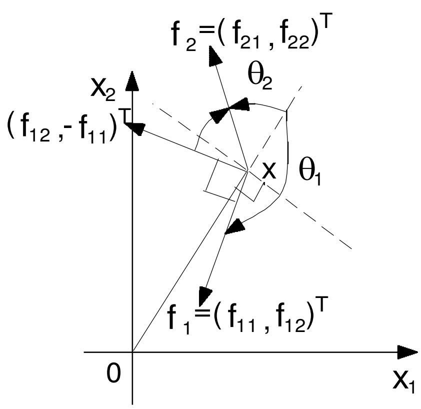

Figure 9. The absolute value of the angle between $(f_{12}, -f_{11})^{\mathrm{T}}$ and $f_2 = (f_{21}, f_{22})^{\mathrm{T}}$ is smaller than or equal to $\pi/2$.

Remark 8: Note that if subsystem 1 is of counter-clockwise direction and subsystem 2 is of clockwise direction, equations (19)–(24) are still valid as definitions for $\hat{\Omega}_1$ to $\hat{\Omega}_6$, but the alternative definitions (25)–(28) need to be modified following the idea of the above discussion; for example, $\hat{\Omega}_3$ will then be $\hat{\Omega}_3 = E_{1s} \cap E_{2u} \cap \{x | f_{12}f_{21} - f_{11}f_{22} \le 0\}$.

Remark 9: Note that $\bigcup_{j=1}^{6} \hat{\Omega}_j = \mathbb{R}^2$ and the interiors of $\hat{\Omega}_j$'s are disjoint.

Assume that l_1 and l_2 are different rays that go through the origin and are in the same conic section of the region $\hat{\Omega}_1$ and l_2 is to the clockwise side of l_1. Suppose $x_0 \neq 0$ is on l_2. Let the switched system follow subsystem 2 until the trajectory intersects l_1 for the first time and then let the system switch to subsystem 1 and evolve following subsystem 1. Suppose x^* is the point on l_2 where the trajectory intersects l_2 for the first time after the switching. Then it can be seen that $\|x^*\|_2 \le \|x_0\|_2$ (see figure 10(a)). Moreover, we have the following Lemma.

Lemma 2: *Consider a switched system* (2) *consisting of two subsystems with unstable foci and of opposite directions. Let l_1 and l_2 be different rays that go through the origin and be in the same conic section of the region $\hat{\Omega}_j$ and l_2 be to the clockwise side of l_1. Let $x_0 \neq 0$ be on l_2. Let the switched system follow subsystem* 2 *until the trajectory intersects l_1 for the first time and then let the system switch to subsystem* 1 *and evolve following subsystem* 1. *Suppose x^* is the point on l_2 where the trajectory intersects l_2 for the first time after the switching, then*

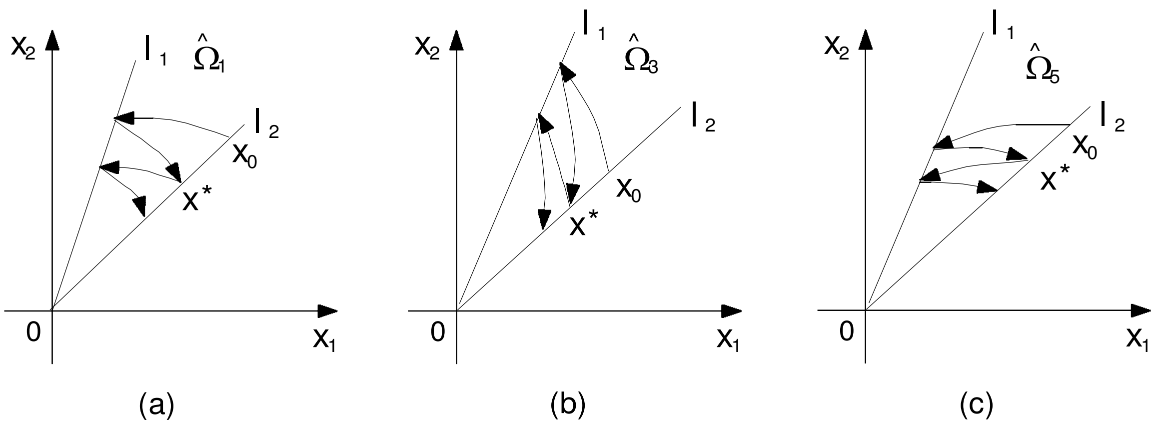

Figure 10. The trajectory of the system, where l_1 and l_2 are in the same conic section of (*a*) $\hat{\Omega}_1$ (*b*) $\hat{\Omega}_3$ (*c*) $\hat{\Omega}_5$.

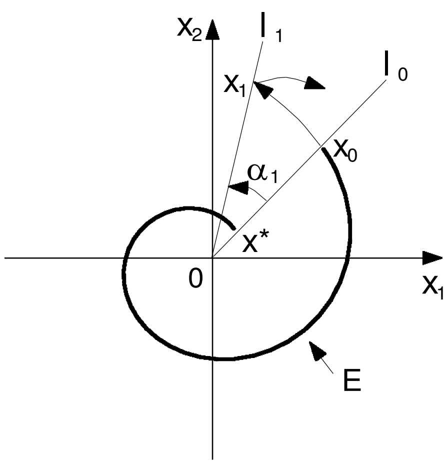

Figure 11. Figure for the proof of Theorem 2.

(a) *for* $j = 1,3,5$, $\|x^*\|_2 \le \|x_0\|_2$,
(b) *for* $j = 2,4,6$, $\|x^*\|_2 \ge \|x_0\|_2$.

The proof of the above lemma is straightforward (see Xu and Antsaklis 1999 c). Figure 10 shows the trajectories in $\hat{\Omega}_1$, $\hat{\Omega}_3$ and $\hat{\Omega}_5$. Next we prove an important theorem that shows a necessary and sufficient condition for the asymptotic stabilizability.

Theorem 2: *A switched system* (2) *consisting of two subsystems with unstable foci and of opposite directions is asymptotically stabilizable if and only if* $\mathrm{Int}\,(\hat{\Omega}_1) \cup \mathrm{Int}\,(\hat{\Omega}_3) \cup \mathrm{Int}\,(\hat{\Omega}_5) \neq \emptyset$.

Proof: If $\exists j \in \{1,3,5\}$ such that $\mathrm{Int}\,(\hat{\Omega}_j) \neq \emptyset$, then we can first force the trajectory to the boundary of a conic section of $\hat{\Omega}_j$ by following subsystem 1. Upon intersecting the boundary, we can switch back and forth between the two subsystems in $\hat{\Omega}_j$ as shown in figure 10. In view of the isocline property of linear systems, we can asymptotically stabilize the system by switching in this way.

Next we show that the 'only if' part is also true. The proof is by contradiction. Assume that the switched system is asymptotically stabilizable but $\mathrm{Int}\,(\hat{\Omega}_1) \cup \mathrm{Int}\,(\hat{\Omega}_3) \cup \mathrm{Int}\,(\hat{\Omega}_5) = \emptyset$. Let $x_0 \neq 0$ be on ray l_0 through the origin (see figure 11). Let α be the angle from l_0 to the ray on which $x(t)$ lies, measured counterclockwise. Assume that s is an arbitrary switching law such that the system switches when α is equal to $0, \alpha_1, \alpha_2, \ldots, \alpha_n, \ldots$ and starting from subsystem 2. And let the corresponding points at switching moment be $x_0, x_1, x_2, \ldots, x_n, \ldots$ and x_1 be on ray l_1 through the origin. (A similar argument can be applied to an arbitrary switching law starting from subsystem 1.)

