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An Approach for Solving General Switched Linear Quadratic Optimal Control Problems

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Abstract

This paper successfully addresses an important class of hybrid optimal control problems of practical significance. It provides a viable general approach to hybrid optimal control based on nonlinear optimization and it shows that when this approach is applied to linear quadratic problems it leads to computationally attractive algorithms. Unlike conventional optimal control problems, optimal control problems for switched systems require the solutions of not only optimal continuous inputs but also optimal switching sequences. Many practical problems only involve optimization where the number of switchings and the sequence of active subsystems are given. This is stage 1 of the two stage optimization method proposed by the authors in previous papers. In order to solve stage 1 problems using efficient nonlinear optimization techniques, the derivatives of the optimal cost with respect to the switching instants need to be known. In this paper, we focus on and solve a special class of optimal control problems, namely, general switched linear quadratic problems. The approach first transcribes a stage 1 problem into an equivalent problem parameterized by the switching instants and then obtains the derivative values based on the solution of an initial value ordinary differential equation formed by the general Riccati equation and its differentiations. Examples illustrate the results.

1 Introduction

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law specifying the active subsystem at each time instant. Many real-world processes such as chemical processes, automotive systems, and electrical circuit systems, etc., can be modeled as switched systems.

Optimal control problems are one of the most challenging and important classes of problems for switched systems. For an optimal control problem of a switched system, one needs to find both an optimal continuous input and an optimal switching sequence since the system dynamics vary before and after every switching instant. The available results in the literature on such problems can be classified as theoretical and practical. [2, 8, 9, 10, 15] contain primarily theoretical results. These results extend the classical maximum principle or the dynamic programming approach to such problems. As to practical results, researchers took advantage of the availability of high speed computers and efficient nonlinear optimization techniques to develop methodologies for solving such problems (see e.g., [3, 4, 5, 6, 7, 11]). Most of the practical methods that we are aware of are based on some discretization of continuous time space and/or discretization of state space into grids and use search methods for the resultant discrete problem to find optimal/suboptimal solutions. But the discretization of time space may lead to computational combinatoric explosion and the solutions obtained may not be accurate enough. In view of this, in some previous papers by the authors (see [12, 13]), approaches that are not based on discretization of continuous time space were explored. In [12], a two stage optimization methodology was proposed. Since in general the two stage optimization is difficult to solve fully and moreover, many practical problems only involve the stage 1 optimization where the number of switchings and the sequence of active subsystems are given (yet the switching instants are unknown), in [13] the authors developed an algorithm for such stage 1 optimization based on the differentiations of the value function with respect to the switching instants.

In this paper, we focus on stage 1 optimization of an important class of optimal control problems, namely, general switched linear quadratic (GSLQ) optimal control problems, where each subsystem is linear and the cost functionals are in general quadratic forms. A conceptual algorithm for solving stage 1 problem is first given. In order to apply it, the derivatives of the optimal cost with respect to the switching instants need to be known. By exploiting the special structure of the problem, an approach for solving GSLQ problems is derived. The approach is more accurate and efficient than that in [13]. The approach first transcribes a GSLQ problem into an equivalent conventional problem parameterized by the switching instants and then obtains the derivative values based on the solution of an initial value ordinary

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differential equation (ODE) formed by the general Riccati equation and its differentiations.

The structure of the paper is as follows. In Section 2, we introduce the model of switched systems and formulate the GSLQ problems. In Section 3, we review the two stage optimization algorithm proposed in [12] and propose a conceptual algorithm for stage 1 optimization. From Section 4 on, we concentrate on stage 1 problems. In Section 4, we propose a method that transcribes a GSLQ problem into an equivalent conventional optimal control problem. In Section 5, it is shown how to obtain the derivatives of the optimal cost with respect to the switching instants based on the solution of an initial value ODE formed by the parameterized Riccati equation and its differentiations. Important additional comments concerning the approach are also given. Section 6 provides some examples to illustrate our approach. Section 7 concludes the paper.

2 Problem Formulation

2.1 Switched Systems

The switched systems we shall consider in this paper are defined as follows.

Definition 1 (Switched System) *A switched system is a tuple $\mathcal{S} = (\mathcal{D}, \mathcal{F})$ where*

- *$\mathcal{D} = (I, E)$ is a directed graph indicating the discrete structure of the system. The node set $I = \{1, 2, \cdots, M\}$ is the set of indices for subsystems. The directed edge set E is a subset of $I \times I - \{(i,i) | i \in I\}$ which contains all valid events. If an event $e = (i_1, i_2)$ takes place, the system switches from subsystem i_1 to i_2.*
- *$\mathcal{F} = \{f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n,\ i \in I\}$ is a collection of vector fields, with f_i describing the vector field for the ith subsystem $\dot{x} = f_i(x, u)$.* □

In view of Definition 1, a switched system is a collection of subsystems whose discrete structure is specified by $\mathcal{D}$. A salient feature of a switched system is that its continuous state x does not exhibit jumps at switching instants.

