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Canonical Forms Under Dynamic Compensation and Zero Structure at Infinity*

by

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ABSTRACT

The exact relation between the interactor and the Hermite normal form of a system P is established and their relation to state feedback compensation is shown. The Smith-McMillan form at infinity of P is then derived from these canonical forms.

INTERACTOR AND HERMITE NORMAL FORM

The interactor ξ_P of a proper plant P(pxm) and its extension, the Hermite normal form H_P, were introduced in [1], [2] respectively as appropriate canonical forms of P under dynamic compensation. It was shown in [3] that $H_P = \xi_P^{-1}$ when P nonsingular. The main difficulty in establishing the relation between ξ_P and H_P in the general case lies in the fact that ξ_P in [1] is defined only when P has full rank. A generalized version of the interactor is introduced here to overcome this difficulty.

If rank P = r = p($\leq$m), the interactor is defined in [1] as the unique polynomial matrix ξ_P (pxp) which satisfies:

$$\lim_{s\to\infty} \xi_P P = K_P \quad , \quad \text{rank } K_P = p \tag{1}$$

with

$$\xi_P = \begin{bmatrix} 1 & 0 \\ u_{ij} & 1 \end{bmatrix} \text{diag } [s^{f_i}] \tag{2}$$

where u_{ij} is divisible by s (or is 0).

The generalized interactor of a proper P, where rank P = r $<$ min (p,m), is defined as follows: Consider the top first r lin. indep. rows of P and let P_r (rxp) denote these rows; let P_{p-r} denote the remaining p-r rows of P. This interchange of rows can be expressed as

$$C\,P = \begin{bmatrix} P_r \\ P_{p-r} \end{bmatrix} \tag{3}$$

where C is nonsingular with entries 0 and 1. Define the interactor ξ_P of P by:

$$\xi_P = \begin{bmatrix} \xi_{P_r} & 0 \\ \gamma_r & \gamma_{p-r} \end{bmatrix} C \tag{4}$$

where ξ_{P_r} is the interactor of Pr defined in (1), (2) and

$$[\gamma_r, \gamma_{p-r}]\, C\, P = [\gamma_r, \gamma_{p-r}] \begin{bmatrix} P_r \\ P_{p-r} \end{bmatrix} = 0 \tag{5}$$

where $[\gamma_r, \gamma_{p-r}] = \gamma$ a minimal basis of the left kernel of CP with γ_{p-r} row proper and in (lower left) Hermite normal form; note that such basis is uniquely specified by CP [4]. The unique ξ_P satisfies:

$$\lim_{s\to\infty} \xi_P P = \lim_{s\to\infty} \begin{bmatrix} \xi_{P_r} P_r \\ 0 \end{bmatrix} = \begin{bmatrix} K_{P_r} \\ 0 \end{bmatrix}, \text{ rank } K_{P_r} = r \tag{6}$$

When rank P = r = min(p,m) and the top r rows of P are lin. indep. then C = I and the above definition reduces to the definition of the interactor in [1].

The Hermite normal form H_P of P, where rank P = r $\leq$ min (p,m) satisfies [2]:

$$P\,\hat{P} = H_P = [\bar{H}, 0] \tag{7}$$

where $\hat{P}$ biproper (i.e. $\hat{P}$, $\hat{P}^{-1}$ proper) and the top first r lin. indep. rows of $\bar{H}$ (pxr) are:

$$H^* = \begin{bmatrix} 1/s^{m_1} & 0 \\ q_{ij} & 1/s^{m_r} \end{bmatrix} \tag{8}$$

where $q_{ij} = 0$ when $m_i = 0$ or $q_{ij} = \alpha/s^{m_{ij}}$ proper when $(m_{ij} <)\ m_i \neq 0$. Here H_P is the Hermite normal form of P over the principal ideal domain of proper transfer fcns (S = all monic polyn. in R[s], Π = s) [2].

<u>Proposition 1</u>:

$$\xi_P H_P = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \tag{9}$$

Proof: ξ_P is defined in (4). $[\gamma_r, \gamma_{p-r}]\, C\, H_P = \gamma\, C\, P\, \hat{P} = 0$ in view of (5), (7). The first r rows of C H_P are $[H^*, 0]$, that is $[\xi_{P_r}, 0]\, C\, H_P = [\xi_{P_r} H^*, 0] = [I_r 0]$ since it has been shown in [3], that $\xi_{P_r} H^* = I$. ΔΔΔ

<u>Linear state feedback (lsf) compensation</u>. It is now shown that H_P can be obtained as the closed loop transfer matrix when appropriate lsf is applied on P. To define lsf, consider the factorization $P = ND^{-1}$ which corresponds to the controllable realization Dz = u, y = Nz [5]. Let D be column proper with column degrees $\partial_{ci} D = d_i$, and define the lsf control law (F,G) by: u = Fz + Gv, $\partial_{ci} F < d_i$ and G real, $|G| \neq 0$; the closed loop transfer matrix is $N(D-F)^{-1}G = (ND^{-1}).(DD_F^{-1}G) = P\,\hat{P}_{F,G}$ ($\hat{P}_{F,G}$ biproper).