Now consider the trajectory of subsystem 2 starting at x_0 and go backward in time, i.e., $x(-t)$. Assume x^* is the point where the trajectory intersects l_1 for the first time (at time $-t^*$) (figure 11). Let

$$E = \{x(-t) \mid -t^* \le -t < 0\}$$

If we consider the trajectory from x_0 to x_1 using s, it is clear that for any point x on the trajectory between x_0 to x_1, $\|x\|_2$ would be greater than or equal to the norm of the corresponding point on E (x' on E corresponds to x when they are on the same ray through the origin).

Since $\mathrm{Int}\,(\hat{\Omega}_1) \cup \mathrm{Int}\,(\hat{\Omega}_3) \cup \mathrm{Int}\,(\hat{\Omega}_5) = \emptyset$, by Lemma 2 and induction we can show that any x on the trajectory between x_k and x_{k+1} ($k \ge 1$) that uses the law s have a norm greater than or equal to the norm of the corresponding points on E. Therefore, any x on the trajectory of the switched system under s would have a norm greater than the minimum value of the norms of the points on E. Hence the switched system is not asymptotically stabilizable by any s, which is a contradiction to our assumption. □

With the help of the above definitions of $\hat{\Omega}_j$ and Theorem 2, if a switched system is asymptotically stabilizable, we propose the following switching law.

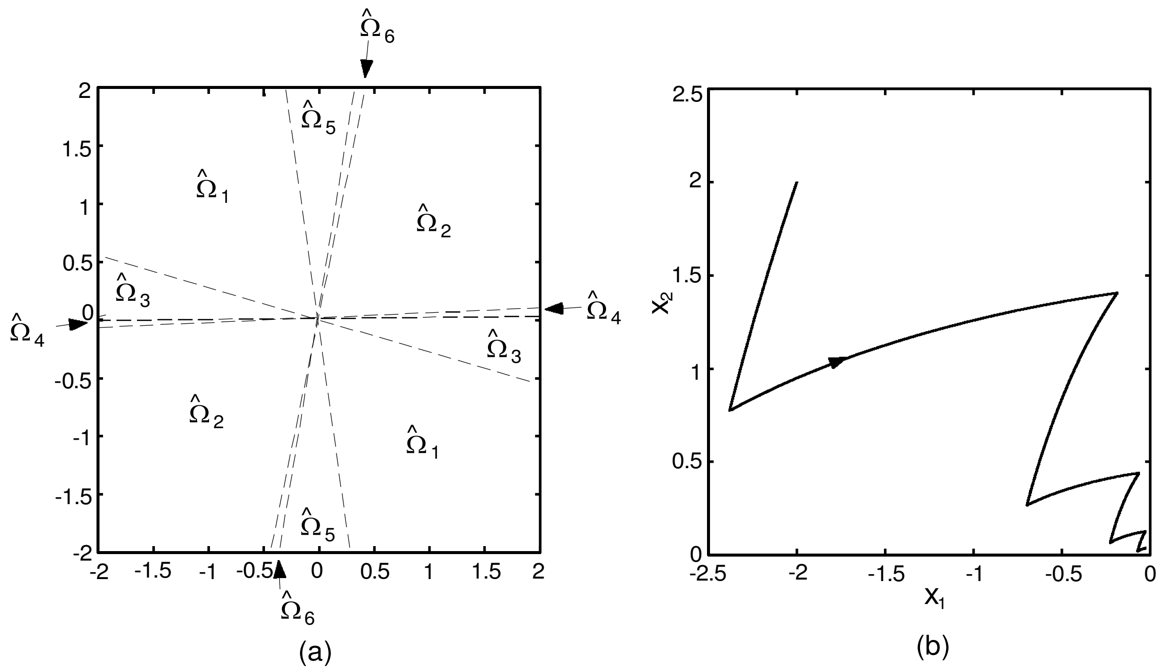

Figure 12. Example 2: (*a*) the conic sections; (*b*) the trajectory in $\hat{\Omega}_1$.

Conic Switching Law II:

1. *Start with subsystem* 1 *or* 2 *and stay with it until the trajectory enters into the interior of one of the conic sections of the regions* $\hat{\Omega}_1$, $\hat{\Omega}_3$ *and* $\hat{\Omega}_5$.
2. *Switch to another subsystem upon intersecting the boundary of the conic section so as to keep the trajectory inside the conic section.*

According to Theorem 2, there must be one of $\hat{\Omega}_1$, $\hat{\Omega}_3$ and $\hat{\Omega}_5$ with non-empty interior for an asymptotically stabilizable switched system. Therefore, by staying with subsystem 1 or 2, the trajectory can always be forced into one of the conic sections as mentioned in Step 1. Conic Switching Law II drives the trajectory closer to the origin in $\hat{\Omega}_1$ or $\hat{\Omega}_3$ or $\hat{\Omega}_5$ as can be seen from Lemma 2.

Example 2: Consider a switched system (2) consisting of two subsystems with unstable foci and of different directions where

$$A_1 = \begin{bmatrix} -2 & 52 \\ -8 & 6 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 11 & -10 \\ 50 & -9 \end{bmatrix}$$

It is asymptotically stabilizable since $\mathrm{Int}(\hat{\Omega}_1) \neq \emptyset$. Figure 12 shows (a) the conic sections of the regions $\hat{\Omega}_j$'s, (b) the trajectory of the system with $x_0 = (-2, 2)^{\mathrm{T}}$. In this example, the system starts with subsystem 2 and when the trajectory first intersects the boundary of the conic section of $\hat{\Omega}_1$ in quadrant II, it switches to subsystem 1 and then to subsystem 2, 1, etc. upon intersecting boundary of the conic section according to Conic Switching Law II.

4. Stabilization of second-order LTI switched systems with nodes

In this section, the asymptotic stabilizability of switched systems (2) consisting of two subsystems with unstable nodes is studied in detail. If a switched system is asymptotically stabilizable, a switching law will also be obtained.

The trajectory of a simple second-order LTI system where

$$A = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}, \qquad c_2 > c_1 > 0$$

is shown in figure 1(*b*). The trajectory travels counterclockwise in quadrant I since $\langle (x_2, -x_1)^{\mathrm{T}}, (c_1 x_1, c_2 x_2)^{\mathrm{T}} \rangle = (c_1 - c_2) x_1 x_2 < 0$. Similarly, the trajectories in quadrants II, III and IV travel clockwise, counterclockwise and clockwise, respectively. Using the notation E_c and E_{cc} in §2, E_{cc} includes quadrants I and III, while E_c includes quadrants II and IV. In general, for a second-order LTI system with unstable node, a linear transformation can be used to transform the system equation into the above simple form (see, e.g. Chapter 2 in Antsaklis and Michel (1997)). So we can analyse the direction of the system at each point in the transformed coordinate system and then translate the result back to the original coordinate system. In this section, we do not consider the special case