For a switched system $\mathcal{S}$, the input of the system consists of both a continuous input $u(t), t \in [t_0, t_f]$ and a switching sequence defined as follows.

Definition 2 (Switching Sequence) *For a switched system $\mathcal{S}$, a switching sequence σ in $[t_0, t_f]$ is defined as*

$$\sigma = ((t_0, i_0), (t_1, e_1), (t_2, e_2), \cdots, (t_K, e_K)), \quad (1)$$

with $0 \le K < \infty$, $t_0 \le t_1 \le t_2 \le \cdots \le t_K \le t_f$, and $i_0 \in I$, $e_k = (i_{k-1}, i_k) \in E$ for $k = 1, 2, \cdots, K$.

We define $\Sigma_{[t_0,t_f]} \triangleq \{\sigma\text{'s in } [t_0, t_f]\}$. □

A switching sequence σ as defined above indicates that subsystem i_k is active in $[t_k, t_{k+1})$. For a switched system to be well-behaved, we exclude the undesirable *Zeno* phenomenon, i.e., infinitely many switchings in finite amount of time. Hence in Definition 2, we only allow nonZeno sequences which switch at most a finite number of times in $[t_0, t_f]$, though different sequences may have different numbers of switchings. We specify $\sigma \in \Sigma_{[t_0,t_f]}$ as a discrete input to the system.

2.2 General Switched Linear Quadratic (GSLQ) Optimal Control Problems

In this paper, we shall consider the following GSLQ optimal control problems.

Problem 1 (GSLQ Problem) *Consider a switched system $\mathcal{S}$ with linear subsystems $\dot{x} = A_i x + B_i u, i \in I$. Given a fixed time interval $[t_0, t_f]$, find a continuous input $u(t), t \in [t_0, t_f]$ and a switching sequence $\sigma \in \Sigma_{[t_0,t_f]}$ such that the cost functional in general quadratic form*

$$\begin{aligned} J &= \frac{1}{2}x(t_f)^T Q_f x(t_f) + M_f x(t_f) + W_f + \int_{t_0}^{t_f} (\frac{1}{2}x^T Q x \\ &\quad + x^T V u + \frac{1}{2}u^T R u + M x + N u + W)\, dt \end{aligned} \quad (2)$$

is minimized. Here t_0, t_f and $x(t_0) = x_0$ are given; $Q_f, M_f, W_f, Q, V, R, M, N, W$ are matrices of appropriate dimensions with $Q_f \ge 0$, $Q \ge 0$ and $R > 0$. □

3 Two Stage Optimization

For general optimal control problems for switched systems, in [12], we proposed a two stage optimization methodology and a two stage algorithm. Here we restate the two stage algorithm as follows.

Algorithm 1 (A Two Stage Algorithm)

Stage 1. (a). Fix the total number of switchings to be K and the sequence of active subsystems and let the minimum value of J with respect to u be a function of the K switching instants, i.e., $J_1 = J_1(t_1, t_2, \cdots, t_K)$ for $K \ge 0$ ($t_0 \le t_1 \le t_2 \le \cdots \le t_K \le t_f$). Find J_1.

(b). Minimize J_1 with respect to $t_1, t_2, \cdots, t_K$.

Stage 2. (a). Vary the order of active subsystems to find an optimal solution under K switchings.

(b). Vary the value of K to find an optimal solution for the optimal control problem. □

The above algorithm has high computational costs. In the following, we concentrate on stage 1 optimization for GSLQ problems. Note that many real world problems are in fact stage 1 optimization problems. For example, the speeding-up of a power train in an automobile only requires switchings from gear 1 to 2 to 3 to 4. As can be seen from Algorithm 1, stage 1 can further be decomposed into two sub-steps (a) and (b) (A similar hierarchical decomposition method can be found in [3]). stage 1(a) is in essence a conventional optimal control problem which seeks the minimum value of J with respect to u under a given switching sequence $\sigma = ((t_0, i_0), (t_1, e_1), \cdots, (t_K, e_K))$. We denote the corresponding optimal cost as a function $J_1(\hat{t})$, where $\hat{t} \triangleq (t_1, t_2, \cdots, t_K)^T$. stage 1(b) is in essence a constrained nonlinear optimization problem

$$\begin{array}{c} \min_{\hat{t}} J_1(\hat{t}) \\ \text{subject to } \hat{t} \in T \end{array} \tag{3}$$

where $T \triangleq \{\hat{t} = (t_1, t_2, \cdots, t_K)^T | t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K \leq t_f\}$, which can be solved using feasible direction methods such as the gradient projection method and the constrained Newton's method [1].

