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Robust Invariant Control Synthesis for Discrete-Time Polytopic Uncertain Linear Hybrid Systems [1]

Hai Lin and Panos J. Antsaklis

Dept. of Electrical Engineering, Univ. of Notre Dame, Notre Dame, IN 46556, USA

Abstract

In this paper, a class of discrete-time uncertain linear hybrid systems, which is affected by both parameter variations and exterior disturbances, is considered. The goal here is to synthesize a hybrid control law, which includes the continuous variable control law and the discrete event control signal, so as to guarantee the controlled invariance for a specific region in the hybrid state space. It turns out that the continuous variable control law is a piecewise linear state feedback control based on the partition of the given region's continuous variable state space projection. A numerical example is given for illustration.

1 Introduction

Most physical systems are subject to state and input/output constraints due to safety requirements or physical limitations. Furthermore, a required certain level of performance might be translated into additional constraints on the controlled system. Invariant set theory is important in analysis and synthesis of such constrained systems, since state and control constraints can be satisfied if and only if the initial state belongs to some proper invariant set for the closed-loop system [12]. Invariant set theory has been studied in the literature for decades, see for example [5, 7, 12] and references therein. [5] gave a comprehensive review of the invariant set theory. [12] brought together some of the main ideas in set invariance theory and placed them in a general, nonlinear setting. In [7], a discrete-time linear system with polyhedral state, control and disturbance constraints was considered, and the controlled invariant set was geometrically and analytically characterized. The authors of [20] considered a class of discrete-time hybrid systems with piecewise linear time-invariant flow function and polyhedral constraints. They also discussed two special classes which made the computation of controlled invariant set decidable. The invariant sets for piecewise affine systems have also been studied in [9] based on convex optimization techniques and linear matrix inequalities. In the literature of hybrid systems, a similar concept, maximal safety set, has been studied for example in [3, 1, 13]. In addition, controller design and verification based on invariant sets has been studied in, for example [19] and [6].

In this paper, we investigate the robust invariant set for a class of discrete-time uncertain linear hybrid systems, which is affected by both parameter variations and exterior disturbances. This model is directly related to piecewise linear (affine) systems, which have been widely studied in the literature, see for example [18, 10, 2, 13] and the references therein. The issues studied include modeling, stability, observability and controllability etc. Piecewise linear systems arise often from linearization of nonlinear systems. However, a large class of practical nonlinear systems with parameter variations are often of interest. To study such uncertain nonlinear systems in a systematic way, we introduce a bundle of linearizations, whose convex hull cover the original uncertain nonlinear dynamics within a region. This explains one of the motivations for our study of uncertain linear hybrid systems.

Another motivation for studying parameter uncertainties and exterior disturbances in hybrid systems comes from the fact that the system parameters are often subject to unknown, possibly time-varying, perturbations and that the real processes are often affected by disturbances. The dynamic uncertainty and robust control of hybrid/ switched systems is an under-explored and highly challenging field [16]. Some reachability analysis results for uncertain hybrid/ switched systems have appeared in [11, 14]; and some work on the induced gain analysis for switched systems has appeared for example in [8, 21].

Our goal here is to synthesize hybrid control laws, including the continuous variable control law and the discrete event control signal, to guarantee the controlled invariance for a specific region. In our earlier work on controlled robust invariance for such hybrid systems [15], we gave a formulation of the maximum permissive controller, which, however, was not constructive. Here we will synthesize an implementable, state feedback based hybrid control law to guarantee the controlled invariance.

The organization of the paper is as follows. The next section defines uncertain linear hybrid systems, and Section 3 formulates the "robust invariant control prob-

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lem". The synthesis procedure of hybrid invariant controller is given in Section 4, and a numerical example is given in Section 5 for illustration. Finally, concluding remarks are made.

2 Uncertain Linear Hybrid Systems

We are interested in the following discrete-time uncertain hybrid dynamical systems:

Definition 2.1 The discrete-time *Uncertain Linear Hybrid Systems* are defined by

$$\begin{aligned} x(t+1) &= A_{q(t)}(w)x(t) + B_{q(t)}(w)u(t) + E_{q(t)}d(t) \\ q(t) &= \delta(q(t-1), \pi(x(t)), \sigma_c(t), \sigma_u(t)) \end{aligned}$$

where $q \in Q = \{q_1, q_2, \cdots, q_s\}$ and Q is the collection of discrete states (modes); $x \in X \subseteq \mathbb{R}^n$ and X stands for the continuous variable state space. For mode q, the continuous variable control $u \in \mathcal{U}_q \subset \mathbb{R}^m$, and the continuous variable disturbance $d \in \mathcal{D}_q \subset \mathbb{R}^p$, where $\mathcal{U}_q$, $\mathcal{D}_q$ are bounded convex polyhedral sets. Denote $\mathcal{U} = \bigcup_{q\in Q} \mathcal{U}_q$, $\mathcal{D} = \bigcup_{q\in Q} \mathcal{D}_q$. And