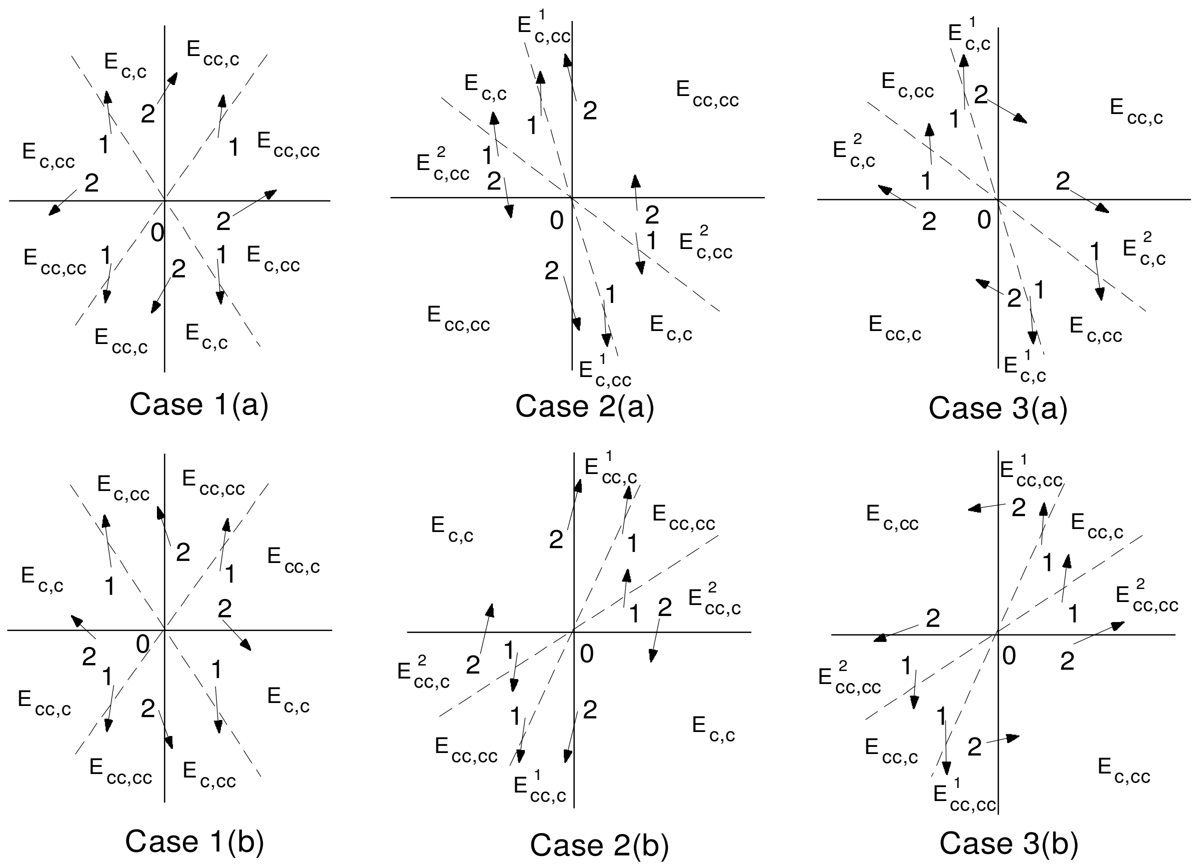

Figure 13. Cases for subsystems with nodes.

$$A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$$

This is because if one of the subsystems has two positive eigenvalues $c_1 = c_2$, it can be shown that the system is not stabilizable.

Now define

$$E_{c,c} = E_{1c} \cap E_{2c} \tag{29}$$

$$E_{c,cc} = E_{1c} \cap E_{2cc} \tag{30}$$

$$E_{cc,c} = E_{1cc} \cap E_{2c} \tag{31}$$

$$E_{cc,cc} = E_{1cc} \cap E_{2cc} \tag{32}$$

$E_{c,c}$ denotes the conic sections in which both the trajectories of subsystem 1 and 2 travel clockwise. $E_{c,cc}$, $E_{cc,c}$ and $E_{cc,cc}$ have analogous properties. Figure 13 shows exhaustively the six possible arrangements of $E_{c,c}$, $E_{c,cc}$, $E_{cc,c}$ and $E_{cc,cc}$. In figure 13, the small arrows indexed by the subsystem numbers on the eigenvector lines indicate the directions of the subsystems crossing the lines. Without loss of generality, we illustrate these cases by fixing E_{1cc} to be in quadrants I and III and E_{1c} to be in quadrants II and IV, respectively. (We can always do so using a linear transformation, which will not affect the applicability of our results.)

Remark 10: In the six cases, we assume that none of the two eigenvector directions of A_2 coincide with any of the eigenvector direction of A_1. This assumption is only for the purpose of discussion. However, in the case that some eigenvectors coincide, it is not difficult to carry out similar analysis.

Next we discuss in detail the different cases as shown in figure 13.

4.1. *Case 1*

We only discuss Case 1(a) (see Case 1(a) in figure 13), since similar argument can be applied to Case 1(b) (similarly we only discuss Cases 2(a) and 3(a) in Cases 2 and 3). First we introduce the notions of Type A, B, C regions.

Type A region. In Case 1(a), the two subsystems are of the same direction in $E_{c,c}$, $E_{cc,cc}$. Any trajectory starting in a conic section of these regions will eventually leave the conic section if the total time the system being active at subsystem 1 and the total time the system being active at subsystem 2 are both long enough. We call a region with this property a *Type A region.* (Of course, if the switched system stays with subsystem 1 in $E_{c,c}$, it will not

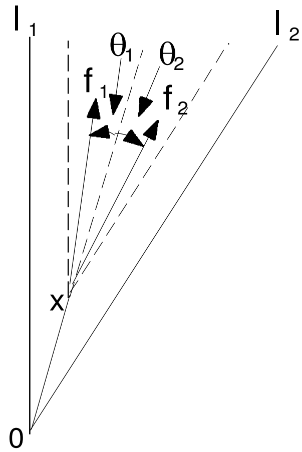

Figure 14. Figure for the proof of Lemma 3.

leave $E_{c,c}$, yet the system will be unstable, so this kind of strategy is not of interest here.)

Type B region. The region $E_{cc,c}$ in Case 1(a) has the property that whenever a trajectory goes into a conic section of it, the trajectory can never leave that conic section again. We call such kind of region *Type B region.*

Type C region. In a conic section of $E_{c,cc}$ in Case 1(a), the trajectory of the systems can always leave the conic section from both boundary lines under appropriate switchings. We call such kind of region *Type C region.*

We now introduce a lemma. Notice that the Ω_j's and $\hat{\Omega}_j$'s mentioned in §§ 4 and 5 are defined in § 3.

Lemma 3: *In Case* 1(a)*, for the Type B region* $E_{cc,c}$*,* $\mathrm{Int}\left(E_{cc,c} \cap \hat{\Omega}_j\right) = \emptyset$, $j = 1,3,5$.

Proof: Consider the conic section of $E_{cc,c}$ in quadrant I and note that similar arguments can be applied to the conic section in quadrant III. Assume that the boundary of the conic section is formed by two rays l_1 and l_2 through the origin. Note that subsystem 1 (resp. 2) is of counterclockwise (resp. clockwise) direction in $E_{cc,c}$ (see figure 14).

From figure 14, $|\theta_1|$ must be smaller than the absolute value of the angle from x to l_1, also $|\theta_2|$ must be smaller than the absolute value of the angle from x to l_2 (all angles are confined to $[-\pi, \pi)$). While the absolute value of the angle between l_1 and l_2 is less than $\pi/2$ for l_2 is in quadrant I, we conclude that $|\theta_1| + |\theta_2| < \pi/2 < \pi$. Therefore by Remark 7 in § 3.2, we know that in $\mathrm{Int}\left(E_{cc,c} \cap \hat{\Omega}_j\right) = \emptyset$, $j = 1,3,5$. □

Once a trajectory goes into $E_{cc,c}$, it cannot leave $E_{cc,c}$ since $E_{cc,c}$ is a Type B region and moreover, in view of Lemma 3, the trajectory cannot be asymptotically stabilized by any switching sequences. So in general, the switched system is *not asymptotically stabilizable* because stabilizability in general requires stabilizability from any initial point.

