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Switched Systems Optimal Control Formulation and a Two Stage Optimization Methodology

Xuping Xu and Panos J. Antsaklis

Abstract— **This paper provides an approach toward solving optimal control problems of switched systems. In general, in such problems one needs to find both optimal continuous inputs and optimal switching sequences, since the system dynamics vary before and after every switching instant. After formulating the optimal control problem, we propose a two stage optimization methodology. Since many practical problems only concern Stage 1 optimization where the number of switchings and the order of active subsystems are given, we concentrate on Stage 1 and develop an algorithm based on differentiations of the value function with respect to the switching instants. The algorithm is also applied to general switched linear quadratic problems and the advantages are discussed.**

Keywords— **Switched Systems, Optimal Control, Nonlinear Optimizations.**

I. Introduction

A switched system is a hybrid system that consists of several subsystems and a switching law specifying the active subsystem at each time instant. Examples of switched systems can be found in chemical processes, automotive systems, and electrical circuit systems, etc.

Recently, optimal control problems of hybrid and switched systems have been attracting researchers from various fields in science and engineering, due to the problems' significance in theory and application. The available results in the literature can be classified into two categories, theoretical and practical. [2], [8], [9], [10], [12], [16] primarily focus on theoretical results. These results extend the classical maximum principle or dynamic programming approaches to such problems. However, because there are no efficient constructive methodologies suggested in these papers for obtaining optimal solutions, there is a significant gap between theoretical results and their applications to real-world examples. As to the second category of practical results, the researchers take advantage of the availability of high speed computers and efficient nonlinear optimization techniques to develop some methodologies for solving such problems (see e.g., [1], [5], [6], [7], [11]).

It is worth noting that because there are many different models and optimal control objectives for hybrid systems, the above papers often differ greatly in their problem formulations and approaches. Switched systems, on the other hand tend to be described by similar models, and similar optimal control problem formulations have appeared in the literature (e.g, [5], [6], [7], [9], [11], [13]). For an optimal control problem of a switched system, one needs to find both an optimal continuous input and an optimal switching sequence since the system dynamics vary before and after every switching instant. Due to the involvement of switching sequences, such a problem is difficult to solve. Most of the methods in the literature that we are aware of are based on some discretization of continuous-time space and/or discretization of state space into grids and use search methods for the resultant discrete model to find optimal/suboptimal solutions. But the discretization approaches may lead to computational combinatoric explosions and the solutions obtained may not be accurate enough. This paper provides an approach for solving optimal control problems of switched systems that is not based on discretizations and employs continuous nonlinear optimization to determine the switching instants.

In Section II, we formulate the optimal control problem and discuss some of the related issues. In Section III, a two stage optimization methodology is proposed under some additional assumptions. Since the two stage optimization is still difficult to implement, we concentrate on Stage 1 optimization where the number of switchings and the order of active subsystems are already given. Focusing on Stage 1 problems is appropriate because in many practical situations, we only need to study problems with fixed number of switchings and fixed order of active subsystems (e.g. the speeding up of an automobile power train only requires switchings from gear 1 to 2 to 3 to 4) and in such cases the solution to Stage 1 is indeed the optimal solution for the problem. On the other hand, Stage 1 optimization itself is already challenging enough and the solution to it is a first step toward solving the general problem which does not possess a good solution up to now. In Section IV, we derive a second-order search algorithm based on differentiations of the value function with respect to the switching instants, which is an extension of the algorithm in [14]. The algorithm is then applied to general switched linear quadratic (GSLQ) problems in Section V with advantages indicated. An example is given to illustrate our approach.

II. Problem Formulation

A. Switched Systems

We define a switched system as follows.

Definition 1 (Switched System) A switched system is a tuple $S = (D, F, L)$ where

• $D = (I, E)$ is a directed graph indicating the discrete structure of the system. The node set $I = \{1, 2, \cdots, M\}$ is the set of indices for subsystems. The directed edge set E is a subset of $I \times I - \{(i,i) | i \in I\}$ which indicates valid events. If an event $e = (i_1, i_2)$ takes place, the system

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switches from subsystem i_1 to i_2.

- $F = \{f_i : R^n \times R^m \times R \to R^n,\ i \in I\}$ with f_i describing the vector field for the ith subsystem $\dot{x} = f_i(x,u,t)$.
- $L = \{\Lambda_e | \Lambda_e \subseteq R^n,\ e \in E\}$ provides a logic constraint which relates the continuous state and the mode switchings. Note for $e = (i_1, i_2) \in E$, $\Lambda_e \neq \emptyset$. Only when $x \in \Lambda_e$, a switching from i_1 to i_2 is possible. □

From Definition 1, a switched system is a collection of subsystems related by a switching logic described by D. The continuous state x and the continuous input u satisfy $x \in R^n$ and $u \in R^m$. If a particular switching law is specified, then the switched system can be described as

$$\dot{x}(t) = f_{i(t)}(x(t), u(t), t) \tag{1}$$

$$i(t) = \varphi(x(t), i(t^-), t), \tag{2}$$

where $\varphi : R^n \times I \times R \to I$ determines the active subsystem at time t. Note that (1)-(2) are used as the definition of switched systems in some of the literature (e.g., [5]). Here we adopt Definition 1 rather than (1)-(2) because in design problems, in general, φ is not defined a priori and it is a designer's task to find a switching law. A salient feature of a switched system is that its continuous state x does not exhibit jumps at switching instants.