- $A_q(w) : \mathcal{W} \to \mathbb{R}^{n\times n}$, $B_q(w) : \mathcal{W} \to \mathbb{R}^{n\times m}$, and $E_q \in \mathbb{R}^{n\times p}$ are the system matrices for the discrete state q. And that the entries of $A_q(w)$ and $B_q(w)$ are continuous function of $w \in \mathcal{W}$, where $\mathcal{W} \subset \mathbb{R}^v$ is an assigned compact set.
- $\pi : X \to X/E_\pi$ partitions the continuous variable state space $X \subset \mathbb{R}^n$ into polyhedral equivalence classes;
- $q(t) \in act(\pi(x(t)))$, where $act : X/E_\pi \to 2^Q$ defines the active mode set;
- $\delta : Q\times X/E_\pi \times \Sigma_c \times \Sigma_u \to Q$ is the discrete state transition function. Here $\sigma_c \in \Sigma_c$ denotes a controllable event and Σ_u the collection of uncontrollable events;
- The *guard* $G(q, q')$ of the transition (q, q') is defined as the set of all continuous variable states x, where $q' \in act(\pi(x(t)))$, such that for every uncontrollable event $\sigma_u \in \Sigma_u$ there exist controllable events $\sigma_c \in \Sigma_c$ s.t. $q' = \delta(q, \pi(x), \sigma_c, \sigma_u)$. The guard of the transition describes the region of the continuous variable state space where the transition can be forced to take place independently of the disturbances generated by the environment.

Remark: Note that in the above definition, we do not consider "state jumps" (reset) for continuous variable state x explicitly. However, the reset function can be easily included in our model by adding some auxiliary modes.

Remark: It is known that often in practice uncertainties enter linearly in the system model and they are linearly constrained. To handle this particular but interesting case, we consider the class of polyhedral sets. Such sets have been considered in previous papers in the literature concerning the control of systems with input and state constraints. Their main advantage is that they are suitable for computation. Therefore, we assume that

$$[A_q(w), B_q(w)] = \sum_{k=1}^{N_q} w_k [A_q^k, B_q^k],$$

where $w_k \geq 0$ and $\sum_{k=1}^{N_q} w_k = 1$. The pair $(A_q(w), B_q(w))$ represents the model uncertainty which belongs to the polytopic set $Conv\{[A_q^k, B_q^k],\ k = 1, \cdots, N_q\}$, which is referred to as polytopic uncertainty and provides a classical description of model uncertainty. Notice that the coefficients w_k are unknown and possibly time varying. Similar hybrid system model with polytopic uncertainty was considered for hybrid tracking and regulation control problems, see [14] and references therein.

In the following we assume the existence of the solution for such uncertain hybrid systems under given initial conditions. And we assume that exact state measurement (q, x) is available. An *admissible control input* (or *law*) is one which satisfies the input constraints $(\Sigma_c, \mathcal{U}_q)$. The elements of an allowable disturbance sequence are contained in $(\Sigma_u, \mathcal{D}_q)$.

3 Controlled Robust Invariance Problem

Given a set $\Omega = (\mathbf{q}, P) \subset Q \times X$ and an initial state $(q_0, x_0) \in \Omega$, it is of interest to determine whether there exist admissible control laws such that the evolution of the system will remain inside the set for all time, despite the presence of structured dynamic uncertainties and disturbances.

Definition 3.1 The set $\Omega \subset Q\times X$ is *controlled robust invariant* for the uncertain hybrid systems of Definition 2.1 if and only if $\forall(q_0, x_0) \in \Omega$, $\forall(\sigma_u, d(t)) \in \Sigma_u \times \mathcal{D}_{q(t)}$ and $\forall[A_{q(t)}(w), B_{q(t)}(w)] \in Conv[A^i_{q(t)}, B^i_{q(t)}]$ there exist admissible control inputs $(\sigma_c, u(t)) \in \Sigma_c \times \mathcal{U}_{q(t)}$, such that the system evolution satisfy $(q(t), x(t)) \in \Omega$, $\forall t \geq 0$.

Note that the continuous variable part P of the region $\Omega = (\mathbf{q}, P)$ does not necessarily coincide with the partitions of π in Definition 2.1, and this gives us more flexibility. However, it is required that the following consistency condition holds.

$$P \subseteq \bigcap_{q_i\in\mathbf{q}} Inv(q_i)$$

where $Inv(q_i) = \{x \in X : q_i \in act(\pi(x))\}$. $Inv(q_i)$ is similar to the concept of invariant set of mode q_i in hybrid automata.

A natural question is how to check whether a given set $\Omega \subset Q\times X$ is controlled robust invariant or not. In [15], we gave a necessary and sufficient geometric condition to check the controlled robust invariance for a given set Ω:

Lemma 3.1 The set $\Omega \subset Q \times X$ is a controlled robust invariant set *if and only if* $\Omega \subseteq pre(\Omega)$.

