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Robust Controlled Invariant Sets for a class of Uncertain Hybrid Systems [1]

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1 Introduction

Invariant set theory has been widely studied in the literature, see for example [1, 2] and reference therein. [1] gives a compressive review of the invariant set theory. [2] brings together some of the main ideas in set invariance theory and places them in a general, non-linear setting. A similar concept, maximal safety set, has been studied in the literature of hybrid systems. The authors of [7] consider a class of discrete time hybrid systems with piecewise linear time-invariant flow function and polyhedral constraints. The invariant sets for piecewise affine systems have also been studied in [3] based on convex optimization techniques and linear matrix inequalities.

However, if the effect of the uncertainty in the model is not taken into account, then it is possible that a controller could drive the system into an unsafe region. A small disturbance or fault could then cause the system to fail. This paper concentrates on the robust controlled invariant sets for a class of uncertain hybrid systems. Our goal is to determine whether a given region in the state space is robust controlled invariant and to compute the maximal robust controlled invariant subset under structured dynamic uncertainty and disturbances.

2 Model

The discussion in this paper assumes the following uncertain, discrete-time hybrid dynamical systems:

Definition 2.1 *Consider the discrete-time Uncertain Piecewise Linear Hybrid Systems defined by*

$$x(t+1) = \tilde{A}_{q(t)}x(t) + \tilde{B}_{q(t)}u(t) + E_{q(t)}d(t) \quad (2.1)$$
$$q(t+1) = \delta(q(t), \pi(x(t)), \sigma_c(t), \sigma_u(t)) \quad (2.2)$$

where $q \in Q = \{q_1, q_2, \cdots, q_s\}$ and Q is the collection of discrete states (modes); $x \in X \subset \mathbb{R}^n$ and X stands for the continuous state space, the continuous control $u \in \mathcal{U} \subset \mathbb{R}^m$, and the continuous disturbance $d \in \mathcal{D} \subset \mathbb{R}^p$, where $\mathcal{U}$, $\mathcal{D}$ are bounded convex polyhedral sets; and

- *$\tilde{A}_q \in \mathbb{R}^{n \times n}$, $\tilde{B}_q \in \mathbb{R}^{n \times m}$, and $E_q \in \mathbb{R}^{n \times p}$ are the*

system matrices for the discrete state q. Assume polytopic uncertainty, that is $[\tilde{A}_q, \tilde{B}_q] = \sum_{i=1}^{N_q} \lambda_i [A_q^i, B_q^i]$, where $\lambda_i \geq 0, \sum_{i=1}^{N_q} \lambda_i = 1$;

- *$\pi : X \to X/E_\pi$ partitions the continuous state space $\mathbb{R}^n$ into polyhedral equivalence classes.*
- *$q(t+1) \in act(\pi(x(t)))$, where $act : X/E_\pi \to 2^Q$ defines the active mode set,*
- *$\delta : Q \times X/E_\pi \times \Sigma_c \times \Sigma_u \to Q$ is the discrete state transition function. Here $\sigma_c \in \Sigma_c$ denotes a controllable event and Σ_u the collection of uncontrollable events.*
- *The guard $G(q, q')$ of the transition (q, q') is defined as the set of all states (q, x) such that $q' \in act(\pi(x(t)))$ and there exist controllable event $\sigma_c \in \Sigma_c$ such that $q' = \delta(q, \pi(x), \sigma_c, \sigma_u)$ for every uncontrollable event $\sigma_u \in \Sigma_u$.*

Assume that exact state measurement (q, x) is available. An *admissible control input* (or *law*) is one which satisfies the input constraints $(\Sigma_c, \mathcal{U})$. The elements of an allowable disturbance sequence are contained in $(\Sigma_u, \mathcal{D})$.

3 Geometric condition for invariance

Given a set $\Omega \subset Q \times X$ and an initial state $(q_0, x_0) \in \Omega$, it is of interest to determine whether there exist admissible control laws such that the evolution of the system will remain inside the set for all time, despite the presence of structured dynamic uncertainty and disturbances.

Definition 3.1 *The set $\Omega \subset Q \times X$ is robust controlled invariant for the uncertain hybrid systems of Definition 2.1 if and only if $\forall (q_0, x_0) \in \Omega$, $\forall (\sigma_u, d(t)) \in \Sigma_u \times \mathcal{D}$ and $\forall [\tilde{A}_{q(t)}, \tilde{B}_{q(t)}] \in Conv[A_{q(t)}^i, B_{q(t)}^i]$ there always exist admissible control input $(\sigma_c, u(t)) \in \Sigma_c \times \mathcal{U}$, such that the system evolution satisfies $(q(t), x(t)) \in \Omega, \forall t \geq 0$.*

A natural question is how to check whether for a given set $\Omega \subset Q \times X$ is robust controlled invariant or not. To answer this question, we first study the robust (one-step) predecessor operator, $pre(\cdot)$, for the uncertain hybrid systems. It is defined as the set of states in $Q \times X$ for which an admissible control input exists which will guarantee that the system will be driven to Ω in one step, for all allowable disturbances and dynamic uncertainties. The following is an important, well-known geometric condition[1] for a set to be control invariant.