For a switched system S, the inputs of the system consist of both a continuous input $u(\cdot)$ and a switching sequence. We define a switching sequence as follows.

Definition 2 (Switching Sequence) For a switched system S, a switching sequence σ in $[t_0, t_f]$ is defined as

$$\sigma = ((t_0, i_0), (t_1, e_1), (t_2, e_2), \cdots, (t_K, e_K)), \tag{3}$$

with $0 \le K < \infty$, $t_0 \le t_1 \le t_2 \le \cdots \le t_K \le t_f$, and $i_0 \in I$, $e_k = (i_{k-1}, i_k) \in E$ for $k = 1, 2, \cdots, K$.

We define $\Sigma_{[t_0,t_f]} \stackrel{\triangle}{=} \{\sigma\text{'s in } [t_0, t_f]\}$. □

A switching sequence σ as defined above indicates that, if $t_k < t_{k+1}$, then subsystem i_k is active in $[t_k, t_{k+1})$ ($[t_K, t_f]$ if $k = K$); if $t_k = t_{k+1}$, then i_k is switched through at instant t_k ('switched through' means that the system switches from subsystem i_{k-1} to i_k and then to i_{k+1} all at instant t_k). For a switched system to be well-behaved, we generally exclude the undesirable *Zeno* phenomenon, i.e., infinitely many switchings in finite amount of time. Hence in Definition 2, we only allow nonZeno sequences which switch at most a finite number of times in $[t_0, t_f]$, though different sequences may have different numbers of switchings. We specify $\sigma \in \Sigma_{[t_0,t_f]}$ as a discrete input.

Example 1 (An automotive system) A manual transmission car with four gears is a good example of a switched system. If we denote the lateral position as x_1 and the velocity x_2, the system dynamics at gear i is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \alpha_i(x_2)u \end{aligned}$$

where $\alpha_i(x_2)$'s are the gear efficiency functions. Here I=$\{1,2,3,4\}$. If the gear can only be shifted one gear up or down, then E=$\{(1,2),(2,1),(2,3),(3,2),(3,4),(4,3)\}$; here $\Lambda_{(1,2)} = \Lambda_{(2,1)} = \{x | x_2 \in [10, 20]\}$, $\Lambda_{(2,3)} = \Lambda_{(3,2)} = \{x | x_2 \in [20, 40]\}$, $\Lambda_{(3,4)} = \Lambda_{(4,3)} = \{x | x_2 \in [40, 60]\}$. The inputs of this system are the continuous input u (the throttle position) and the switching sequence (gear shifting). □

B. An Optimal Control Problem

Problem 1: Consider a switched system $S = (D, F, L)$. Find a pair of admissible $\sigma^* \in \Sigma_{[0,t_f]}$ and $u^* \in U = \{$piecewise continuous function u on $[0, t_f]$ with $u(t) \in \Omega \subseteq R^m$, $\forall t \in [0, t_f]\}$ that have the properties:
(a) the system state trajectory is driven from x_0 at $t = 0$ to an $(n - l_f)$-dimensional smooth manifold $S_f = \{x | \phi_f(x) = 0, \phi_f : R^n \to R^{l_f}\}$ at t_f (t_f is given) and
(b) $J = \psi(x(t_f)) + \int_0^{t_f} L(x(t), u(t), t)dt$ is minimized. □

Problem 1 is a basic optimal control problem with fixed final time, and final state on a smooth manifold. In the following, we further assume that f, ψ, L, ϕ_f possess enough smoothness properties we need in our derivations.

The fixed final time formulation of Problem 1 is mainly for the convenience of subsequent studies in this paper. For a general problem with non-fixed final time, we can introduce an additional state variable and translate the problem into one with fixed final time (for details, see [15]).

C. The Maximum Principle and HJB Equations

Extensions of the maximum principle (MP) and the Hamilton-Jacobi-Bellman (HJB) equations for problems similar to Problem 1 have appeared in the literature. For example, [8], [10], [12] provide the MP for hybrid and switched systems, while [13], [16] derive the HJB equations for such systems. Note however these results are not readily useful for finding solutions for practical problems.

D. A Related Issue

A related issue is the existence of solutions. Even for simple switched systems consisting of linear subsystems, the optimal solution may not exist because of the nonZenoness requirement, as shown in the following example.

Example 2: Consider a switched system $S = (D, F, L)$, where $D = (I, E)$ with $I = \{1, 2\}$, $E = \{(1, 2), (2, 1)\}$, $F = \{f_1 = x + u,\ f_2 = -x + u\}$ and $\Lambda_{(1,2)} = \Lambda_{(2,1)} = R$. Find an optimal control (σ, u) such that $x(0) = 1$, $x(2) = 1$ and $J = \int_0^2 [(x(t) - 1)^2 + u^2(t)]dt$ is minimized.