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The set $pre(\Omega)$ is called the one step predecessor set for Ω. In particular, $pre(\Omega)$ is the set of states in $Q \times X$, for which, despite disturbances and dynamic uncertainties, admissible control inputs exist and guarantee that the system will be driven to Ω in one step. To calculate $pre(\Omega)$, we first calculate the predecessor set for Ω either purely by discrete event transition, $pre_d(\Omega)$, or purely by continuous variable transition at mode q, $pre_c^q(P)$. Then an algorithm is given for $pre(\Omega)$ in [14] by considering the coupling between $pre_d(\Omega)$ and $pre_c^q(P)$, which is based on linear programming techniques.

Here we recall some notations necessary for the development of the feedback invariant controller in the next section. The continuous variable predecessor operator under mode q, which is denoted as $pre_c^q : 2^X \to 2^X$, computes the set of continuous variable states for which there exist control inputs $u \in \mathcal{U}_q$ so that the continuous variable state will be driven into the set P through the transition $A_q(w)x + B_q(w)u + E_q d$ despite the disturbance $d \in \mathcal{D}_q$ and uncertainty $w \in \mathcal{W}$. The action of the operator is described by

$$pre_c^q(P) = \{x \in Inv(q) | \forall d \in \mathcal{D}_q, \forall w \in \mathcal{W}, \exists u \in \mathcal{U}_q, \\ s.t.\ A_q(w)x + B_q(w)u + E_q d \in P\}$$

In general, a given set Ω is not controlled robust invariant. However, some subsets of Ω are likely to be controlled robust invariant. In addition, it follows immediately from the definition that the union of two controlled robust invariant sets is controlled robust invariant. However, the same statement cannot be made about the intersection of two controlled robust invariant sets, even in the absence of disturbances and uncertainties. So we have the following definition.

Definition 3.2 The set $\tilde{\mathcal{C}}_\infty(\Omega)$ is the *maximal controlled robust invariant set* contained in $\Omega \subset Q \times X$ for the uncertain hybrid systems of Definition 2.1 if and only if $\tilde{\mathcal{C}}_\infty(\Omega)$ is controlled robust invariant and contains all the robustly controlled invariant sets contained in Ω.

It can be shown that the maximal controlled robust invariant set is unique [15]. Then the next question is how to find the maximal controlled robust invariant set $\tilde{\mathcal{C}}_\infty(\Omega)$. In [15], a procedure was given to determine the maximal controlled robust invariant subset in Ω.

After checking the controlled robust invariance and calculating the maximal controlled robust invariant subset, we know whether there exist admissible control laws such that the specific region (or subset of the region) is invariant with respect to a given uncertain piecewise linear hybrid systems. However we do not know the controller itself. So another interesting problem is to construct a *control law*, $c : Q \times X \to 2^{\Sigma_c \times \mathcal{U}}$, which guarantees that the states remain within the region (assume proper initial conditions) despite the uncertainties and disturbances, while it satisfies certain input/output constraints. The problem can be formulated as the following "*Robust Invariant Control Problem*".

Problem 1 *Given a controlled robust invariant set $\Omega \subset Q \times X$ with respect to polytopic uncertain linear hybrid systems of Definition 2.1, construct a* control law, *$c : Q \times X \to 2^{\Sigma_c \times \mathcal{U}}$, that guarantees the robust invariance of Ω.*

In our earlier work [15], we gave a formulation of the maximum permissive controller for such purpose, which was not easy to construct for implementation. Here we will develop a systematic way to design implementable, state feedback based hybrid control laws to guarantee the controlled invariance for a specific region Ω.

4 Hybrid Invariant Control

In this section, we present a systematic procedure for *hybrid invariant controller* design. It is assumed that the given region $\Omega = (\mathbf{q}, P) \subset Q \times X$ is controlled invariant (if not, we turn to consider its nonempty maximal invariant subset), and assume that the initial condition is within Ω. Our objective is to build a control law, $c : Q \times X \to 2^{\Sigma_c \times \mathcal{U}}$, which robustly drives the system to guarantee that the states remain within Ω despite the uncertainties and disturbances.

In general, the discrete part of $\Omega = (\mathbf{q}, P)$, $\mathbf{q}$, contains more than one mode, that is $\mathbf{q} = \{q_1, q_2, \cdots, q_r\}$. In this case, we first calculate the one step continuous variable predecessor set of P under mode q_i, i.e. $pre_c^{q_i}(P) \bigcap P$, which is denoted as $P_{q_i}^1$. Then we can derive the relationship of $P_{q_i}^1$ and P as the following proposition.