The following algorithm provides a framework for stage 1 optimization in the subsequent sections.

Algorithm 2 (A Conceptual Algorithm for Stage 1 Optimization)

(1). Set the iteration index $j = 0$. Choose an initial $\hat{t}^j$.

(2). By solving an optimal control problem (stage 1(a)), find $J_1(\hat{t}^j)$.

(3). Find $\frac{\partial J_1}{\partial \hat{t}}(\hat{t}^j)$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}(\hat{t}^j)$.

(4). Use the gradient projection method or the constrained Newton's method to update $\hat{t}^j$ to be $\hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$ (here the stepsize α^j is chosen using the Armijo's rule [1]). Set the iteration index $j = j + 1$.

(5). Repeat Steps (2), (3), (4) and (5), until a prespecified termination condition is satisfied. □

It should be pointed out that the key elements of the above algorithm are

(a). An optimal control algorithm for Step (2).

(b). The derivations of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$ for Step (3).

(c). A nonlinear optimization algorithm for Step (4).

Note that (a) can be dealt with by using numerical methods for conventional optimal control problems and (c) can be dealt with by using for example feasible direction methods for constrained nonlinear optimization. However, (b) poses an obstacle because the analytical expressions of $J_1(\hat{t})$ are almost impossible to obtain except for very few classes of problems. The unavailability of analytical expressions of $J_1(\hat{t})$ hence makes the values of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$ difficult to obtain. It is the task of the subsequent sections to address (b) and derive an approach for deriving the values of $\frac{\partial J_1}{\partial \hat{t}}$ and $\frac{\partial^2 J_1}{\partial \hat{t}^2}$.

4 An Equivalent Problem Formulation

Henceforth, we focus on the stage 1 optimization for GSLQ problems and develop an approach for finding the derivative values of J_1 so that Algorithm 2 can be applied. For the general nonlinear case see [14]. In this section, we transcribe a GSLQ problem into an equivalent conventional optimal control problem parameterized by the unknown switching instants. A specific feature of the equivalent problem is that the independent time variable has the property that the switching instants are fixed with respect to it.

For convenience of notation and clarity of the presentation of the main idea of our approach, in Sections 4 and 5, we will concentrate on the case of two subsystems (see Problem 2) where subsystem 1 is active in the interval $[t_0, t_1)$ and subsystem 2 is active in the interval $[t_1, t_f]$ (t_1 is the switching instant to be determined). The approach works similarly for more than one switchings, and at the end of Section 5 we will comment on this.

Problem 2 *For a switched system*

$$\dot{x} = A_1 x + B_1 u, \ t_0 \leq t < t_1, \tag{4}$$

$$\dot{x} = A_2 x + B_2 u, \ t_1 \leq t \leq t_f, \tag{5}$$

find an optimal switching instant t_1 and an optimal $u(t)$ such that the cost functional in general quadratic form (2) is minimized. Here t_0, t_f and $x(t_0) = x_0$. □

In [16], a methodology used in solving boundary value problems with unknown end-time is applied to similar problem formulations to transcribe them into equivalent problems. Here we outline similar transcription for Problem 2 in the followings.

We introduce a new variable x_{n+1} corresponding to the switching instant t_1. Let x_{n+1} satisfy

$$\frac{dx_{n+1}}{dt} = 0, \ x_{n+1}(0) = t_1. \tag{6}$$

Next a new independent time variable τ is introduced. A piecewise linear correspondence relationship between t and τ is established as follows.

$$t = \begin{cases} t_0 + (x_{n+1} - t_0)\tau, & 0 \leq \tau \leq 1, \\ x_{n+1} + (t_f - x_{n+1})(\tau - 1), & 1 \leq \tau \leq 2. \end{cases} \tag{7}$$

Note $\tau = 0$ corresponds to $t = t_0$, $\tau = 1$ to $t = t_1$, and $\tau = 2$ to $t = t_f$. By introducing x_{n+1} and τ, Problem 2 can be transcribed into

Problem 3 (Equivalent Problem) *For a system with dynamics*

$$\frac{dx(\tau)}{d\tau} = (x_{n+1} - t_0)(A_1 x + B_1 u), \tag{8}$$

$$\frac{dx_{n+1}}{d\tau} = 0, \tag{9}$$

for $\tau \in [0, 1)$ and

$$\frac{dx(\tau)}{d\tau} = (t_f - x_{n+1})(A_2 x + B_2 u), \tag{10}$$

$$\frac{dx_{n+1}}{d\tau} = 0, \tag{11}$$

for $\tau \in [1, 2]$, find optimal x_{n+1} and $u(t)$ such that the cost functional

$$J = \tfrac{1}{2} x(2)^T Q_f x(2) + M_f x(2) + W_f + \int_0^1 x_{n+1} L(x, u) d\tau + \int_1^2 (t_f - x_{n+1}) L(x, u) d\tau \tag{12}$$

where

$$L(x, u) = \frac{1}{2} x^T Q x + x^T V u + \frac{1}{2} u^T R u + M x + N u + W \tag{13}$$

is minimized. Here $x(0) = x_0$ is given. □

Remark 1 Problem 3 and 2 are equivalent in the sense that a solution for Problem 3 is also a solution for Problem 2 by a proper change of independent variables as in (7) and by regarding $x_{n+1} = t_1$, and vice versa. □