Consider the switching sequence $\sigma_K = ((0,1),(1/K,(1,2)),(2/K,(2,1)),\cdots,((2K-1)/K,(1,2)))$ and $u(t) = 0$ for all $t \in [0, 2]$, then as $K \to \infty$, $J \to 0$. But $J = 0$ cannot be achieved because it requires infinite switchings in finite time. So the problem has no optimal solution in $\sigma \in \Sigma_{[0,2]}$ and $u \in U$. □

As seen from Example 2, the Zeno phenomenon may prevent us from finding an optimum. Two additional requirements which may be introduced to avoid Zenoness are proposed in [13]. They are the minimum dwell time switching requirement and the costs for switchings requirement.

III. Two Stage Optimization

In general, we need to find an optimal control input (σ^*, u^*) for Problem 1 such that

$$J(\sigma^*, u^*) = \min_{\sigma \in \Sigma_{[0,t_f]},\ u \in U} J(\sigma, u). \tag{4}$$

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Notice that for any fixed switching sequence σ, Problem 1 reduces to a conventional optimal control problem for which we only need to find an optimal continuous input u which minimizes $J_\sigma(u) = J(\sigma, u)$. This observation leads us toward solving Problem 1 using a two stage optimization methodology under some additional assumptions.

Lemma 1: For Problem 1, if
(a) an optimal solution (σ^*, u^*) exists and
(b) for any given switching sequence σ, there exists a corresponding $u^* = u^*(\sigma)$ such that $J_\sigma(u)$ is minimized, then the following equation holds

$$\min_{\sigma\in\Sigma_{[0,t_f]},\ u\in U} J(\sigma,u) = \min_{\sigma\in\Sigma_{[0,t_f]}} \min_{u\in U} J(\sigma,u). \qquad (5)$$

Proof: First we prove that $\min_{\sigma\in\Sigma_{[0,t_f]},\ u\in\mathcal{U}} J(\sigma,u) \le \inf_{\sigma\in\Sigma_{[0,t_f]}} \min_{u\in\mathcal{U}} J(\sigma,u)$. This is because for any fixed σ, there exists a $u^* = u^*(\sigma)$ such that $J(\sigma,u^*(\sigma)) = \min_{u\in\mathcal{U}} J(\sigma,u)$. But for this pair of $(\sigma,u^*(\sigma))$, we must have $J(\sigma^*,u^*) \le J(\sigma,u^*(\sigma))$, therefore we must have

$$J(\sigma^*,u^*) \le \inf_{\sigma\in\Sigma_{[0,t_f]}} J(\sigma,u^*(\sigma)) = \inf_{\sigma\in\Sigma_{[0,t_f]}} \min_{u\in\mathcal{U}} J(\sigma,u). \qquad (6)$$

While we also have the inequality

$$\inf_{\sigma\in\Sigma_{[0,t_f]}} \min_{u\in\mathcal{U}} J(\sigma,u) \le \min_{u\in\mathcal{U}} J(\sigma^*,u) = J(\sigma^*,u^*(\sigma^*)). \qquad (7)$$

And we can choose $u^*(\sigma^*) = u^*$, since for any other u, we must have $J(\sigma^*,u^*) \le J(\sigma^*,u)$ by the optimality of (σ^*,u^*). Hence combining (6) and (7) we have

$$J(\sigma^*,u^*) \le \inf_{\sigma\in\Sigma_{[0,t_f]}} \min_{u\in\mathcal{U}} J(\sigma,u) \le J(\sigma^*,u^*(\sigma^*)) = J(\sigma^*,u^*). \qquad (8)$$

Hence all inequalities in (8) must be equalities and the $\inf_{\sigma\in\Sigma_{[0,t_f]}}$ can be replaced by $\min_{\sigma\in\Sigma_{[0,t_f]}}$ so we obtain

$$J(\sigma^*,u^*) = \min_{\sigma\in\Sigma_{[0,t_f]},\ u\in\mathcal{U}} J(\sigma,u) = \min_{\sigma\in\Sigma_{[0,t_f]}} \min_{u\in\mathcal{U}} J(\sigma,u). \square$$

The right hand side of (5) needs twice the minimization process. This supports the validity of the following two stage optimization methodology.

Two stage optimization methodology

Stage 1 Fixing σ, solve the inner minimization problem.
Stage 2 Regarding the optimal cost for each σ as a function $J_1 = J_1(\sigma) = \min_{u\in U} J(\sigma,u)$, minimize J_1 with respect to $\sigma \in \Sigma_{[0,t_f]}$. □

We can implement the above methodology by the following algorithm.

A Two Stage Algorithm

Stage 1 *(a)* Fix the total number of switchings to be K and the order of active subsystems, let the minimum value of J with respect to u be a function of the switching instants, i.e., $J_1 = J_1(t_1, t_2, \cdots, t_K)$, and find J_1.
(b) Minimize J_1 with respect to $t_1, t_2, \cdots, t_K$.

Stage 2 *(a)* Vary the order of active subsystems to find an optimal solution under K switchings.
(b) Vary the number of switchings K to find an optimal solution for Problem 1. □

The above algorithm has high computational cost. In practice, many problems only require the solutions of the optimal continuous input and the optimal switching instants for Stage 1 where a fixed number of switchings and a fixed order of active subsystems are given. In general, explicit expressions of J_1 are difficult to obtain or quite complicated even for very simple problems (see e.g., Example 1 in [13]). Therefore it is necessary to use optimization methods that do not require explicit expression of J_1 as a function of t_k's to develop algorithms.