Proposition 4.1 $\bigcup_{q_i \in \mathbf{q}} P_{q_i}^1 = P$

Proof : Because $P_{q_i}^1 = pre_c^{q_i}(P) \bigcap P$, so $P_{q_i}^1 \subseteq P$ for all $i = 1, \cdots, r$. So $\bigcup_{q_i \in \mathbf{q}} P_{q_i}^1 \subseteq P$.

On the other hand, assume that $\bigcup_{q_i \in \mathbf{q}} P_{q_i}^1 \neq P$, and let $E = P - \bigcup_{q_i \in \mathbf{q}} P_{q_i}^1$. For the continuous variable state $x(t) \in E$, because $x(t) \notin \bigcup_{q_i \in \mathbf{q}} P_{q_i}^1$, therefore $x(t) \notin P_q^1$ for any possible $q \in \mathbf{q}$. Then, for $x(t) \in E$, there does not exist a continuous variable control signal $u(t) \in \mathcal{U}_q$ to guarantee that the next step continuous variable state $x(t+1) = A_q(w)x(t) + B_q(w)u(t) + E_q d(t)$ remains in P, i.e. $x(t+1) \notin P$ (from the definition of $pre_c^q(\cdot)$) for some $d \in \mathcal{D}_q$, $w \in \mathcal{W}$, and for all possible modes q. Therefore, for such initial hybrid state $(q, x(t)) \in \Omega$, there does not exist hybrid control signal (σ_c, u) to make the next state $(q', x(t+1))$ remains in Ω. This leads to a contradiction to the assumption that $\Omega = (\mathbf{q}, P)$ is controlled invariant. Therefore, $\bigcup_{q_i \in \mathbf{q}} P_{q_i}^1 = P$ holds. $\Box$

In other words, the continuous variable part of the region $\Omega = (\{q_1, q_2, \cdots, q_r\}, P)$ is partitioned into r subsets $P_{q_i}^1$, $i = 1, \cdots, r$. For simplicity, we first assume that P is a convex polyhedron and $P = \{x : Fx \leq g\}$, while the case when P is not convex will be discussed later in this section. When P is convex, it is straightforward to show that $P^1_{q_i} = pre^{q_i}_c(P) \bigcap P$ is convex polyhedral set as well. Denote its vertex set as $vert\{P^1_{q_i}\}$. For the finite number of its vertices $x^j_{q_i} \in vert\{P^1_{q_i}\}$, we define the cost functional, $J : Q \times [0,1]^{N_q} \times \mathcal{U}_q \to \mathbb{R}^+$

$$\begin{aligned} J(q_i, w, u) &= \|F[A_{q_i}(w)x^j_{q_i} + B_{q_i}(w)u(t)]\|_\infty \\ &= \|F \sum_{l=1}^{N_{q_i}} [w_l A^l_{q_i}, w_l B^l_{q_i}] \begin{pmatrix} x^j_{q_i} \\ u(t) \end{pmatrix} \|_\infty \end{aligned}$$

where $\|\cdot\|_\infty$ stands for the infinite norm. The control signal for the vertex $x^j_{q_i} \in vert\{P^1_{q_i}\}$ is selected as the solution to the following minmax optimization problem:

$$\min_{u \in \mathcal{U}_{q_i}} \max_{w \in [0,1]^{N_{q_i}}} J(q_i, w, u)$$
$$s.t. \begin{cases} A_{q_i}(w)x^j_{q_i} + B_{q_i}(w)u(t) + E_{q_i}d(t) \in P \\ u \in \mathcal{U}_{q_i}, \quad d \in \mathcal{D}_{q_i} \end{cases}$$

The optimal action of the controller is one that tries to minimize the maximum cost, and try to counteract the worst disturbance and the worst model uncertainty. It should be pointed out that the above optimization problem must be feasible for the vertices of $P^1_{q_i}$. This claim comes from the predecessor definition and the construction of $P^1_{q_i} = pre^{q_i}_c(P) \bigcap P$. For the case of polytopic uncertainty, the constraints can be equivalently transformed into the following form because of linearity and convexity.

$$\min_{u \in \mathcal{U}_{q_i}} \max_{w \in [0,1]^{N_{q_i}}} J(q_i, w, u)$$
$$s.t. \begin{cases} F[A^1_{q_i} x^j_{q_i} + B^1_{q_i} u(t)] \leq g - \delta_{q_i} \\ F[A^2_{q_i} x^j_{q_i} + B^2_{q_i} u(t)] \leq g - \delta_{q_i} \\ \cdots\cdots \\ F[A^{N_{q_i}}_{q_i} x^j_{q_i} + B^{N_{q_i}}_{q_i} u(t)] \leq g - \delta_{q_i} \\ u \in \mathcal{U}_{q_i} \end{cases}$$

where δ_{q_i} is a vector whose components are given by $\delta_{q_i}(j) = \max_{d \in \mathcal{D}_{q_i}} F_j E_{q_i} d$, and F_j is the j-th row of the matrix F. The above minmax optimization problem can be equivalently transformed to the following linear programming problem [4].