For the Type C region $E_{c,cc}$, it can also be proved that $\mathrm{Int}\left(E_{c,cc} \cap \hat{\Omega}_j\right) = \emptyset$, $j = 1,3,5$, by showing $|\theta_1| + |\theta_2| < \pi$.

Consequently in Case 1(a)*, the switched system cannot be asymptotically stabilized regardless of the initial point.*

4.2. *Case 2*

We only discuss Case 2(a) (see Case 2(a) in figure 13). Here $E_{c,c}$ and $E_{cc,cc}$ are Type A regions. $E^1_{c,cc}$ is a Type B region and $E^2_{c,cc}$ is a Type C region.

By applying similar arguments as in Lemma 3, it can be shown that $|\theta_1| + |\theta_2| < \pi$ so that $\mathrm{Int}\left(E^1_{c,cc} \cap \hat{\Omega}_j\right) = \emptyset$, $j = 1,3,5$. Hence we conclude that the system is *not asymptotically stabilizable*. However, it is still possible for the trajectory starting in the Type C region to be driven asymptotically toward the origin as long as the region satisfies the condition of the following lemma.

Lemma 4: *In Case* 2(a)*, trajectories starting in* $\mathrm{Int}\left(E^2_{c,cc}\right)$ *can be driven asymptotically toward the origin if and only if* $\mathrm{Int}\left(E^2_{c,cc} \cap \hat{\Omega}_1\right) \cup \mathrm{Int}\left(E^2_{c,cc} \cap \hat{\Omega}_3\right) \cup \mathrm{Int}\left(E^2_{c,cc} \cap \hat{\Omega}_5\right) \neq \emptyset$.

Proof: If $\mathrm{Int}\left(E^2_{c,cc} \cap \hat{\Omega}_1\right) \cup \mathrm{Int}\left(E^2_{c,cc} \cap \hat{\Omega}_3\right) \cup \mathrm{Int}\left(E^2_{c,cc} \cap \hat{\Omega}_5\right) \neq \emptyset$, then we can adopt Conic Switching Law II as in § 3.2 to drive trajectories starting in $\mathrm{Int}\left(E^2_{c,cc}\right)$ asymptotically toward the origin.

Next we show that the 'only if' part is also true. The proof is by contradiction. Assume that trajectories starting in $\mathrm{Int}\left(E^2_{c,cc}\right)$ can be driven asymptotically toward the origin but $\mathrm{Int}\left(E^2_{c,cc} \cap \hat{\Omega}_j\right) = \emptyset$, $j = 1,3,5$. Assume the boundary of the conic section of $E^2_{c,cc}$ in quadrant II is formed by two rays l_1 and l_2 through the origin. Note that subsystem 1 (resp. 2) is of clockwise (resp. counterclockwise) direction in $E^2_{c,cc}$ as in figure 15.

Now let $x_0 \neq 0$ be in the interior of the conic section. Let F be a region formed by l_1, l_2 and the trajectory T_1 and T_2 (the shaded region in figure 15). Here T_i ($i = 1,2$) is the trajectory if the system starts at x_0 and follows subsystem i until it intersects l_i. We can show that for

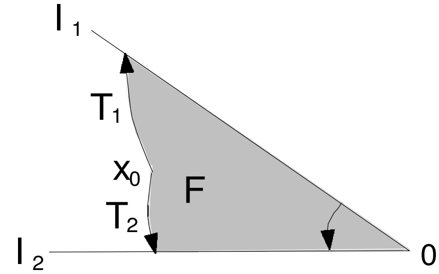

Figure 15. The region F.

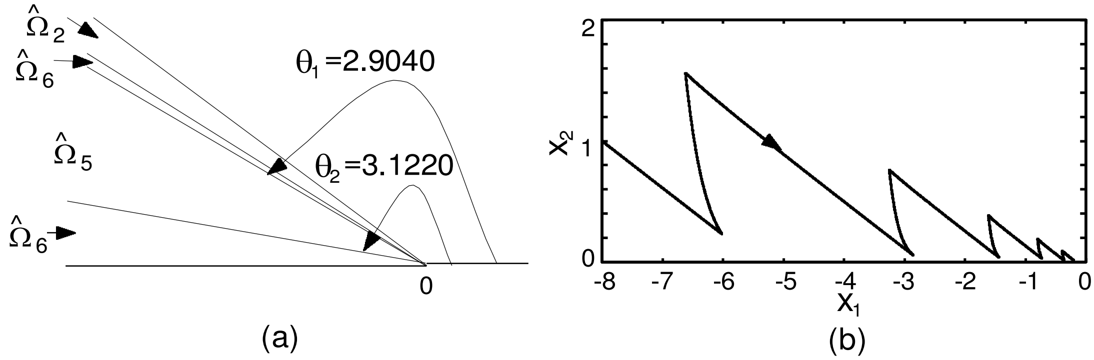

Figure 16. Example 3: (*a*) the conic sections; (*b*) the trajectory in $\hat{\Omega}_5$ in $E^2_{c,cc}$.

any switching law, the trajectory of the switched system starting with x_0 will never enter $\mathrm{Int}(F)$ (this can be proved by using similar techniques as in the proof of Theorem 2), however this is a contradiction to the assumption. □

In this case, if the conditions in Lemma 4 are satisfied, then trajectories starting in $\mathrm{Int}(E^2_{c,cc})$ can be driven asymptotically toward the origin using Conic Switching Law II.

Example 3: Consider a switched system (2) consisting of two subsystems with unstable nodes where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 20 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -37 & -152 \\ 14.25 & 58 \end{bmatrix}$$

It can be shown that $\mathrm{Int}(E^2_{c,cc} \cap \hat{\Omega}_5) \neq \varnothing$, so trajectories starting in $E^2_{c,cc}$ can be driven asymptotically toward the origin. Figure 16 shows (a) the conic sections in $E^2_{c,cc}$, (b) the trajectory of the system with $x_0 = (-8, 1)^{\mathrm{T}}$. Here, the system starts with subsystem 2 and when the trajectory first intersects the boundary of the conic section of $\hat{\Omega}_5$ in $E^2_{c,cc}$ in quadrant II, it switches to subsystem 1 and then to subsystem 2, 1, etc. upon intersecting boundary of the conic section according to Conic Switching Law II.

4.3. *Case 3*

We only discuss Case 3(a) (see Case 3(a) in figure 13). In this case $E^1_{c,c}$ and $E^2_{c,c}$ are Type A regions. We first claim that $\mathrm{Int}(E_{c,cc} \cap \hat{\Omega}_j) = \varnothing$ and $\mathrm{Int}(E_{cc,c} \cap \hat{\Omega}_j) = \varnothing$, $j = 1, 3, 5$. This can be proved using arguments similar to the proof of Lemma 3 by showing $|\theta_1| + |\theta_2| < \pi$. Hence, the trajectory must not always be inside $E_{c,cc}$ ($E_{cc,c}$) in order to be driven asymptotically toward the origin.

In fact, we can adopt a similar test as in the case of two subsystems with unstable foci of the same direction in § 3.1. In particular, Conic Switching Law I can be modified as follows.