IV. A Second-Order Approach

In the following we focus on the problem where the number of switchings and the order of active subsystems are given (Stage 1) and we develop a second-order approach to determine the optimal switching instants. This is an extension of the algorithm in [14] which is motivated by the approach in [3], [4]. Note that in the following, the value functions we use may not be the optimal value functions under fixed switching sequences. We assume that the number of switchings is K and the order of subsystems is $i_0, i_1, \cdots, i_K$. The optimal switching instants $t_1, \cdots, t_K$ need to be determined (i.e., Step 1(b)).

In the derivations in this section, we assume that we have a nominal control input $u(\cdot)$ and nominal switching instants $t_1, t_2, \cdots, t_K$ (if possible, choose $u(\cdot)$ to be an optimal input corresponding to the current values of switching instants, but this is not mandatory). Assume $u(\cdot)$ not varying. We can regard the value function V^0 at t_0 (may not be optimal) as a function of $x_0, t_0, t_1, \cdots, t_K$ only. Similarly, the value function V at t_i will depend on $x(t_i), t_i, t_{i+1}, \cdots, t_K$ only. In the following we assume that the final set $S_{t_f} = R^n$. We denote $\frac{\partial V}{\partial x}$ as a row vector V_x, $\frac{\partial^2 V}{\partial x^2}$ as V_{xx} and so on.

A. Single Switching

Let us first consider the case of a single switching. We write a function with a superscript 0 whenever it is evaluated at t_0 and a superscript $1-$ (resp. $1+$) whenever it is evaluated at t_1- (resp. t_1+). It is not difficult to see that

$$V^0(x_0,t_0,t_1) = V^{1+}(x(t_1),t_1) + \int_{t_0}^{t_1} L(x,u,t)dt.$$

For a small variation dt_1 of t_1, we have

$$\begin{aligned}
& V^0(x_0,t_0,t_1+dt_1) \\
= \ & V^{1+}(x(t_1+dt_1),t_1+dt_1) + \int_{t_0}^{t_1+dt_1} L(x,u,t)dt \\
= \ & V^{1+}(x(t_1),t_1) + \int_{t_0}^{t_1} L(x,u,t)dt + V_x^{1+}dx(t_1) \\
& + V_{t_1}^{1+}dt_1 + L^{1-}dt_1 + \frac{1}{2}(dx(t_1))^T V_{xx}^{1+}dx(t_1) \\
& + \frac{1}{2}V_{t_1t_1}^{1+}dt_1^2 + dt_1 V_{t_1x}^{1+}dx(t_1) + \frac{1}{2}dt_1 L_x^{1-}dx(t_1)
\end{aligned}$$

$$+\frac{1}{2}dt_1 L_u^{1-} du(t_1) + \frac{1}{2}L_t^{1-} dt_1^2 + o(dt_1^2) \quad (9)$$

where

$$dx(t_1) = f^{1-}dt_1 + \tfrac{1}{2}(f_t^{1-} + f_x^{1-}f^{1-} + f_u^{1-}\dot{u}^{1-})dt_1^2 + o(dt_1^2)$$
$$du(t_1) = \dot{u}^{1-}dt_1 + o(dt_1)$$

By substituting $dx(t_1)$ and $du(t_1)$ into (9), we obtain

$$\begin{aligned}
&V^0(x_0, t_0, t_1 + dt_1) \\
&= V^0(x_0, t_0, t_1) + (V_x^{1+}f^{1-} + V_{t_1}^{1+} + L^{1-})dt_1 + \\
&\tfrac{1}{2}[(f^{1-})^T V_{xx}^{1+} f^{1-} + V_x^{1+}(f_t^{1-} + f_x^{1-}f^{1-} + f_u^{1-}\dot{u}^{1-}) + \\
&2V_{t_1x}^{1+}f^{1-} + V_{t_1t_1}^{1+} + L_x^{1-}f^{1-} + L_u^{1-}\dot{u}^{1-} + L_t^{1-}]dt_1^2 + o(dt_1^2) \\
&\triangleq V^0(x_0, t_0, t_1) + V_{t_1}^0 dt_1 + \tfrac{1}{2}V_{t_1t_1}^0 dt_1^2 + o(dt_1^2)
\end{aligned}$$

Now since $V^{1+}(x(t_1), t_1)$ is the value function for fixed $u(\cdot)$, we have the relationship

$$V_{t_1}^{1+} + V_x^{1+}f^{1+} + L^{1+} = 0 \quad (10)$$

By differentiating (10), we obtain

$$\begin{aligned}
&V_{t_1x}^{1+} = -(f^{1+})^T V_{xx}^{1+} - V_x^{1+}f_x^{1+} - L_x^{1+} \\
&V_{t_1t_1}^{1+} = -V_{t_1x}^{1+}f^{1+} - V_x^{1+}f_t^{1+} - L_t^{1+} - (V_x^{1+}f_u^{1+} + L_u^{1+})\dot{u}^{1+} \\
&= (f^{1+})^T V_{xx}^{1+} f^{1+} + (V_x^{1+}f_x^{1+} + L_x^{1+})f^{1+} - V_x^{1+}f_t^{1+} \\
&\quad -L_t^{1+} - (V_x^{1+}f_u^{1+} + L_u^{1+})\dot{u}^{1+}
\end{aligned}$$