$$\min_{u \in \mathcal{U}_{q_i}} \zeta$$
$$s.t. \begin{cases} F[A^1_{q_i} x^j_{q_i} + B^1_{q_i} u(t)] \leq \zeta \\ F[A^2_{q_i} x^j_{q_i} + B^2_{q_i} u(t)] \leq \zeta \\ \cdots\cdots \\ F[A^{N_{q_i}}_{q_i} x^j_{q_i} + B^{N_{q_i}}_{q_i} u(t)] \leq \zeta \\ F[A^1_{q_i} x^j_{q_i} + B^1_{q_i} u(t)] \leq g - \delta_{q_i} \\ F[A^2_{q_i} x^j_{q_i} + B^2_{q_i} u(t)] \leq g - \delta_{q_i} \\ \cdots\cdots \\ F[A^{N_{q_i}}_{q_i} x^j_{q_i} + B^{N_{q_i}}_{q_i} u(t)] \leq g - \delta_{q_i} \\ u \in \mathcal{U}_{q_i} \end{cases}$$

Because of the guaranteed feasibility of the above linear programming problem for each vertex of the polytope $P^1_{q_i}$, the admissible control for each vertex $x^j_{q_i}$ exists, and it is denoted as $u^j_{q_i}$. In the next step, we will construct the continuous variable control signals for the continuous variable state contained in region $P^1_{q_i}$ through convexity. Note that any $x \in P^1_{q_i}$ can be (not uniquely) written as the convex combination of the vertex of $P^1_{q_i}$, $x = \sum_j \alpha^j_{q_i}(x) x^j_{q_i}$, where the convex combination coefficients $\alpha^j_{q_i}(x) \geq 0$ and $\sum_j \alpha^j_{q_i}(x) = 1$. We set the control signal $u_{q_i}(x)$ for state x simply as the convex combination of the control signals at the vertex $u^j_{q_i}$. In particular,

$$u_{q_i}(x) = \sum_j \alpha^j_{q_i}(x) u^j_{q_i} \tag{4.1}$$

and $u_{q_i}(x) \in \mathcal{U}_{q_i}$ comes from the convexity of $\mathcal{U}_{q_i}$. And

$$\begin{aligned} & F[A_{q_i}(w)x + B_{q_i}(w)u_{q_i}] \\ = \; & F[A_{q_i}(w) \sum_j \alpha^j_{q_i} x^j_{q_i} + B_{q_i}(w) \sum_j \alpha^j_{q_i} u^j_{q_i}] \\ = \; & \sum_j \alpha^j_{q_i} F[A_{q_i}(w) x^j_{q_i} + B_{q_i}(w) u^j_{q_i}] \\ \leq \; & \sum_j \alpha^j_{q_i} [g - \delta_{q_i}] \\ = \; & g - \delta_{q_i} \end{aligned}$$

for all $[A_{q_i}(w), B_{q_i}(w)]$. In other words, for any $x \in P^1_{q_i}$, the control signal $u_{q_i}(x)$ will drive the next state remaining in P despite the uncertainty and disturbance. Therefore, it is easy to show that the control law of the form (4.1) solves the invariance problem. Then the candidate control input is selected as $(\sigma_c(t), u(t))$, where σ_c makes $q(t) = q_i$ and $u(t)$ is of the form (4.1).

When P is not convex, we know that the non-convex piecewise linear set P can be written as finite union of convex piecewise linear sets P_j, that is $P = \cup^m_{j=1} P_j$ [17]. From the definition of the continuous variable predecessor operator $pre^q_c(\cdot)$, we have $pre^q_c(P) = pre^q_c(\cup^m_{j=1} P_j) = \cup^m_{j=1} pre^q_c(P_j)$. Note that the set P_j and $pre^q_c(P_j)$ are both convex polyhedral sets. Define the set

$$P^{q_i}_{j,k} = P_k \cap pre^q_c(P_j), \quad \text{for } j,k = 1, \cdots, m$$

obviously, $P^{q_i}_{j,k}$ is convex polyhedral. Consider the polytopic subregion as $(q_i, pre^{q_i}_c(P_j) \cap P_k)$, for $j,k = 1, \cdots, m$, then the controller synthesis method developed previously in this section can be used directly to the subregion $\Omega^{q_i}_{j,k} = (q_i, P^{q_i}_{j,k})$. In particular, the control signal for each vertex of $P^{q_i}_{j,k}$ can be solved by the following minmax optimization problem.