Conic Switching Law III: *Switch to subsystem* 1 *in* $E_{c,cc}$ *and subsystem* 2 *in* $E_{cc,c}$. *While in* $E^1_{c,c}$ *and* $E^2_{c,c}$, *switch to subsystem* 1 *whenever the system trajectory enters the conic sections of* $E^1_{c,c} \cap \Omega_j$ *or* $E^2_{c,c} \cap \Omega_j$, $j = 1, 3, 5$ *and switch to subsystem* 2 *whenever the system trajectory enters the conic sections of* $E^1_{c,c} \cap \Omega_j$ *or* $E^2_{c,c} \cap \Omega_j$, $j = 2, 4, 6$.

The following theorem provides necessary and sufficient conditions for the asymptotical stabilizability of the switched system.

Theorem 3: *Consider a switched system* (2) *consisting of two subsystems with unstable nodes as in Case* 3(a). *Let* l *be a ray that goes through the origin. Let* $x_0 \neq 0$ *be on* l. *Let* x^* *be the point on* l *where the trajectory intersects* l *for the first time after leaving* x_0, *when the switched system evolves according to Conic Switching Law III. Then the switched system is asymptotically stabilizable if and only if* $\|x^*\|_2 < \|x_0\|_2$.

The conditions in the above theorem are not easy to check. The following corollary provides a simplified sufficient condition to check whether a switched system is not asymptotically stabilizable.

Corollary 1 (The Parallelogram Sufficient Condition)**:** *Consider a switched system* (2) *consisting of two subsystems with unstable nodes as in Case* 3(a). *If the boundaries of the regions are denoted as* l_1, l_2, l_3, l_4 *as shown in figure* 17. *Let* x_0 *be a non-zero point on* l_3. *If* $x_0x_1x_2x_3$ *is a parallelogram on* $\mathbb{R}^2$, *where* $x_1 \in l_2$, $x_2 \in l_4$, $x_3 \in l_3$ *and* $x_0x_1 \parallel x_2x_3 \parallel l_1$, $x_1x_2 \parallel x_0x_3$. *Then the switched system is not asymptotically stabilizable if* $\|x_3\|_2 \geq \|x_0\|_2$.

Proof: Consider the trajectory starting at x_0 and evolving according to Conic Switching Law III. Notice that in $E_{c,cc}$, any point x on the trajectory is always farther from the origin than the corresponding point on the line segment x_0x_1 which is on the same ray through the origin, so the trajectory is farther away

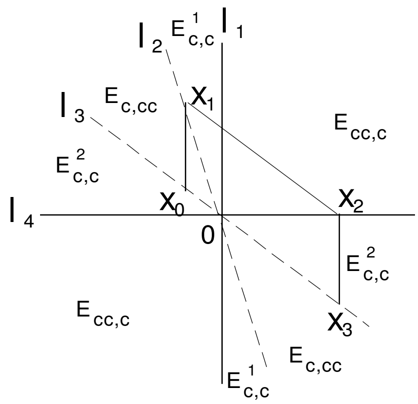

Figure 17. Figure for Corollary 1.

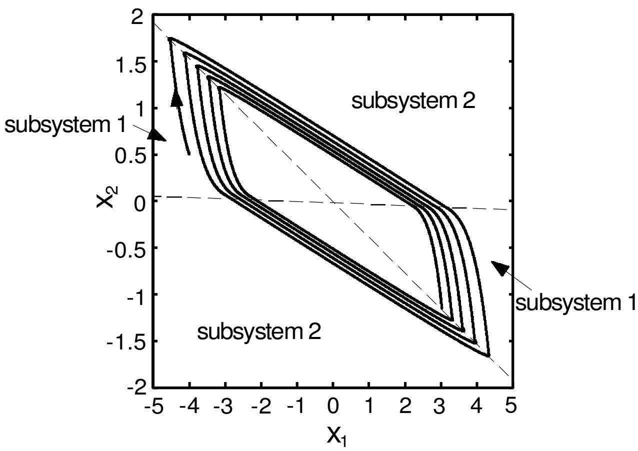

Figure 18. The trajectory for Example 4.

from the origin than x_0x_1 in $E_{c,cc}$. Similarly, in $E^1_{c,c}$, we can also show that the points on the trajectory are farther from the origin than the corresponding points on x_1x_2. And we can have the similar results for $E_{cc,c}$, $E^2_{c,c}$. Therefore, the trajectory of the system starting at x_0 until intersecting l_3 again for the first time must be outside the parallelogram, so if $\|x_3\|_2 \geq \|x_0\|_2$, this would imply $\|x'\|_2 \geq \|x_0\|_2$, where x' is the intersecting point on l_3 when the system trajectory evolves according to Conic Switching Law III for half round. Moreover, this implies that $\|x^*\|_2 \geq \|x_0\|_2$ after one round. So the switched system is not asymptotic stabilizable by Theorem 3. $\square$

Example 4: Consider a switched system (2) consisting of two subsystems with unstable nodes where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 58 & 152 \\ -14.25 & -37 \end{bmatrix}$$

The switched system is asymptotically stabilizable using Conic Switching Law III. Figure 18 shows the trajectory of the system with $x_0 = (-4, 0.5)^{\mathrm{T}}$. Here the system starts with subsystem 1 and when the trajectory first intersects a dashed line, it switches to subsystem 2 and then to subsystem 1, 2, etc. upon intersecting the dashed lines according to Conic Switching Law III.

5. Stabilization of second-order LTI switched systems with saddle points

In this section, we study the asymptotic stabilizability of switched systems (2) consisting of two subsystems with saddle points.

The trajectory of a simple second-order LTI system with saddle point where

$$A = \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}, \qquad c_1 > 0 > c_2$$

is shown in figure 1(c). The trajectory travels clockwise in quadrant I since $\langle (x_2, -x_1)^{\mathrm{T}},\ (c_1x_1, c_2x_2)^{\mathrm{T}} \rangle = (c_1 - c_2)x_1x_2 > 0$. Similarly, the trajectories in quadrants II, III and IV travel counterclockwise, clockwise and counterclockwise, respectively. Hence E_c includes quadrants I and III, while E_{cc} includes quadrants II and IV. While on the vertical axis, the trajectory tends toward the origin. On the horizontal axis, the trajectory tends toward ∞. In general, for a second-order system with saddle point, linear transformation techniques can used to determine the direction of the system at a point.

Figure 19 shows exhaustively the six possible arrangements of $E_{c,c}$, $E_{c,cc}$, $E_{cc,c}$ and $E_{cc,cc}$. As in §4, we illustrate these cases by fixing E_{1c} to be in the I, III quadrants and E_{1cc} to be in the II, IV quadrants.

Remark 11: Similar to §4, in the six cases, we assume that none of the two eigenvector directions of A_2 coincides with any eigenvector direction of A_1. Again this assumption is only for the purpose of discussion. In the case that some eigenvector directions do coincide, it is not difficult to carry out similar analysis as to the one discussed in the following.

5.1. *Case 1*

We only discuss Case 1(a) (see Case 1(a) in figure 19). Case 1(b) is analogous. Similarly we only discuss Cases 2(a) and 3(a) in Cases 2 and 3. In this case, $E_{c,c}$, $E_{cc,cc}$ are Type A regions. $E_{cc,c}$ is a Type B region and $E_{c,cc}$ is a Type C region. In this case, we have the following lemma.

Lemma 5: *In Case* 1(a), *for the Type C region* $E_{c,cc}$, *we have*

$$\mathrm{Int}\left(E_{c,cc} \cap \hat{\Omega}_1\right) \cup \mathrm{Int}\left(E_{c,cc} \cap \hat{\Omega}_3\right) \cup \mathrm{Int}\left(E_{c,cc} \cap \hat{\Omega}_5\right) \neq \varnothing$$

Proof: See figure 20, assume that rays l_1, l_2, l_3, l_4 through the origin are in the eigenvector directions of the two subsystems, respectively. So the conic section of $E_{c,cc}$ in quadrant I is bounded by l_1 and l_2.