With the help of the expressions of $V_{t_1x}^1$ and $V_{t_1t_1}^1$, we can write $V_{t_1}^0$ and $V_{t_1t_1}^0$ in the following form:

$$\begin{aligned}
&V_{t_1}^0 = L^{1-} - L^{1+} + V_x^{1+}(f^{1-} - f^{1+}) \\
&V_{t_1t_1}^0 = (f^{1-} - f^{1+})^T V_{xx}^{1+}(f^{1-} - f^{1+}) - (V_x^{1+}f_x^{1+} \\
&\quad +L_x^{1+})(f^{1-} - f^{1+}) + (V_x^{1+}(f_x^{1-} - f_x^{1+}) + L_x^{1-} - L_x^{1+})f^{1-} \\
&\quad +V_x^{1+}(f_t^{1-} - f_t^{1+}) + L_t^{1-} - L_t^{1+} + (V_x^{1+}f_u^{1-} + L_u^{1-})\dot{u}^{1-} \\
&\quad -(V_x^{1+}f_u^{1+} + L_u^{1+})\dot{u}^{1+}
\end{aligned}$$

If we know $V_{t_1}^0$, $V_{t_1t_1}^0$, we can use second-order nonlinear optimization methods to find the optimal switching instant.

B. Two or More Switchings

For switched systems with two or more switchings, we need to have more information to construct a second-order optimization algorithm. Assume that a system switches from subsystem 1 to 2 at t_1 and from subsystem 2 to 3 at t_2 ($t_0 \le t_1 \le t_2 \le t_f$). The value function then is

$$\begin{aligned}
V^0(x_0, t_0, t_1, t_2) &= V^{1+}(x(t_1), t_1) + \textstyle\int_{t_0}^{t_1-} L(x, u, t)dt \\
&= V^{2+}(x(t_2), t_2) + \textstyle\int_{t_0}^{t_2-} L(x, u, t)dt
\end{aligned}$$

By similar derivation to the above equations as in Section 4.1, $V_{t_1}^0$, $V_{t_1t_1}^0$, $V_{t_2}^0$, $V_{t_2t_2}^0$ can be obtained. To form a second-order search algorithm, the additional information we need is $V_{t_1t_2}^0$. The following important observation reveals some intrinsic relationship among different switching instants is helpful in the derivation of $V_{t_1t_2}^0$. In particular

(a) If u does not vary and $dt_1 = 0$, then $dx(t_1) = 0$ regardless of the change dt_2.

(b) However if u does not vary and $dt_2 = 0$, the state at t_2 will still have a nonzero variation $\delta x(t_2)$ which is the propagated variation due to the variation dt_1 at t_1. □

In the following, we refer to observations (a) and (b) as the forward decoupling principle.

Lemma 2: To the first order, $\delta x(t_2)$ is

$$\delta x(t_2) = A(t_2, t_1)(f^{1-} - f^{1+})dt_1 \quad (11)$$

where $A(t_2, t_1)$ is the state transition function for the variational time-varying equation in $[t_1, t_2]$

$$\delta\dot{x} = \frac{\partial f(x, u, t)}{\partial x}\delta x.$$

Proof: It can be readily obtained from the fact that (also see figure 1)

$$\delta x(t_2) = A(t_2, t_1 + dt_1)[(f^{1-} - f^{1+})dt_1 + o(dt_1)] + o(dt_1). \square$$

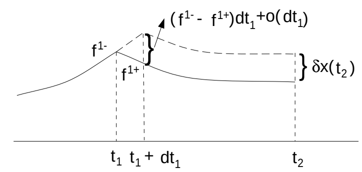

Fig. 1. The variation $\delta x(t_2)$.

Note that Lemma 2 is a special case of the needle-like variations in the proof of Pontryagin's MP.

In order to obtain $V_{t_1t_2}^0$, we need to consider dt_1, dt_2 at the same time and expand $V^0(x_0, t_0, t_1 + dt_1, t_2 + dt_2)$ to second-order to find the expression of the coefficient of dt_1dt_2 which is $V_{t_1t_2}^0$. The three terms that contribute to the coefficient of dt_1dt_2 are:

$$\frac{1}{2}(dx(t_2))^T V_{xx}^{2+} dx(t_2),\ dt_2 V_{t_2x}^{2+} dx(t_2),\ \frac{1}{2}dt_2 L_x^{2-} dx(t_2).$$

Using the forward decoupling principle, to first order

$$dx(t_2) = A(t_2, t_1)(f^{1-} - f^{1+})dt_1 + f^{2-}dt_2 \quad (12)$$

The coefficient of the dt_1dt_2 term can then be obtained by substituting (12) into the three terms and sum them.

$$\begin{aligned}
&V_{t_1t_2}^0 = (f^{1-} - f^{1+})^T A^T(t_2, t_1) V_{xx}^{2+} f^{2-} \\
&\quad +V_{t_2x}^{2+} A(t_2, t_1)(f^{1-} - f^{1+}) + \tfrac{1}{2}L_x^{2-} A(t_2, t_1)(f^{1-} - f^{1+}).
\end{aligned}$$

The forward decoupling principle can also be similarly extended to the case of K switchings to relate $\delta x(t_l)$ and dt_k ($k < l$) and the expression for $V_{t_kt_l}^0$ can be obtained. We summarize and extend the results obtained in this section by the following theorem.