<u>Lemma 2</u>. Let rank P = r = p ($\leq$ m) and let Q be a (pxp) polyn. matrix such that

$$\lim_{s\to\infty} QP = K_P \quad , \quad \text{rank } K_P = p \tag{10}$$

Then there exists lsf (F,G) such that

$$Q\,P\,\hat{P}_{F,G} = [I_p, 0] \tag{11}$$

If p = m, (F,G) is unique.
Proof: Find F and G so that $QN = K_P(D-F)$ and $K_P G = [I_p, 0]$. It can be shown that such (F,G) always exists; it is unique when p = m. ΔΔΔ

<u>Proposition 3</u> Let rank P = r $<$ min (p,m). There exists lsf (F,G) such that

$$P\,\hat{P}_{F,G} = \xi_P^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = H_P \tag{12}$$

If r = m, (F,G) is unique.

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Proof: Use Lemma 2 to find $\hat{P}_{F,G}$ such that $\xi_{Pr}\, P_r\, \hat{P}_{F,G} = [I_r, 0]$ $\Delta\Delta\Delta$

A special case of this result (r=p) has been shown in [6]; note also that $P\hat{P}_{F,G} = H_P$ is implied in [2] and used in [3] and elsewhere. Here (F,G) is easily derived and it is shown to be unique when rank P = m.

ZERO STRUCTURE AT INFINITY

P(s) has an infinite zero of order k if $P(1/\lambda)$ has a finite zero of order k at $\lambda = 0$ [4,7,8]. The infinite zero orders n_i of P are directly available if the Smith-McMillan factorization at infinity is known, namely

$$P = B_1 \begin{bmatrix} \Lambda_r & 0 \\ 0 & 0 \end{bmatrix} B_2 \tag{13}$$

where B_1, B_2 biproper and $\Lambda_r = \operatorname{diag}[s^{-n_i}]$ $n_i \leq n_{i+1}$ [9-12]. In (12), $\hat{P}_{F,G}$ is biproper which implies that the infinite zeros of P are exactly the infinite zeros of H_P. Even though all the information about the zero structure at infinity of P exists in H_P or ξ_P only partial results have been available, namely: when rank P = p = m, the highest degree in ξ_P is the largest infinite zero order of P [13]; if in addition ξ_P is diagonal the infinite orders are the degrees of the diagonal entries [14].

The following results establish the relation between ξ_P, H_P (or Q) and the Smith-McMillan form at infinity of P. Note that a version of Silverman's structure algorithm was used in [15] to derive B_1, B_2 of (13).

Proposition 4. Assume rank P = p and let Q satisfy (10). If Q is row proper, its row degrees are the infinite zero orders of P.

Proof: Interchange rows so that KQ(s) has row degrees $n_i \leq n_{i+1}$ and write $KQ = [\operatorname{diag} s^{n_i}]\,\hat{Q}$; $\hat{Q}$ is biproper.

In view of (11), (13) is derived with $B_1 = \hat{Q}^{-1}$, $B_2 = \begin{bmatrix} K & 0 \\ 0 & I_{m-p} \end{bmatrix} \hat{P}_{F,G}^{-1}$. $\Delta\Delta\Delta$

Proposition 5. Let rank P = p. $H_P(1/\lambda)$ is a polynomial matrix in Hermite normal form. If $S(\lambda)$ is its Smith form, S(1/s) is the Smith McMillan form at infinity of P.

Lemma 6. Let rank P = p. There exists real nonsingular matrix C so that ξ_{CP} is row proper.

Let rank P = r < min (p,m) and choose C in (4) as follows: Find row proper minimal basis γ of left kernel of P and collect p-r columns of γ to obtain γ_{p-r} row proper with row degrees those of γ; this specifies P_{p-r}. Note that $\gamma_{p-r}^{-1}\gamma_r$ proper. $H^*(1/\lambda)$ of Pr specifies the zero orders at infinity (Prop. 5). If the remaining r rows of P are rearranged to satisfy Lemma 6 then the row degrees of ξ_{Pr} are the zero orders at infinity of P. Having established the relation between the zeros at infinity and $H_P(\xi_P)$, it is straight forward to study the effect of feedback and cascade compensation on these zeros.

Example

$$P = \begin{bmatrix} 1/s+1 & 1/s+2 \\ 1/s+3 & 1/s+4 \end{bmatrix}; \quad \xi_P = \begin{bmatrix} s & 0 \\ -s^3+2s^2 & s^3 \end{bmatrix}$$

row proper with row degrees 1, 3 the infinite zero orders of P (Prop. 4). The Smith McMillan form at infinity is given by (13) with

$$\Lambda_P = \begin{bmatrix} 1/s & 0 \\ 0 & 1/s^3 \end{bmatrix}, \quad B_1 = \hat{\xi}_P^{-1} = \begin{bmatrix} 1 & 0 \\ (s-2)/s & 1 \end{bmatrix}$$

where ξ_P^{-1} $(=H_P) = \hat{\xi}_P^{-1}\Lambda_P$. Also

$$B_2 = \hat{P}_{F,G}^{-1} = \begin{bmatrix} s/s+1 & s/s+2 \\ 6s^2/(s+1)(s+3) & 8s^2/(s+2)(s+4) \end{bmatrix}$$

Note that $H_P(1/\lambda) = \begin{bmatrix} \lambda & 0 \\ \lambda-2\lambda^2 & \lambda^3 \end{bmatrix}$ (Prop. 5)

If P as above but with s on the second row numerators then

$$\xi_P = \begin{bmatrix} s & 0 \\ -s^3+2s^2 & s^2 \end{bmatrix}$$

which is not row proper. Interchange rows of P, that is

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad \text{Then} \quad \xi_{CP} = \begin{bmatrix} 1 & 0 \\ -s^2-2s & s^3 \end{bmatrix}$$

which is row proper with row degrees the infinite zero orders of P (Lemma 6, Prop. 4).

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