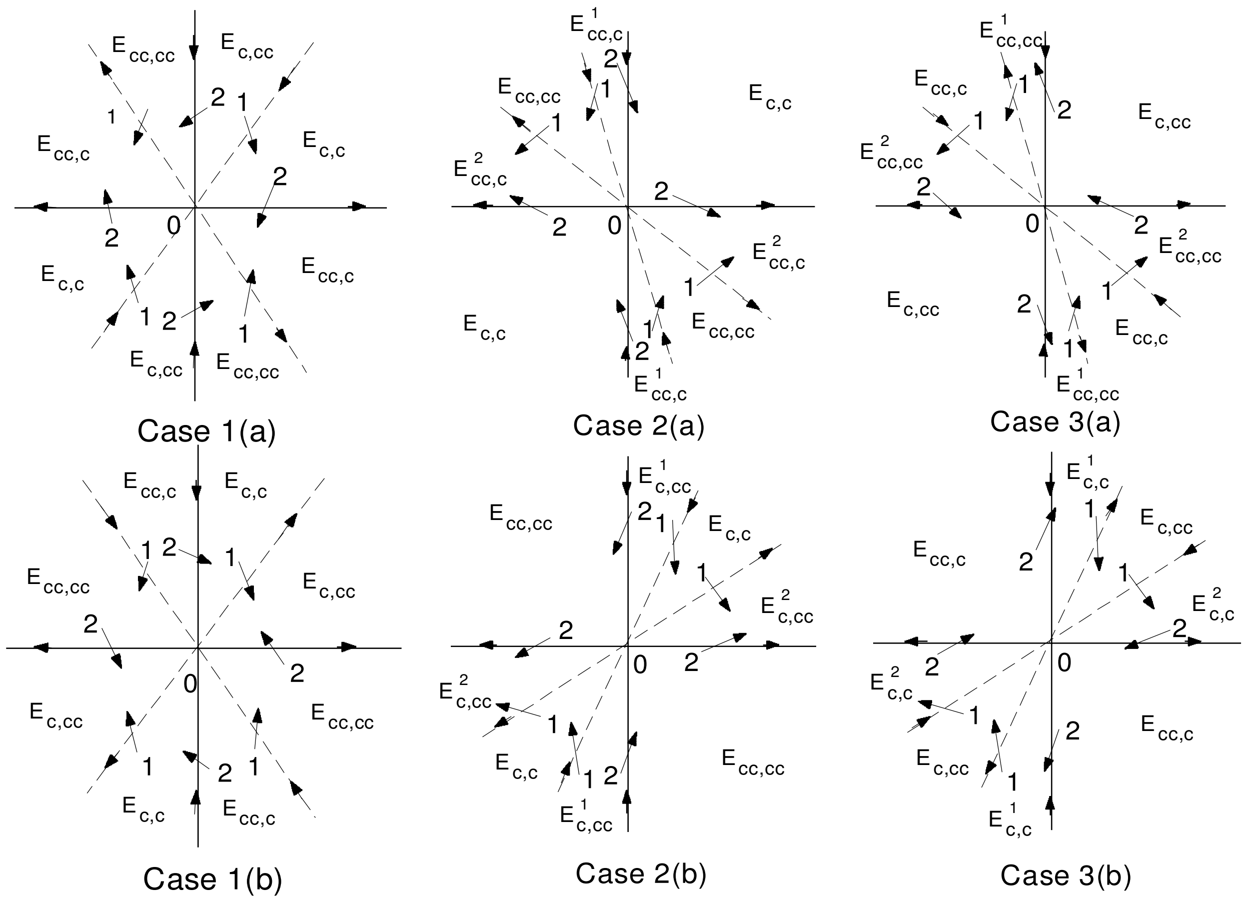

Figure 19. Cases for subsystems with saddle points.

At a point x close enough to l_1, we must have $|\theta_1|$ very close to the absolute value of the angle from x to the negative direction of l_1, i.e. $|\alpha|$ very close to 0 as in figure 20 (all angles are confined to $[-\pi, \pi)$). Also note that $|\theta_2|$ is greater than the angle from x to l_4. Therefore as $|\alpha|$ is small enough, $|\theta_1| + |\theta_2|$ will be greater than or equal to the addition of absolute value of the angle from x to l_4 and the absolute value of the angle from x to the negative direction of l_1, which is greater than π. So by Remark 7 in §3.2, we conclude that $\operatorname{Int}(E_{c,cc} \cap \hat{\Omega}_1) \cup \operatorname{Int}(E_{c,cc} \cap \hat{\Omega}_3) \cup \operatorname{Int}(E_{c,cc} \cap \hat{\Omega}_5) \neq \emptyset$. $\square$

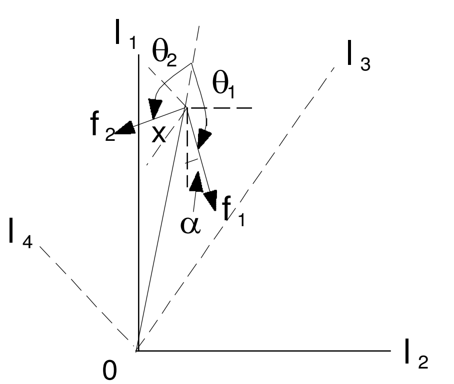

Figure 20. Figure for proof of Lemma 5.

For the Type B region $E_{cc,c}$, we have the following lemma.

Lemma 6: *If*

$$\operatorname{Int}(E_{cc,c} \cap \hat{\Omega}_1) \cup \operatorname{Int}(E_{cc,c} \cap \hat{\Omega}_3) \cup \operatorname{Int}(E_{cc,c} \cap \hat{\Omega}_5) \neq \emptyset$$

then trajectories starting in $E_{cc,cc}$, $E_{c,c}$ and $E_{cc,c}$ can be driven asymptotically toward the origin.

By Lemmas 5 and 6, if the condition of Lemma 6 holds, then the system can be asymptotically stabilized from any initial point on $\mathbb{R}^2$; Conic Switching Law II as in §3.2 can then be used.

Example 5: Consider a switched system (2) consisting of two subsystems with saddle points where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -10 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4.5 & -5.5 \\ -5.5 & -4.5 \end{bmatrix}$$

It can be shown that $\operatorname{Int}(E_{cc,c} \cap \hat{\Omega}_j) \neq \emptyset$, $j = 1, 3, 5$, so trajectories starting in $E_{cc,c}$ can be driven asymptotically

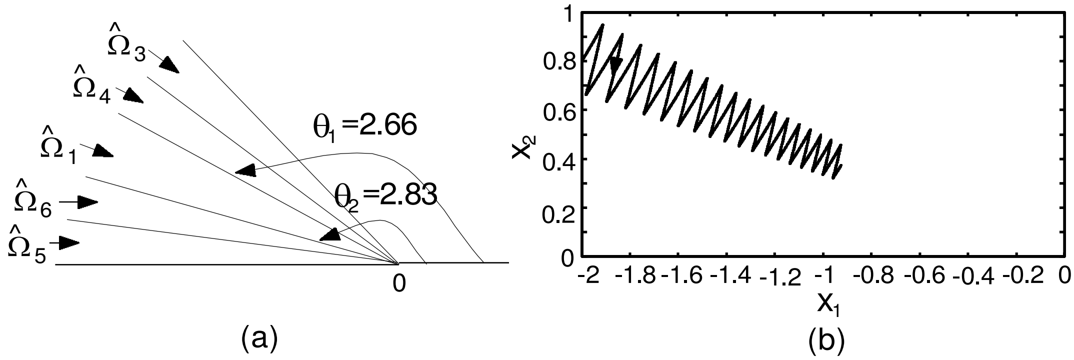

Figure 21. Example 5: (*a*) the conic sections; (*b*) the trajectory in $\hat{\Omega}_1$ in $E^2_{cc,c}$.

toward the origin. Figure 21 shows (*a*) the conic sections in $E_{cc,c}$, (*b*) the trajectory of the system with $x_0 = (-2, 0.8)^{\mathrm{T}}$. Here, the system starts with subsystem 2 and when the trajectory first intersects the boundary of the conic section of $\hat{\Omega}_1$ in $E^2_{cc,c}$ in quadrant II, it switches to subsystem 1 and then to subsystem 2, 1, etc. upon intersecting boundary of the conic section according to Conic Switching Law II.