Theorem 1: For a switched system with K switchings,

$$\begin{aligned}
&V^0(x_0, t_0, t_1 + dt_1, t_2 + dt_2, \cdots, t_K + dt_K) \\
&= V^0(x_0, t_0, t_1, t_2, \cdots, t_K) + \textstyle\sum_{k=1}^K V_{t_k}^0 dt_k + \\
&\tfrac{1}{2}\textstyle\sum_{k=1}^K V_{t_kt_k}^0 dt_k^2 + \sum_{1\le k<l\le K} V_{t_kt_l}^0 dt_k dt_l + o(\sum_{k=1}^K dt_k^2)
\end{aligned}$$

where

$$\begin{aligned}
&V_{t_k}^0 = L^{k-} - L^{k+} + V_x^{k+}(f^{k-} - f^{k+}) \\
&V_{t_kt_k}^0 = (f^{k-} - f^{k+})^T V_{xx}^{k+}(f^{k-} - f^{k+}) - (V_x^{k+}f_x^{k+} + \\
&\quad L_x^{k+})(f^{k-} - f^{k+}) + (V_x^{k+}(f_x^{k-} - f_x^{k+}) + L_x^{k-} - L_x^{k+})f^{k-} \\
&\quad +V_x^{k+}(f_t^{k-} - f_t^{k+}) + L_t^{k-} - L_t^{k+} + (V_x^{k+}f_u^{k-} + L_u^{k-})\dot{u}^{k-} \\
&\quad -(V_x^{k+}f_u^{k+} + L_u^{k+})\dot{u}^{k+} \\
&V_{t_kt_l}^0 = (f^{k-} - f^{k+})^T A^T(t_l, t_k) V_{xx}^{l+} f^{l-} - ((f^{l+})^T V_{xx}^{l+} \\
&\quad +V_x^{l+}f_x^{l+} + L_x^{l+})A(t_l, t_k)(f^{k-} - f^{k+}) \\
&\quad +\tfrac{1}{2}L_x^{l-} A(t_l, t_k)(f^{k-} - f^{k+}).
\end{aligned}$$

□

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With all the first and second derivatives of V^0 with respect to t_k's, the following second-order search algorithm can be used to update the switching instants.

A Second-Order Search Algorithm

Step 1 Choose a nominal $u(\cdot)$ and nominal switching instants $t_1, t_2, \cdots, t_K$.
Step 2 Fix $u(\cdot)$ and calculate $V^0_{t_k}$, $V^0_{t_k t_k}$ and $V^0_{t_k t_l}$.
Step 3 Find the descent direction $-[V^0_{t_k t_l}]^{-1}[V^0_{t_1}, V^0_{t_2}, \cdots, V^0_{t_K}]^T$ where $[V^0_{t_k t_l}]$ is the square matrix whose (k,l)-th component is $V^0_{t_k t_l}$. Update $[t_1, t_2, \cdots, t_K]^T$ by using line search methods along this direction (If t_k^{new} is outside $[t_0, t_f]$, enforce it to be on the boundary of $[t_0, t_f]$).
Step 4 Update $u(\cdot)$ by finding the optimal (or suboptimal) control input for the new switching instants.
Step 5 Repeat Step 2 to Step 4 until $\|[V^0_{t_k t_l}]^{-1}[V^0_{t_1}, V^0_{t_2}, \cdots, V^0_{t_K}]^T\|_2 < \epsilon$ (ϵ given). □

Remark: This approach has advantages but also certain drawbacks which are as follows.
(a) The conditions are derived under the assumption that $u(\cdot)$ is not varying during a search iteration. Yet in most cases, when the switching instants vary, the control input needs to vary correspondingly. Therefore, this approach can only give us the optimal switching instants for the nominal $u(\cdot)$.
(b) In general, $V_x^{1+}, V_{xx}^{1+}, \dot{u}^{1-}, \dot{u}^{1+}$ can only be determined after significant computational effort has been made.
(c) The computation of $A(t_l, t_k)$ is in general not easy.

Problem (a) may be partly resolved by updating the $u(\cdot)$ to be the corresponding optimal input (or near optimal) for the new switching instant at each new iteration. For problem (b), we can find out the values for V^{1+} at $(x(t_1), t_1)$ by integration and obtain a numerical approximation of V_x^1, (resp. V_{xx}^1) by observing the variation of V^1 (resp V_{xx}^{1+}) with respect to small variation of x. Similarly we can also find approximations for $\dot{u}^0, \dot{u}^{1+}$.

V. Application to General Switched Linear Quadratic Problems

In this section, we consider a special class of optimal control problems for switched systems, i.e., general switched linear quadratic (GSLQ) problems. For this class of problems, the above difficulties can be successfully addressed.