Remark 2 Problem 3 provides us with the advantage that it no longer has a varying switching instant. Actually, because x_{n+1} is actually an unknown constant throughout $\tau \in [0,2]$, Problem 3 can be regarded as a conventional optimal control problem with an unknown parameter x_{n+1}. The problem is conventional because it has fixed time instant when the system dynamics changes. In the subsequent discussions, we regard x_{n+1} as an unknown parameter for optimal control problem with cost (12) and subsystems (8) and (10), i.e., we regard Problem 3 as an optimal control problem parameterized by the switching instant x_{n+1}. □

5 The Development of the Approach

In this section, based on the equivalent problem formulation in Section 4, we develop an approach for finding $\frac{\partial J_1}{\partial t_1}$ by studying the equivalent Problem 3.

As indicated in Remark 2, the equivalent Problem 3 can be regarded as a GSLQ problem parameterized by the switching instant x_{n+1}. Assume we are given a fixed x_{n+1} and assume the optimal value function is

$$V^*(x,\tau,x_{n+1}) = \frac{1}{2}x^T P(\tau,x_{n+1})x + S(\tau,x_{n+1})x + T(\tau,x_{n+1}) \quad (14)$$

where $P^T(\tau,x_{n+1}) = P(\tau,x_{n+1})$. By using the dynamic programming approach and solving the resultant HJB equation for $\tau \in [0,1)$ and $\tau \in [1,2]$ (see [14] for details), we can obtain the optimal control for $\tau \in [0,1)$ as

$$u(x,\tau,x_{n+1}) = -K(\tau,x_{n+1})x(\tau,x_{n+1}) - E(\tau,x_{n+1}) \quad (15)$$

where

$$\begin{aligned} K(\tau,x_{n+1}) &= R^{-1}(B_1^T P(\tau,x_{n+1}) + V^T), && (16)\\ E(\tau,x_{n+1}) &= R^{-1}(B_1^T S^T(\tau,x_{n+1}) + N^T), && (17)\end{aligned}$$

and $P(\tau,x_{n+1})$, $S(\tau,x_{n+1})$ and $T(\tau,x_{n+1})$ (abbreviated as P, S and T) satisfy the following parameterized general Riccati equation (parameterized by x_{n+1})

$$\begin{aligned} -\frac{\partial P}{\partial \tau} &= (x_{n+1}-t_0)(Q + PA_1 + A_1^T P - (PB_1 \\ &\quad +V)R^{-1}(B_1^T P + V^T)), && (18)\\ -\frac{\partial S}{\partial \tau} &= (x_{n+1}-t_0)(M + SA_1 - (N \\ &\quad +SB_1)R^{-1}(B_1^T P + V^T)), && (19)\\ -\frac{\partial T}{\partial \tau} &= (x_{n+1}-t_0)(W - \frac{1}{2}(N \\ &\quad +SB_1)R^{-1}(B_1^T S^T + N^T)). && (20)\end{aligned}$$

The optimal control for $\tau \in [1,2]$ is

$$u(x,\tau,x_{n+1}) = -K(\tau,x_{n+1})x(\tau,x_{n+1}) - E(\tau,x_{n+1}) \quad (21)$$

where

$$\begin{aligned} K(\tau,x_{n+1}) &= R^{-1}(B_2^T P(\tau,x_{n+1}) + V^T), && (22)\\ E(\tau,x_{n+1}) &= R^{-1}(B_2^T S^T(\tau,x_{n+1}) + N^T), && (23)\end{aligned}$$

and P, S and T satisfies the following parameterized general Riccati equation

$$\begin{aligned} -\frac{\partial P}{\partial \tau} &= (t_f - x_{n+1})(Q + PA_2 + A_2^T P - (PB_2 \\ &\quad +V)R^{-1}(B_2^T P + V^T)), && (24)\\ -\frac{\partial S}{\partial \tau} &= (t_f - x_{n+1})(M + SA_2 - (N \\ &\quad +SB_2)R^{-1}(B_2^T P + V^T)), && (25)\\ -\frac{\partial T}{\partial \tau} &= (t_f - x_{n+1})(W - \frac{1}{2}(N \\ &\quad +SB_2)R^{-1}(B_2^T S^T + N^T)). && (26)\end{aligned}$$