5.2. *Case 2*

We only discuss Case 2(a) (see Case 2(a) in figure 19). In this case, $E_{c,c}$ and $E_{cc,cc}$ are Type A regions, $E^2_{cc,c}$ is a Type B region and $E^1_{cc,c}$ is a Type C region.

By adopting similar arguments as in Lemma 5, we can show that

$$\mathrm{Int}\,(E^1_{cc,c} \cap \hat{\Omega}_1) \cup \mathrm{Int}\,(E^1_{cc,c} \cap \hat{\Omega}_3) \cup \mathrm{Int}\,(E^1_{cc,c} \cap \hat{\Omega}_5) \neq \varnothing$$

Furthermore, it can be shown that $\mathrm{Int}\,(E^2_{cc,c} \cap \hat{\Omega}_j) = \varnothing$, $j = 1, 3, 5$.

Therefore, in this case, the system is *not asymptotically stabilizable*. Yet for any initial point in $E^1_{cc,c}$, we can use Conic Switching Law II to keep the trajectory in $E^1_{cc,c}$ and to drive the trajectory asymptotically toward the origin.

Example 6: Consider a switched system (2) consisting of two subsystems with saddle points where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -10 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.6667 & 0.8889 \\ -2.0000 & -1.6667 \end{bmatrix}$$

It can be shown that all the points in $E^1_{cc,c}$ belong to $\hat{\Omega}_1$. Specifically, we keep the trajectory between the rays l_1 with angle 1.6 and l_2 with angle 1.8 (Figure 22(*a*)). Figure 22 shows (a) l_1 and l_2 in $E^1_{cc,c}$, (b) the trajectory of the system with $x_0 = (-2, 10)^{\mathrm{T}}$. In this example, the stabilizing control law is derived using a slight modification of Conic Switching Law II. The system starts with subsystem 2 and when the trajectory first intersects the l_1, it switches to subsystem 1 and then to subsystem 2, 1,

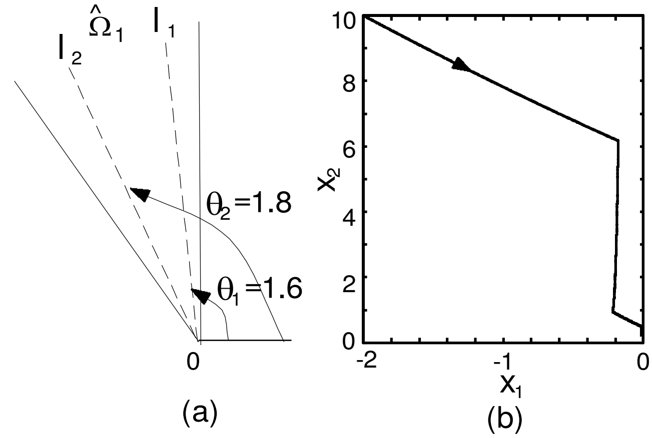

Figure 22. Example 6: (*a*) l_1 and l_2; (*b*) the trajectory in $\hat{\Omega}_1$ in $E^2_{cc,c}$.

etc. upon intersecting l_2, l_1, etc. similarly to Conic Switching Law II.

5.3. *Case 3*

We only discuss Case 3(a) (see Case 3(a) in figure 19). In this case, $E^1_{cc,cc}$ and $E^2_{cc,cc}$ are Type A regions.

In this case, we claim that

$$\mathrm{Int}\,(E_{c,cc} \cap \hat{\Omega}_1) \cup \mathrm{Int}\,(E_{c,cc} \cap \hat{\Omega}_3) \cup \mathrm{Int}\,(E_{c,cc} \cap \hat{\Omega}_5) \neq \varnothing$$

In fact, here especially $\mathrm{Int}\,(E_{c,cc} \cap \hat{\Omega}_3) \neq \varnothing$. This can be proved using similar arguments as in the proof of Lemma 5 by considering points close enough to the vertical axis.

Furthermore, we claim that the system is always *asymptotically stabilizable* in this case. This is because for any initial point $x \in \mathbb{R}^2$, we can always first choose appropriate switchings such that the system trajectory is driven into $E_{c,cc}$ and then adopt Conic Switching Law II so as to keep the system trajectory in one conic section of $\mathrm{Int}\,(E_{c,cc} \cap \hat{\Omega}_j), j = 1, 3, 5$ (there must be one available). In this way, the system can be asymptotically stabilized.

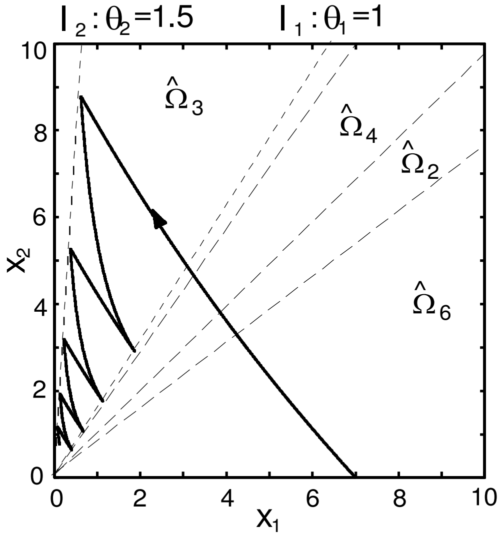

Figure 23. The trajectory in $\hat{\Omega}_3$ in $E_{c,cc}$ and the conic sections for Example 7.

Example 7: Consider a switched system (2) consisting of two subsystems with saddle points where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.5714 & -0.8571 \\ 1.7143 & 1.5714 \end{bmatrix}$$

This system is asymptotically stabilizable as discussed above. Here we keep the trajectory between ray l_1 with angle 1 and ray l_2 with angle 1.5. Figure 23 shows the trajectory of the system with $x_0 = (7,0)^{\mathrm{T}}$ and the conic sections. Here, the stabilizing control law is derived using a slight modification of Conic Switching Law II. The system starts with subsystem 2 and when the trajectory first intersects the l_2, it switches to subsystem 1 and then to subsystem 2, 1, etc. upon intersecting l_1, l_2, etc. similarly to Conic Switching Law II.

6. Several subsystems

Having studied switched systems consisting of two subsystems in detail in §§ 3, 4, 5, now we turn to switched systems (2) consisting of $M \geq 2$ several subsystems, i.e. $I = \{1, 2, \ldots, M\}$ in (2). The methodology based on the geometric properties of vector fields developed in the previous three sections can be similarly extended and applied to several subsystems. Here we will concentrate only on the case when all subsystems are with unstable foci for the purpose of illustration. There are two possible cases.