A. General Switched Linear Quadratic Problems

Problem 2 (GSLQ Problem) Consider a switched system $S = (D, F, L)$ with $E = I \times I - \{(i,i) | i \in I\}$, $\Lambda_e = R^n, \forall e \in E$ and linear subsystems $\dot{x} = A_i x + B_i u, i \in I$. If the order of active subsystems $i_0, i_1, \cdots, i_K$ is given, find the optimal switching instants $t_1, \cdots, t_K$ and the optimal control input $u(\cdot)$ such that the cost functional in general quadratic form

$$J = \tfrac{1}{2}x(t_f)^T Q_f x(t_f) + M_f x(t_f) + L_f + \int_{t_0}^{t_f} (\tfrac{1}{2}x^T Q x + x^T V u + \tfrac{1}{2}u^T R u + M x + N u + W) dt$$

is minimized, where t_0, t_f and $x(t_0) = x_0$ are given. $Q_f, M_f, L_f, Q, V, R, M, N, W$ are matrices of appropriate dimensions, with $Q_f, Q \geq 0$ and $R > 0$. □

Note that for the general quadratic control of a single linear system $\dot{x} = Ax + Bu$, we can use the dynamic programming approach to obtain the following results.

The optimal value function is

$$V^*(x,t) = \frac{1}{2}x^T P(t) x + S(t) x + T(t)$$

where $P(t) = P^T(t)$ and

$$\begin{aligned}
-\dot{P}(t) &= Q + P(t)A + A^T P(t) \\
&\quad -(P(t)B + V)R^{-1}(B^T P(t) + V^T) \\
-\dot{S}(t) &= M + S(t)A - (N + S(t)B)R^{-1}(B^T P(t) + V^T) \\
-\dot{T}(t) &= W - \tfrac{1}{2}(N + S(t)B)R^{-1}(B^T S^T(t) + N^T)
\end{aligned}$$

and the optimal control is in the feedback form

$$u(x(t), t) = -K(t)x(t) - E(t) \tag{13}$$

where

$$K(t) = R^{-1}(B^T P(t) + V^T) \tag{14}$$
$$E(t) = R^{-1}(B^T S^T(t) + N^T) \tag{15}$$

B. Second-Order Search Algorithm for General Switched Linear Quadratic Problems

Now the second-order search algorithm developed in Section IV is to be used. Unlike Section IV, here we choose the nominal $K(\cdot)$ and $E(\cdot)$ rather than $u(\cdot)$ to be fixed at each iteration (but be updated after the iteration). This can give us the flexibility of letting $u(\cdot)$ vary as a function of x since here u depends on x (see (13)). We now have

$$\begin{aligned}
&V^{k+}_{t_k x} = -(f^{k+})^T V^{k+}_{xx} - V^{k+}_x f^{k+}_x - L^{k+}_x \\
&\quad -(V^{k+}_x f^{k+}_u + L^{k+}_u) u^{k+}_x, \\
&V^{k+}_{t_k t_k} = -V^{k+}_{t_k x} f^{k+} - V^{k+}_x f^{k+}_t - L^{k+}_t - (V^{k+}_x f^{k+}_u + L^{k+}_u) u^{k+}_t \\
&= (f^{k+})^T V^{k+}_{xx} f^{k+} + (V^{k+}_x f^{k+}_x + L^{k+}_x) f^{k+} - V^{k+}_x f^{k+}_t \\
&\quad -L^{k+}_t - (V^{k+}_x f^{k+}_u + L^{k+}_u)\dot{u}^{k+}.
\end{aligned}$$

Similar to the derivation in Section IV, it can be shown that $V^0_{t_k}$ is of the same form as in Theorem 1 and

$$\begin{aligned}
&V^0_{t_k t_k} = (f^{k-} - f^{k+})^T V^{k+}_{xx}(f^{k-} - f^{k+}) - (V^{k+}_x f^{k+}_x \\
&\quad + L^{k+}_x)(f^{k-} - f^{k+}) + (V^{k+}_x (f^{k-}_x - f^{k+}_x) + L^{k-}_x - L^{k+}_x) f^{k-} \\
&\quad + V^{k+}_x (f^{k-}_t - f^{k+}_t) + L^{k-}_t - L^{k+}_t + (V^{k+}_x f^{k-}_u + L^{k-}_u)\dot{u}^{k-} \\
&\quad -(V^{k+}_x f^{k+}_u + L^{k+}_u)(2u^{k+}_x f^{k-} + \dot{u}^{k+}).
\end{aligned}$$

$V^0_{t_k t_l}$ can also be derived similarly to the derivation in Section IV-B. However, there are some differences in this case. Although the forward decoupling principle gives us the same expression for $\delta x(t_2)$ as (11), yet here

$$\delta \dot{x} = \frac{\partial f(x,u,t)}{\partial x}\delta x + \frac{\partial f(x,u,t)}{\partial u}\delta u = (A(t) - B(t)K(t))\delta x.$$