Once we have solved (18-20) and (24-26) (for a fixed x_{n+1}), we can obtain the parameterized optimal cost at $\tau = 0$, i.e., the optimal J_1 under fixed x_{n+1} as

$$J_1(t_1) = J_1(x_{n+1}) = V^*(x_0,0,x_{n+1}) = \tfrac{1}{2}x_0^T P(0,x_{n+1})x_0 + S(0,x_{n+1})x_0 + T(0,x_{n+1}). \quad (27)$$

From (27), we have

$$\frac{dJ_1}{dx_{n+1}}(x_{n+1}) = \frac{\partial V^*}{\partial x_{n+1}}(x_0,0,x_{n+1}) = \tfrac{1}{2}x_0^T \frac{\partial P}{\partial x_{n+1}}(0,x_{n+1})x_0 + \frac{\partial S}{\partial x_{n+1}}(0,x_{n+1})x_0 + \frac{\partial T}{\partial x_{n+1}}(0,x_{n+1}). \quad (28)$$

In order to obtain the value of $\frac{dJ_1}{dx_{n+1}}$ from (28), we need to know $\frac{\partial P}{\partial x_{n+1}}$, $\frac{\partial S}{\partial x_{n+1}}$ and $\frac{\partial T}{\partial x_{n+1}}$ at $(0,x_{n+1})$. To obtain these values, we differentiate (18-20) and (24-26) with respect to x_{n+1} to obtain

$$\begin{aligned} -\tfrac{\partial}{\partial \tau}(\tfrac{\partial P}{\partial x_{n+1}}) &= (Q + PA_1 + A_1^T P - (PB_1 +V)R^{-1}(B_1^T P + V^T)) + (x_{n+1}-t_0)(\tfrac{\partial P}{\partial x_{n+1}}A_1 \\ &\quad + A_1^T \tfrac{\partial P}{\partial x_{n+1}} - (\tfrac{\partial P}{\partial x_{n+1}}B_1)R^{-1}(B_1^T P + V^T)) \\ &\quad -(PB_1 + V)R^{-1}(B_1^T \tfrac{\partial P}{\partial x_{n+1}}))) \end{aligned} \quad (29)$$

$$\begin{aligned} -\tfrac{\partial}{\partial \tau}(\tfrac{\partial S}{\partial x_{n+1}}) &= (M + SA_1 - (N + SB_1)R^{-1}(B_1^T P \\ &\quad +V^T)) + (x_{n+1}-t_0)(\tfrac{\partial S}{\partial x_{n+1}}A_1 - (\tfrac{\partial S}{\partial x_{n+1}}B_1)R^{-1}(B_1^T P \\ &\quad +V^T) - (N + SB_1)R^{-1}(B_1^T \tfrac{\partial P}{\partial x_{n+1}})) \end{aligned} \quad (30)$$

$$\begin{aligned} -\tfrac{\partial}{\partial \tau}(\tfrac{\partial T}{\partial x_{n+1}}) &= (W - \tfrac{1}{2}(N + SB_1)R^{-1}(B_1^T S^T + N^T)) \\ &\quad +(x_{n+1}-t_0)(-\tfrac{1}{2}(\tfrac{\partial S}{\partial x_{n+1}}B_1)R^{-1}(B_1^T S^T + N^T) \\ &\quad -\tfrac{1}{2}(N + SB_1)R^{-1}(B_1^T(\tfrac{\partial S}{\partial x_{n+1}})^T)) \end{aligned} \quad (31)$$

for $\tau \in [0,1)$ and

$$\begin{aligned} -\tfrac{\partial}{\partial \tau}(\tfrac{\partial P}{\partial x_{n+1}}) &= -(Q + PA_2 + A_2^T P - (PB_2 \\ &\quad +V)R^{-1}(B_2^T P + V^T)) + (t_f - x_{n+1})(\tfrac{\partial P}{\partial x_{n+1}}A_2 \\ &\quad + A_2^T \tfrac{\partial P}{\partial x_{n+1}} - (\tfrac{\partial P}{\partial x_{n+1}}B_2)R^{-1}(B_2^T P + V^T)) \\ &\quad -(PB_2 + V)R^{-1}(B_2^T \tfrac{\partial P}{\partial x_{n+1}}))) \end{aligned} \quad (32)$$