6.1. *Case 1. All subsystems of the same direction*

Without loss of generality, assume that all the M subsystems are with unstable foci and of clockwise direction. We adapt Conic Switching Law I in § 3.1 to be

Conic Switching Law IV: *Associate with each point on* $\mathbb{R}^2$ *a subsystem* k *where* $|\theta_k|$ *is the biggest among all* $|\theta_i|$*'s (the* θ_i*'s here are all negative) (if some* $|\theta_{k_1}| = |\theta_{k_2}|$, *we may choose subsystem according to Remark* 6 *in* § 3.1).

Similar arguments as in § 3.1 can be used for several subsystems to show the following theorem.

Theorem 4: *Consider a switched system* (2) *consisting of* $M \geq 2$ *subsystems with unstable foci and of the same direction. Let* l *be a ray that goes through the origin. Let* $x_0 \neq 0$ *be on* l. *Let* x^* *on* l *be the point where the trajectory intersects* l *for the first time after leaving* x_0, *when the switched system evolves according to Conic Switching Law IV. The switched system is asymptotically stabilizable if and only if* $\|x^*\|_2 < \|x_0\|_2$.

6.2. *Case 2. Not all subsystems of the same direction*

Assume that $K\,(K > 0)$ subsystems are of clockwise direction and $L\,(L > 0)$ subsystems are of counterclockwise direction ($K + L = M$). The following theorem can now be shown.

Theorem 5: *A switched system* (2) *consisting of* $K\,(K > 0)$ *subsystems* $S_1^-, \ldots, S_K^-$ *with unstable foci and of clockwise direction and* $L\,(L > 0)$ *subsystems* $S_1^+, \ldots, S_L^+$ *with unstable foci and of counterclockwise direction is asymptotically stabilizable if and only if at least one of the following three conditions holds:*

(1). *The switched system consisting of* $S_1^-, \ldots, S_K^-$ *is asymptotically stabilizable.*

(2). *The switched system consisting of* $S_1^+, \ldots, S_L^+$ *is asymptotically stabilizable.*

(3). *There exist* i *and* j *with* $1 \leq i \leq K$ *and* $1 \leq j \leq L$ *such that the switched system consisting of the two subsystems* S_i^- *and* S_j^+ *is asymptotically stabilizable.*

Proof: If any one of statement 1, 2 holds, then we can use Conic Switching Law IV to asymptotically stabilize the switched system. Or if statement 3 holds, then we can use Conic Switching Law II for the two subsystems to asymptotically stabilize the system.

Next we show that the 'only if' part is also true. First of all, if the switched system is asymptotically stabilizable, and if statement 1 or 2 holds, then the only if part is also true. So in the following, we assume that conditions 1 and 2 do not hold and we want to prove that condition 3 must hold. The proof is by contradiction. Assume that in this case, switched system consisting of any two subsystems S_i^- and S_j^+ is not asymptotically stabilizable. So it is implied that

$$\mathrm{Int}\,(\hat{\Omega}_1) \cup \mathrm{Int}\,(\hat{\Omega}_3) \cup \mathrm{Int}\,(\hat{\Omega}_5) = \varnothing$$

for any S_i^- and S_j^+.

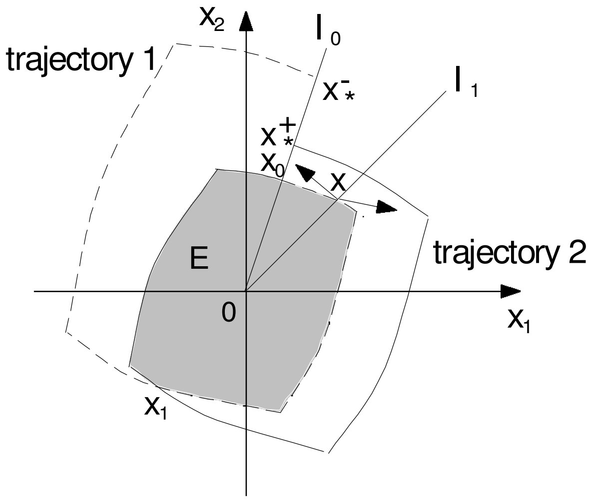

Figure 24. Figure for the proof of Theorem 3.

Now consider figure 24 and assume that x_0 is a non-zero initial point on ray l_0 in $\mathbb{R}^2$. Let trajectory 1 (dotted curve) be the trajectory starting from x_0 and switching among $S_1^-, \ldots, S_K^-$ using Conic Switching Law IV. Trajectory 1 intersects l_0 at x_*^- for the first time. Since statement 1 does not hold in this case, we have $\|x_*^-\|_2 \geq \|x_0\|_2$. Similarly, trajectory 2 is the trajectory under Conic Switching Law IV for $S_1^+, \ldots, S_L^+$, and it intersects l_0 at x_*^+ with $\|x_*^+\|_2 \geq \|x_0\|_2$.

It is clear that trajectory 1 intersects trajectory 2 at some point x_1. Now we argue that any trajectory starting at x_0 cannot enter the interior of the region E bounded by trajectory 1 and trajectory 2 (the shaded region in figure 24).

Assume that x is on the boundary of E, e.g. x is on trajectory 1. Notice that the vector field of any one of the subsystems $S_1^-, \ldots, S_K^-$ cannot lead the state into E since trajectory 1 is already the 'best' trajectory using Conic Switching Law IV. While for vector fields of $S_1^+, \ldots, S_L^+$, since we have

$$\mathrm{Int}\,(\hat{\Omega}_1) \cup \mathrm{Int}\,(\hat{\Omega}_3) \cup \mathrm{Int}\,(\hat{\Omega}_5) = \varnothing$$

for any S_i^- and S_j^+ by our assumption, $S_1^+, \ldots, S_L^+$ also cannot lead the state into E. The similar arguments can be applied to points on the boundary of E which are on trajectory 2.

Therefore, any trajectory starting at x_0 cannot enter $\mathrm{Int}\,(E)$, so the switched system is not asymptotically stabilizable. This is a contradiction to our assumption of the asymptotic stabilizability of the switched system. So condition 3 must hold. □

7. Conclusions

In this paper, we study the problem of asymptotically stabilizing switched systems consisting of second-order LTI subsystems. Necessary and sufficient conditions are obtained for the asymptotic stabilizability of such switched systems. Conic switching laws are proposed which asymptotically stabilize a switched system if the system is asymptotically stabilizable. Complete studies have been carried out for two second-order LTI subsystems with foci, nodes and saddle points. Based on the angles of vector fields and the geometric properties of $\mathbb{R}^2$, our approach decomposes $\mathbb{R}^2$ into different regions and carries out thorough case studies. As is mentioned in §6, the approach is also applicable to more than two subsystems. It is also worth mentioning that for subsystems of mixed type (e.g. one subsystem with focus and another with node), similar region decomposition and design approach can be adopted. The approach in this paper has been implemented via computer software as a systematic way for asymptotic stabilization of second-order LTI switched systems. Higher order LTI switched systems may also be studied using the results obtained in this paper by projecting the trajectory of the system to some 2-dimensional subspaces, however this has not yet been done. In Hu *et al.* (1999), the conic switching law proposed in this paper is shown to be robust and be applicable to local stabilization of non-linear second-order switched systems. Finally, it is worth noting that earlier results of this paper have appeared in Xu and Antsaklis (1999 a, b). Note that further details about the approach in this paper may be obtained in Xu and Antsaklis (1999 c) or directly from the authors.

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