And in addition to the three terms in Section IV-B, there is one more term $\frac{1}{2}dt_l L^{l-}_u du(t_l)$ contributing to the coefficient of $dt_k dt_l$. Hence we now have

$$\begin{aligned}
&V^0_{t_k t_l} = (f^{k-} - f^{k+})^T A^T(t_l, t_k) V^{l+}_{xx} f^{l-} - ((f^{l+})^T V^{l+}_{xx} \\
&\quad + V^{l+}_x f^{l+}_x + L^{l+}_x + \tfrac{1}{2}L^{l-}_x + \tfrac{1}{2}L^{l-}_u u^{l-}_x) A(t_l, t_k)(f^{k-} - f^{k+}).
\end{aligned}$$

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It can now be seen from the expressions of $V^0_{t_k}$, $V^0_{t_k t_k}$ and $V^0_{t_k t_l}$ that all terms necessary for the evaluation of them are readily available. In this case,

$$\begin{aligned}\dot{u}^{k-} &= -\dot{K}^{k-}x - K^{k-}f^{k-} - \dot{E}^{k-}\\ \dot{u}^{k+} &= -\dot{K}^{k+}x - K^{k+}f^{k+} - \dot{E}^{k+}\\ u^{k+}_x = -K^{k+},\ u^{l-}_x &= -K^{l-},\ V^{k+}_x = x^T P + S,\ V^{k+}_{xx} = P\end{aligned}$$

where x, P, S are continuous at t_k; $\dot{K}^{k-}$,$\dot{K}^{k+}$,$\dot{E}^{k-}$, $\dot{E}^{k+}$ are functions of P, S obtainable by substituting the expressions of $\dot{P}$ and $\dot{S}$ into the differentiation of (14) and (15). $A(t_l, t_k)$ is the state transition matrix for the time varying linear system $\delta\dot{x} = (A(t) - B(t)K(t))\delta x$ which can be calculated easily by numerical integrations.

Now that we have the expressions for $V^0_{t_k}$, $V^0_{t_k t_k}$ and $V^0_{t_k t_l}$, we can use the algorithm in Section IV to find the optimal switching instants. Note here $K(\cdot)$ and $E(\cdot)$ are assumed to be fixed at each iteration, but $u(\cdot)$ varies as a function of x. The advantages of applying the algorithm to GSLQ problems are that the difficulty (a) is partly resolved while difficulties (b) and (c) are easily addressed.

C. An Example

Example 3: Consider a switched system consisting of

$$\begin{aligned}\text{subsystem 1: } \dot{x} &= \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u,\\ \text{subsystem 2: } \dot{x} &= \begin{bmatrix} 0.5 & 5.3 \\ -5.3 & 0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u,\\ \text{subsystem 3: } \dot{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.\end{aligned}$$

Assume that $t_0 = 0$, $t_f = 3$ and the system switches at $t = t_1$ from subsystem 1 to 2 and at $t = t_2$ from subsystem 2 to 3 ($0 \le t_1 \le t_2 \le 3$). We want to find optimal t_1, t_2 and an optimal input u such that $x(0) = [4\ 4]^T$ and $x(3)$ is close to $[-4.1437\ 9.3569]^T$ and the cost functional $J = \frac{1}{2}\int_0^2 u^2(t)dt$ is minimized.

For this problem, we add to J a penalty term $[(x_1(3) + 4.1437)^2 + (x_2(3) - 9.3569)^2]$ and consider the expanded cost functional J_{exp}. Using the second-order search algorithm with initial values $t_1 = 0.8$, $t_2 = 1.8$, after 6 iterations we find that the optimal switching instant is $t_1 = 1.0035$, $t_2 = 2.0040$ and the corresponding optimal cost is 0.0135 (it is faster and more accurate than the result in [14]). The corresponding state trajectory is shown in Figure 2(a). This numerical solution is close to the theoretical optimal solution $t_1^{opt} = 1$, $t_2^{opt} = 2$, $J_{exp}^{opt} = 0$ and $u^{opt} \equiv 0$. (Figure 2(b) shows the optimal cost for different $t_1 < t_2$.) □

VI. Conclusion

In this paper, we first formulated an optimal control problem for switched systems. A two stage optimization methodology was then proposed and a second-order search algorithm was developed to implement it. The search algorithm is an extension of [14] which was motivated by the method in [3], [4]. The difficulties of the application of the algorithm are pointed out. For the special class of GSLQ problems, some of the difficulties can be addressed

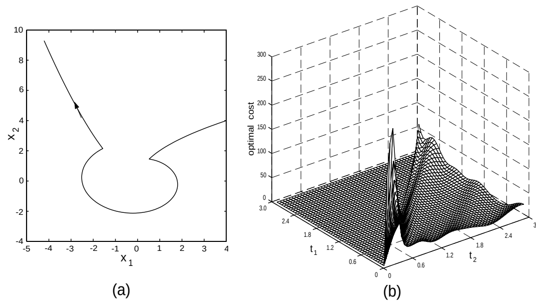

Fig. 2. Example 3: (a). The state trajectory. (b). The optimal cost for different $t_1 < t_2$.

efficiently as shown in Section V. Note that similar earlier results have appeared in [13], [14]. From the example, we find that even for GSLQ problems, the optimal cost as a function of switching instants may not be convex, which adds into the difficulties of the problem. The problems of characterizing such a function and finding the global solution for optimal switching instants are to be further explored.

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