$$\begin{aligned} -\tfrac{\partial}{\partial \tau}(\tfrac{\partial S}{\partial x_{n+1}}) &= -(M + SA_2 - (N + SB_2)R^{-1}(B_2^T P \\ &\quad +V^T)) + (t_f - x_{n+1})(\tfrac{\partial S}{\partial x_{n+1}}A_2 - (\tfrac{\partial S}{\partial x_{n+1}}B_2)R^{-1}(B_2^T P \\ &\quad +V^T) - (N + SB_2)R^{-1}(B_2^T \tfrac{\partial P}{\partial x_{n+1}})) \end{aligned} \quad (33)$$

$$-\frac{\partial}{\partial \tau}(\frac{\partial T}{\partial x_{n+1}}) = -(W - \frac{1}{2}(N + SB_2)R^{-1}(B_2^T S^T + N^T)) \\ +(t_f - x_{n+1})(-\frac{1}{2}(\frac{\partial S}{\partial x_{n+1}} B_2)R^{-1}(B_2^T S^T + N^T) \\ -\frac{1}{2}(N + SB_2)R^{-1}(B_2^T(\frac{\partial S}{\partial x_{n+1}})^T)) \tag{34}$$

for $\tau \in [1, 2]$.

The equations (18-20) and (29-31) for $\tau \in [0, 1)$ and the equations (24-26) and (32-34) for $\tau \in [1, 2]$ together with the following boundary conditions at $\tau = 2$

$$\begin{array}{ll} P(2, x_{n+1}) = Q_f, & S(2, x_{n+1}) = M_f, \\ T(2, x_{n+1}) = W_f, & \frac{\partial P}{\partial x_{n+1}}(2, x_{n+1}) = 0, \\ \frac{\partial S}{\partial x_{n+1}}(2, x_{n+1}) = 0, & \frac{\partial T}{\partial x_{n+1}}(2, x_{n+1}) = 0, \end{array} \tag{35}$$

form an initial value ordinary differential equation (ODE) for P, S, T, $\frac{\partial P}{\partial x_{n+1}}$, $\frac{\partial S}{\partial x_{n+1}}$ and $\frac{\partial T}{\partial x_{n+1}}$ which can be solved efficiently using the function ode45 in MATLAB. From the solution of this ODE, values of $\frac{\partial P}{\partial x_{n+1}}$, $\frac{\partial S}{\partial x_{n+1}}$ and $\frac{\partial T}{\partial x_{n+1}}$ at $(0, x_{n+1})$ can be obtained and substituted into (28) to obtain the value of $\frac{dJ_1}{dt_1}$. Algorithm 2 can then be applied.

Remark 3 (Several Subsystems and More Than One Switchings) For GSLQ problems with K subsystems and more than one switchings, we can similarly transcribe the problem into an equivalent problem in $\tau \in [0, K+1]$. It is then straightforward to differentiate the Riccati equation parameterized by $x_{n+1}, \cdots, x_{n+K}$ (i.e., $t_1, \cdots, t_K$) to obtain additional differential equations for $\frac{\partial P}{\partial x_{n+k}}$'s, $\frac{\partial S}{\partial x_{n+k}}$'s and $\frac{\partial T}{\partial x_{n+k}}$'s. Along with the boundary conditions $P = Q_f$, $S = M_f$, $T = W_f$, $\frac{\partial P}{\partial x_{n+k}} = 0$, $\frac{\partial S}{\partial x_{n+k}} = 0$ and $\frac{\partial T}{\partial x_{n+k}} = 0$ all at $(K+1, x_{n+1}, \cdots, x_{n+K})$ for all $1 \le k \le K$, we can solve the resultant initial value ODE backwards in τ to find their values at $\tau = 0$. Once we have their values at $\tau = 0$, we can substitute them into

$$\frac{\partial J_1}{\partial x_{n+k}} = \frac{\partial V^*}{\partial x_{n+k}}(x_0, 0, x_{n+1}, \cdots, x_{n+K}) \\ = \frac{1}{2}x_0^T \frac{\partial P}{\partial x_{n+k}}(0, x_{n+1}, \cdots, x_{n+k})x_0 + \frac{\partial S}{\partial x_{n+k}}(0, x_{n+1}, \\ \cdots, x_{n+k})x_0 + \frac{\partial T}{\partial x_{n+k}}(0, x_{n+1}, \cdots, x_{n+k}) \tag{36}$$

to derive the accurate values of $\frac{\partial J_1}{\partial t_k}$'s. □

Remark 4 (Second Order Derivatives) If we take second order partial derivatives of equation (27), we obtain

$$\frac{d^2 J_1}{dx_{n+1}^2}(t_1) = \frac{\partial^2 V^*}{\partial x_{n+1}^2}(x_0, 0, x_{n+1}) = \frac{1}{2}x_0^T \frac{\partial^2 P}{\partial x_{n+1}^2}(0, x_{n+1})x_0 \\ + \frac{\partial^2 S}{\partial x_{n+1}^2}(0, x_{n+1})x_0 + \frac{\partial^2 T}{\partial x_{n+1}^2}(0, x_{n+1}). \tag{37}$$