$$\min_{u \in \mathcal{U}_{q_i}} \max_{w \in [0,1]^{N_{q_i}}} \|F_j \sum_{l=1}^{N_{q_i}} [w_l A^l_{q_i}, w_l B^l_{q_i}] \begin{pmatrix} x^r_{q_i} \\ u(t) \end{pmatrix} \|_\infty$$
$$s.t. \begin{cases} A_{q_i}(w)x^r_{q_i} + B_{q_i}(w)u(t) + E_{q_i}d(t) \in P_j \\ u \in \mathcal{U}_{q_i}, \quad d \in \mathcal{D}_{q_i} \end{cases}$$

where it is assumed that $P_j = \{F_j x \leq g_j\}$. Similarly, the above minmax optimization problem can be transformed into a linear programming problem, and the feasibility of which is guaranteed from the definition of the predecessor operator $pre_c^q(\cdot)$ and the construction of $P_{j,k}^{q_i} = P_k \cap pre_c^q(P_j)$. And we have

$$\begin{aligned}\bigcup_{j,k=1}^{m} (pre_c^{q_i}(P_j) \cap P_k) &= pre_c^{q_i}(\cup_{j=1}^m P_j) \bigcap (\cup_{k=1}^m P_k) \\ &= pre_c^{q_i}(P) \bigcap P = P_{q_i}^1\end{aligned}$$

According to the previous proposition, we conclude that $\bigcup_{q_i \in \mathbf{q}} (\bigcup_{j,k} (pre_c^{q_i}(P_j) \cap P_k)) = P$. In other words, the continuous variable part of the region $\Omega = (\mathbf{q}, P)$, P is partitioned into a finite number of polytopic subregions $P_{j,k}^{q_i}$, for $j,k = 1, \cdots, m$, $q_i \in \mathbf{q} = \{q_1, q_2, \cdots, q_r\}$.

In summary, the invariant controller for the region $\Omega = (\{q_1, q_2, \cdots, q_r\}, P)$ is given as follows. For $x(t) \in P_{j,k}^{q_i}$, the discrete control signal σ_c is selected as the one that makes $q(t) = q_i$. Because of the controlled invariance assumption of Ω and the definition of guard set, the existence of σ_c can be shown. Secondly, the continuous variable control signal, $u_{q_i}(t)$, is of the form $u_{q_i}(t) = \sum_r \alpha_{q_i}^r(x) u_{q_i}^r$. In this expression, $\alpha_{q_i}^r(x)$ is the convex combination coefficients of $x(t)$ by the vertices of $P_{j,k}^{q_i}$, and $u_{q_i}^r$ is the control signal for the corresponding vertices of $P_{j,k}^{q_i}$. It has been shown that the vertex control signal $u_{q_i}^r$ can be derived by solving a linear programming problem. A control of the above form can be implemented as a *piecewise linear state feedback controller* as follows. Let $X_{j,k}^{q_i}$ be a matrix whose columns are formed by the vertex vector of $P_{j,k}^{q_i}$. The columns of matrix $U_{j,k}^{q_i}$ are the calculated continuous variable control vector, u_q^i, corresponding to each vertex of $P_{j,k}^{q_i}$. A piecewise linear state feedback controller is then obtained by applying the control

$$u_{q_i}(x) = \sum_r \alpha_{q_i}^r(x) u_{q_i}^r = U_{j,k}^{q_i} (X_{j,k}^{q_i^T} X_{j,k}^{q_i})^{-1} X_{j,k}^{q_i^T} x \tag{4.2}$$

where $(\cdot)^T$ stands for transpose, and $(\cdot)^{-1}$ inverse of matrix. The convex combination coefficients $\alpha_{q_i}^r(x)$ can be calculated as $(X_{j,k}^{q_i^T} X_{j,k}^{q_i})^{-1} X_{j,k}^{q_i^T} x$ if $(X_{j,k}^{q_i^T} X_{j,k}^{q_i})$ is invertible. Otherwise another procedure is needed to generate the convex combination vector coefficients $\alpha_{q_i}(x)$. Note that all the calculations to derive the matrix $X_{j,k}^{q_i}$ and $U_{j,k}^{q_i}$ can be done off line, which can be efficiently calculated by linear programming techniques. And the implementation of the control law only needs to calculate the convex combination coefficients vector $\alpha_{q_i}(x)$, which can be easily done by solving some linear equations. Therefore, this computational advantage makes the above method a good candidate to deal with high dimensional hybrid systems.

There still exist one point to be clarified, namely the case when some states $(q(t), x(t)) \in \Omega$ may have more than one permissive control law. Then some criteria should be designed for the selection of $(\sigma_c(t), u(x))$, e.g. the magnitude or energy of $u(x)$ etc. And this kind freedom may also lead to optimization with respect to some kind of cost function of the control signals.