Theorem 3.1 *The set $\Omega \subseteq \mathbb{R}^n$ is a robust controlled invariant set if and only if $\Omega \subseteq pre(\Omega)$.*

It follows immediately that the set Ω is robust control invariant if and only if $pre(\Omega) \cap \Omega = \Omega$, since $\Omega \subseteq pre(\Omega) \Leftrightarrow pre(\Omega) \cap \Omega = \Omega$. Testing for invariance need to: compute $pre(\Omega)$, which can be efficiently done by the predecessor operator algorithm described in [5, 6]; test whether $\Omega \subseteq pre(\Omega)$, this can be done by a feasibility of a linear programming problem[4]. So this condition can be efficiently tested by solving a finite number of linear programming problems that depends on the number of regions and discrete states of the system.

4 Maximal Robust Controlled Invariant Set

We have answered how to check the robust controlled invariance for a given set $\Omega \subset Q \times X$. And in general, a given set Ω is not robust controlled invariant. However, some subsets of Ω are likely to be robust controlled invariant. In addition, it follows immediately from the definition that the union of two robust controlled invariant sets is robust controlled invariant. So we have the following definition.

Definition 4.1 *The set $\tilde{C}_\infty(\Omega)$ is the maximal robust positively invariant set contained in $\Omega \subset Q \times X$ for the uncertain hybrid systems of Definition 2.1 if and only if $\tilde{C}_\infty(\Omega)$ is robust controlled invariant and contains all the robust controlled invariant sets contained in Ω.*

It can be shown that the maximal robust positively invariant set is unique. Then the next question is how to find the maximal robust controlled invariant set $\tilde{C}_\infty(\Omega)$. We have the following algorithm for computing the maximal robust control invariant set.

INPUT: $\Omega = (q, P)$, $\tilde{C}_0(\Omega) = X$, $\tilde{C}_1(\Omega) = \Omega$, $i = 1$;
while $\tilde{C}_i(\Omega) \neq \tilde{C}_{i-1}(\Omega)$
  $\tilde{C}_{i+1}(\Omega) = pre(\tilde{C}_i(\Omega)) \cap \Omega$
  if $\tilde{C}_{i+1}(\Omega) = \emptyset$
    Terminate and **OUTPUT:** $\tilde{C}_\infty(\Omega) = \emptyset$.
  end if
  $i = i + 1$
end while
OUTPUT: $\tilde{C}_\infty(\Omega) = \tilde{C}_i(\Omega)$

Remark: The i-step robust admissible set $\tilde{C}_i(\Omega)$ have properties: $\tilde{C}_{i+1}(\Omega) \subseteq \tilde{C}_i(\Omega)$, $\tilde{C}_i(\Omega) = \bigcap_{k=1}^i \tilde{C}_k(\Omega)$.

5 Maximum Permissive Control Law

Another interesting problem is to construct a (memoryless) *control law*, $c : Q \times X \to 2^{\Sigma_c \times \mathcal{U}}$, that robustly drives the system to guarantee that the states remain within some region (assume proper initial conditions) despite the uncertainties and disturbances, while satisfying some certain input/output constraints. If such control law exists and is *non-blocking*, i.e. $c(q, x) \neq \emptyset$ for all $(q, x) \in Q \times X$, then we say the control law c,

solves the robust controlled invariance problem. Many control laws may be able to solve a particular problem. It is often preferred to find a control law that imposes fewer restrictions on the inputs allowed. The *least restrictive* (or *maximum permissive*) [7] control law is the maximal element among all the control laws that solve a given robust controlled invariance problem in the partial order defined by $\preceq$, where $c_1 \preceq c_2$ if $\forall (q, x) \in Q \times X$, $c_1(q, x) \subseteq c_2(q, x)$.