Following similar ideas of differentiation of the parameterized Riccati equation, we can take first and second-order differentiations of (18)-(20) and (24)-(26) with respect to x_{n+1} and form a set of ODEs. Along with the initial conditions (35) and 0's at $\tau = 2$ for $\frac{\partial^2 P}{\partial x_{n+1}^2}$, $\frac{\partial^2 S}{\partial x_{n+1}^2}$ and $\frac{\partial^2 T}{\partial x_{n+1}^2}$, the resultant initial value ODE for P, S, T, $\frac{\partial P}{\partial x_{n+1}}$, $\frac{\partial S}{\partial x_{n+1}}$, $\frac{\partial T}{\partial x_{n+1}}$, $\frac{\partial^2 P}{\partial x_{n+1}^2}$, $\frac{\partial^2 S}{\partial x_{n+1}^2}$ and $\frac{\partial^2 T}{\partial x_{n+1}^2}$ can be solved and hence the accurate value of $\frac{d^2 J_1}{dx_{n+1}^2}$ can be obtained and the constrained Newton's method can be applied in Step (4) in Algorithm 2. □

6 Some Examples

The approach developed above is applied to the following examples.

Example 1 Consider a switched system consisting of

$$\text{subsystem 1: } \dot{x} = \begin{bmatrix} 0.6 & 1.2 \\ -0.8 & 3.4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u, \tag{38}$$

$$\text{subsystem 2: } \dot{x} = \begin{bmatrix} 4 & 3 \\ -1 & 0 \end{bmatrix} x + \begin{bmatrix} 2 \\ -1 \end{bmatrix} u. \tag{39}$$

Assume that $t_0 = 0$, $t_f = 2$ and the system switches once at $t = t_1$ $(0 \le t_1 \le 2)$ from subsystem 1 to 2. We want to find optimal t_1 and u such that

$$J = \frac{1}{2}(x_1(2)-4)^2 + \frac{1}{2}(x_2(2)-2)^2 + \frac{1}{2}\int_0^2 (x_2(t)-2)^2 + u^2(t)\, dt$$

is minimized. Here $x(0) = [0,\ 2]^T$.

We use the approach in this paper to obtain the value of $\frac{dJ_1}{dt_1}$. From an initial nominal $t_1 = 1.0$, by using Algorithm 2 with the gradient projection method, after 12 iterations we find that the optimal switching instant is $t_1 = 0.1897$ and the corresponding optimal cost is 9.7667. After translating the result into the form suitable for the original problem, we show the corresponding continuous control and state trajectory in Figure 1 (a) and (b). Figure 2 shows the optimal cost for different t_1's. □

Example 2 Consider a switched system consisting of

$$\text{subsystem 1: } \dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \tag{40}$$

$$\text{subsystem 2: } \dot{x} = \begin{bmatrix} 0.5 & 5.3 \\ -5.3 & 0.5 \end{bmatrix} x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u, \tag{41}$$

$$\text{subsystem 3: } \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1.5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u. \tag{42}$$

Assume that $t_0 = 0$, $t_f = 3$ and the system switches at $t = t_1$ from subsystem 1 to 2 and at $t = t_2$ from subsystem 2 to 3 $(0 \le t_1 \le t_2 \le 3)$. We want to find optimal t_1, t_2 and u such that $J = [(x_1(3) + 4.1437)^2 + (x_2(3) - 9.3569)^2] + \frac{1}{2}\int_0^3 u^2(t)\ dt$ is minimized. Here $x(0) = [4,\ 4]^T$.

We use the approach in this paper to obtain the values of $\frac{\partial J_1}{\partial t_1}$ and $\frac{\partial J_1}{\partial t_2}$. From initial nominal values $t_1 = 0.8$, $t_2 = 1.8$, by using Algorithm 2 with the gradient projection method, after 20 iterations we find that the optimal switching instant is $t_1 = 0.9982$, $t_2 = 1.9983$ and the corresponding optimal cost is 4.4087×10^{-5}. After translating the result into the form suitable for the original problem, we show the corresponding continuous control and state trajectory in Figure 3 (a) and (b).

Note that the theoretical optimal solutions for this problem are $t_1^{opt} = 1$, $t_2^{opt} = 2$, $u^{opt} \equiv 0$ and $J^{opt} = 0$, so the result we obtained is quite accurate. Figure 4 shows the optimal cost for different $t_1 < t_2$. □

It can be observed from Figure 4 that the function $J_1(t_1, t_2)$ has several ripples. Hence it is not convex even for this simple GSLQ problem; that is why such problems pose significant difficulties.