5 Numerical Example

Consider the following discrete-time uncertain linear hybrid systems:

$$\begin{aligned} x(t+1) &= \begin{cases} A_0(w)x(t) + B_0(w)u(t) + E_0 d(t), & q = q_0 \\ A_1(w)x(t) + B_1(w)u(t) + E_1 d(t), & q = q_1. \end{cases} \\ q(t) &= \delta(q(t-1), \pi(x(t)), \sigma_c(t), \sigma_u(t)) \end{aligned}$$

where

$$\begin{aligned} A_0(w) &= \begin{bmatrix} 1+w & 1 \\ 0 & 1 \end{bmatrix}, \; B_0(w) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; E_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ A_1(w) &= \begin{bmatrix} 0 & 1 \\ w-1 & -1 \end{bmatrix}, \; B_1(w) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; E_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

We assume that the time varying uncertain parameter w is subjected to the constraint $-0.1 \leq w \leq 0.1$. For simplicity, we assume that $act(q_0)$, $act(q_1)$ and all the *guard* set for discrete modes transition are the whole $\mathbb{R}^2$. Assume $u \in \mathcal{U} = [-1, 1]$, $d \in \mathcal{D} = [-0.1, 0.1]$.

First, consider the region $\Omega = \{q_0, q_1\} \times \{x : Fx \leq g\}$, where

$$F = \begin{bmatrix} -0.3547 & -0.9350 \\ -0.7214 & -0.6925 \\ -0.6892 & -0.7246 \\ 0.7214 & 0.6925 \\ 0.3547 & 0.9350 \\ 0.6892 & 0.7246 \end{bmatrix}, g = \begin{bmatrix} 0.5165 \\ 0.1587 \\ 0.1516 \\ 0.1587 \\ 0.5165 \\ 0.1516 \end{bmatrix}$$

The above hybrid region Ω turns out to be controlled invariant, which, in fact, is a maximal controlled invariant set contain in the polytopic region

$$\Omega_0 = \{q_0, q_1\} \times \{x : \begin{bmatrix} -0.225 \\ -0.225 \end{bmatrix} \leq \begin{bmatrix} 1 & 0.9 \\ 1 & 1.1 \end{bmatrix} x \leq \begin{bmatrix} 0.225 \\ 0.225 \end{bmatrix}\},$$

calculated by the procedure for $\tilde{C}_\infty(\Omega_0)$ described in [15]. Notice that the continuous variable part $P = \{x : Fx \leq g\}$ in this example is a convex polyhedron. Our next step is to design the invariant controller. For such purpose, we calculate the one step predecessor set $P_{q_0}^1 = pre_c^{q_0}(P) \cap P$ for mode q_0 and $P_{q_1}^1 = pre_c^{q_1}(P) \cap P$ for mode q_1. We have $P_{q_0}^1 \bigcup P_{q_1}^1 = P$. Then we calculate the vertices control vectors for each subregion, $P_{q_i}^1$, by solving the above induced linear programming problem. The coordinate of vertices and their corresponding control vector for $P_{q_0}^1$ may be used to form the matrices X_{q_0} and U_{q_0} respectively. Similarly, we get the matrices X_{q_1} and U_{q_1} from $P_{q_1}^1$. For example, the coordinate of vertices and their corresponding control vector for $P_{q_1}^1$ is calculated, and we obtain the matrices X_{q_1} and U_{q_1} for the calculation of the feedback gain matrix in

(4.2) for the subregion $P^1_{q_1}$ as follows.

$$X^T_{q_1} = \begin{pmatrix} -0.2200 & 0.0000 \\ 0.2200 & 0.0000 \\ -0.6355 & 0.4328 \\ 0.6355 & -0.4328 \\ -0.2350 & 0.4328 \\ 0.2350 & -0.4328 \end{pmatrix}, \quad U^T_{q_1} = \begin{pmatrix} -0.2200 \\ 0.2200 \\ -0.5820 \\ 0.5820 \\ -0.1815 \\ 0.1815 \end{pmatrix}$$

In summary, for any state $(q, x) \in \Omega$, the hybrid control law is designed as follows:

- **Case** $q = q_0$**:**
  - If $x \in P^1_{q_0}$, then the u is given by Equation (4.2) with $X_q = X_{q_0}$ and $U_q = U_{q_0}$. And the discrete control signal σ_c is selected in such a way that $q_0 = \delta_q(q_0, \pi(x), \sigma_c(t), \Sigma_u)$.
  - If $x \notin P^1_{q_0}$, then x must be contained in $P^1_{q_1}$. In this case, the u is given by Equation (4.2) with $X_q = X_{q_1}$ and $U_q = U_{q_1}$. And the discrete control signal σ_c is selected in such a way that $q_1 = \delta_q(q_0, \pi(x), \sigma_c(t), \Sigma_u)$.
- **Case** $q = q_1$**:**
  - If $x \in P^1_{q_1}$, then the u is given by Equation (4.2) with $X_q = X_{q_1}$ and $U_q = U_{q_1}$. And the discrete control signal σ_c is selected in such a way that $q_1 = \delta_q(q_1, \pi(x), \sigma_c(t), \Sigma_u)$.
  - If $x \notin P^1_{q_1}$, then x must be contained in $P^1_{q_0}$. In this case, the u is given by Equation (4.2) with $X_q = X_{q_0}$ and $U_q = U_{q_0}$. And the discrete control signal σ_c is selected in such a way that $q_0 = \delta_q(q_1, \pi(x), \sigma_c(t), \Sigma_u)$.