For a given region $\Omega \subset Q \times X$, if there exists an unique nonempty maximal robust controlled invariant subset $\tilde{C}_\infty(\Omega)$, then there exists a unique non-blocking least restrictive control law that solves the robust controlled invariance problem for $\tilde{C}_\infty(\Omega)$ with proper initial conditions. The least restrictive controller can be given as follows:

$$c(q, x) = \begin{cases} \{(\sigma_c, u) | \forall \sigma_u \in \Sigma_u, d \in \mathcal{D}, \\ \quad \forall [\tilde{A}_q, \tilde{B}_q] \in Conv[A_q^i, B_q^i], \quad (q, x) \in \tilde{C}_\infty(\Omega) \\ \quad \tilde{A}_q x + \tilde{B}_q u + E_q d \in \tilde{C}_\infty(\Omega)\}, \\ \\ \Sigma_c \times \mathcal{U}, \quad\quad\quad\quad\quad\quad\quad (q, x) \notin \tilde{C}_\infty(\Omega) \end{cases}$$

6 Conclusion

In this paper, we put our group's recent progress in the analysis and synthesis of uncertain piecewise linear hybrid systems into the framework of invariant set theory. We develop an implementable procedure to check the robust controlled invariance and compute the maximal robust controlled invariant subset under structured dynamic uncertainty and disturbances. A formulation of least restrictive control law that solves a given robust controlled invariance problem is presented. In general, the maximal robust control invariant set can not be determined in finite number of steps. So the above algorithm and the lest restrictive control law synthesis is *semi-decidable*[7], because the termination of the algorithm is not guaranteed. In [4] two termination conditions were proposed to formulate a constructive algorithm.

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Synthesis of State-Feedback Optimal Controllers
for Continuous-Time Switched Linear Systems

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Abstract

The paper deals with the optimal control of switched piece-wise linear autonomous systems, where the objective is to minimize a performance index over an infinite time horizon. We assume that the switching sequence has a finite length: the unknown switching times and the switching sequence are the optimization parameters. We also assume that a cost may be associated to each switch.

The optimal control for this class of systems takes the form of a state feedback, i.e., it is possible to identify a set of regions of the state space such that an optimal switch should occur if and only if the present state belongs to one of them. We show how the tables containing these regions can be computed off-line through a numerical procedure.

1 Introduction

Switched systems are a particular class of hybrid systems that switch between many operating modes, where each mode is governed by its own characteristic dynamical law [6]. Typically, mode transitions are triggered by variables crossing specific thresholds (state events), by the elapse of certain time periods (time events), or by external inputs (input events). The problem of determining optimal control laws for this class of hybrid systems has been widely investigated in the last years and many results can be found in the control and computer science literature. For continuous-time hybrid systems, most of the literature is focused on the study of necessary conditions for a trajectory to be optimal [11, 14], and on the computation of optimal/suboptimal solutions by means of dynamic programming or the maximum principle [4, 5, 10, 12, 13, 15]. For determining the optimal feedback control law some of these techniques require the discretization of the state space in order to solve the corresponding Hamilton-Jacobi-Bellman equations. In [9] the authors use a hierarchical decomposition approach to break down the overall problem into smaller ones. In so doing, discretization is not involved and the main computational complexity arises from a higher-level nonlinear programming problem.

In the continuous-time case, and in particular for switched linear systems composed by stable autonomous dynamics, by assuming that the switching sequence is pre-assigned

(and, therefore, that the only decision variables to be optimized are the switching instants), in [7, 8] we proved an important result, namely that the control law is a state-feedback and there exists a numerically viable procedure to compute the switching tables $C_{k,N}$ showing the points of the state space where the k-th switch of a sequence of length N should occur. A similar state-feedback result was also proved in [1] for optimal control problems based on discrete-time hybrid models and linear cost functions, which leads to piecewise affine optimal control laws.

In [2, 3] we generalized the optimization problem of [7, 8] by taking both the switching instants and the switching sequence as decision variables. The procedure we used to solve the generalized problem exploits a synergy of discrete-time and continuous-time techniques, by alternating between a "master" procedure that finds an optimal switching sequence and a "slave" procedure that finds the optimal switching instants still using the approach described in [7, 8].

In this paper we present a different solution to the generalized optimization problem that is still based on the construction of switching tables. Using a simple procedure inspired by dynamic programming, we show how it is possible to avoid the exponential growth of the computational cost as the length of the switching sequence is increased.

To motivate the interest for pursuing two different approaches to solve essentially the same problem, we compare the procedure presented in this paper with the master-slave algorithm in [2, 3]. More details about the computational complexity of both procedures will be given in Section 4.

The procedure presented in this paper is based on the generation of a set of "switching tables" and has the following properties:

- is guaranteed to find the optimal solution;

- has a computational cost of the order $O(Ns^2)$, where N is the length of the switching sequence and s is the number of possible operating modes (or dynamics);

- provides a "global" closed-loop solution, i.e., the tables may be used to determine the optimal state feedback law for all initial states.

On the other hand, the master-slave algorithm presented in [2, 3]:

- is not guaranteed to converge to a global optimum;