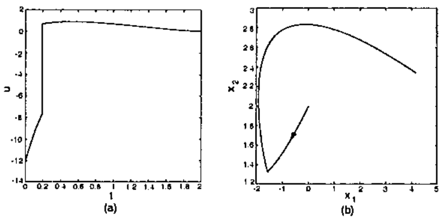

Figure 1: Example 1: (a) The control input. (b) The state trajectory.

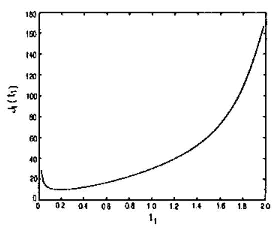

Figure 2: The optimal cost for Example 1 for different t_1's.

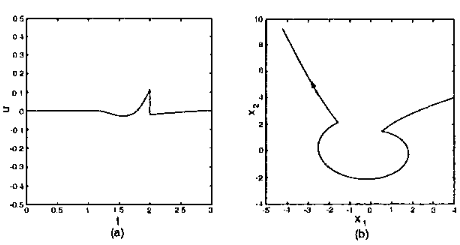

Figure 3: Example 2: (a) The control input. (b) The state trajectory.

7 Conclusion

In this paper, an approach for solving GSLQ optimal control problems is proposed. The approach is mainly developed in Sections 4 and 5 and it is applicable to GSLQ problems with many subsystems and more than one switchings as pointed out in Remark 3. The approach is based on solving the parameterized Riccati equations and their differentiations. Derivatives of the optimal cost with respect to the switching instants can be obtained accurately, therefore nonlinear optimization algorithms can be used to find the optimal switching instants for the original GSLQ problem.

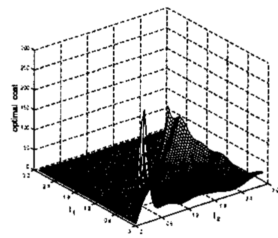

Figure 4: The optimal cost for Example 2 for different (t_1, t_2)'s.

References

[1] D. P. Bertsekas, *Nonlinear Programming*, Second Edition, Athena Scientific, 1999.

[2] M. S. Branicky, V. S. Borkar, S. K. Mitter, *A unified framework for hybrid control: model and optimal control theory*, In IEEE Trans. on Auto. Contr., **43**(1), pp. 31-45, January, 1998.

[3] K. Gokbayrak, C. G. Cassandras, *A hierarchical decomposition method for optimal control of hybrid systems*, In Proc. of the 39th IEEE CDC, 2000.

[4] S. Hedlund, A. Rantzer, *Optimal control of hybrid systems*, In Proc. of the 38th IEEE CDC, pp. 3972-3977, 1999.

[5] M. Johansson, *Piecewise Linear Control Systems*, Ph.D. Dissertation, Lund Inst. of Tech., Sweden, 1999.

[6] B. Lincoln, B. M. Bernhardsson, *Efficient pruning of search trees in LQR control of switched linear systems*. In Proc. of the 39th IEEE CDC, pp. 1828-1833, 2000.

[7] J. Lu, L. Liao, A. Nerode, J. H. Taylor, *Optimal control of systems with continuous and discrete states*, In Proc. of the 32nd IEEE CDC, pp. 2292-2297.

[8] B. Piccoli, *Hybrid systems and optimal control*, In Proc. of the 37th IEEE CDC, pp. 13-18, 1998.

[9] T. I. Seidman, *Optimal control for switching systems*, In Proc. of the 21st Annual Conf. on Infor. Sci. and Sys., pp. 485-489, Baltimore, Maryland, 1987.

[10] H. Sussmann, *A maximum principle for hybrid optimal control problems*, In Proc. of the 38th IEEE CDC, 1999.

[11] L. Y. Wang, A. Beydoun, J. Cook, J. Sun, I. Kolmanovsky, *Optimal hybrid control with applications to automotive powertrain systems*, in LNCIS 222, Springer-Verlag, 1997.

[12] X. Xu, P. J. Antsaklis, *Optimal control of switched systems: new results and open problems*, In Proc. of the 2000 ACC, pp. 2683-2687, 2000.

[13] X. Xu, P. J. Antsaklis, *A dynamic programming approach for optimal control of switched systems*, In Proc. of the 39th IEEE CDC, pp. 1822-1827, 2000.

[14] X. Xu, *Analysis and design of switched systems*, Ph.D. Dissertation, Department of Electrical Engineering, University of Notre Dame, 2001, URL: `http://www.nd.edu/~isis/oap.html`.

[15] J. Yong, *Systems governed by ordinary differential equations with continuous, switching and impulse controls*, Appl. Math. Optim., **20**(1989), pp. 223-235.

[16] M. Zefran, *Continuous methods for motion planning*, Ph.D. Dissertation, Department of Computer and Information Science, University of Pennsylvania, 1996.