6 Conclusion

In this paper, we put our group's recent progress in the analysis and synthesis of uncertain piecewise linear hybrid systems into the framework of invariant set theory. We developed an implementable procedure to synthesize the robust invariant controller. The continuous variable controller designed is a piecewise linear state feedback control law, based on the partition of the given controlled invariant region into finite polytopic subregions. The state feedback gain matrix is constant within each subregion. The matrix was determined by the vertex coordinates of the subregion's polytope and their corresponding control vectors, which could be determined off-line by solving some linear programming problems. Therefore, the online implementation of the state feedback control law only needs to first determine which subregion the current state belongs to and then multiply the state coordinate with the corresponding gain matrix. Comparing with the online optimization based controller design for tracking and regulation problems in [14], this method has the advantage of less online computational burden. And this computational benefit makes the method developed here a promising candidate to deal with high dimensional hybrid systems.

References

[1] E. Asarin, O. Bournez, T. Dang, O. Maler, and A. Pnueli, “Effective synthesis of switching controllers for linear systems,” *Proceedings of IEEE*, 88(7), 1011-1025, 2000.
[2] A. Bemporad, G. Ferrari-Trecate and M. Morari. “Observability and controllability of piecewise affine and hybrid systems,” *IEEE Tran. on Automat. Contr.*, 45(10), 1864-1876, 2000.
[3] L. Berardi, E. De Santis, M. D. Di Benedetto, and G. Pola, “Approximations of Maximal Controlled Safe Sets for Hybrid Systems,” *Workshop on Hybrid Control and Automotive Applications*, 2001.
[4] D. P. Bertsekas, *Nonlinear Programing*, 2nd edition, Athena Scientific, 1999.
[5] F. Blanchini, “Set invariance in control,” *Automatica*, 35, 1747-1767, 1999.
[6] P. E. Caines, and Yuan-Jun Wei, “Hierarchical Hybrid Control Systems: A Lattice Theoretic Formulation,” *IEEE Trans. Automat. Contr.*, 43(4), 501-508, 1998.
[7] C. E. T. Dórea and J.C. Hennet, “On (A, B)-invariance of polyhedral domains for discrete-time systems,” *CDC 1996*, 4319-4324, 1996.
[8] J. P. Hespanha, “Computation of Root-Mean-Square Gains of Switched Linear Systems,” *HSCC 2002*, 239-252.
[9] M. Jirstrand, “Invariant sets for a class of hybrid systems,” *CDC 1998*, 3699-3704, 1998.
[10] M. Johansson, *Piecewise Linear Control Systems*, Ph.D. Thesis, Lund Institute of Technology, Sweden, 1999.
[11] Ulf T. Jönsson, “On Reachability Analysis of Uncertain Hybrid Systems,” *CDC 2002*, 2397-2402, 2002.
[12] E.C. Kerrigan, and J.M. Maciejowski, “Invariant sets for constrained nonlinear dicretetime systems with application to feasibility in model predictive control,” *CDC*, 2000.
[13] X. D. Koutsoukos, *Analysis and Design of Piecewise Linear Hybrid Dynamical Systems*, PhD thesis, Univ. of Notre Dame, Notre Dame, IN, 2000.
[14] H. Lin, and P. J. Antsaklis, “Controller Synthesis for a class of Uncertain Piecewise Linear Hybrid Dynamical Systems,” *CDC*, 2002.
[15] H. Lin, and P. J. Antsaklis, “Controlled Invariant Sets for a class of Uncertain Piecewise Linear Hybrid Dynamical Systems,” *CDC*, 2002.
[16] T. Moor, and J. M. Davoren, “Robust controller synthesis for hybrid systems using modal logic,” *HSCC 2001*, 433-446.
[17] E. Sontag, “Remarks on piecewise-linear algebra,” *Pacific Journal of Mathematics*, 92(1), 183-210, 1982.
[18] E. Sontag. “Interconnected automata and linear systems: A theoretical framework in discrete-time,” *Hybrid Systems III*, Vol.1066 of Lecture Notes in Computer Science, 436-448, Springer, 1996.
[19] J. A. Stiver, X. D. Koutsoukos, and P. J. Antsaklis, “An Invariant Based Approach to the Design of Hybrid Control Systems,” *International Journal of Robust and Nonlinear Control*, 11(5), 453-478, 2001.
[20] R. Vidal, S. Schaffert, O. Shakernia, J. Lygeros, S. Sastry, “Decidable and Semi-decidable Controller Synthesis for Classes of Discrete-time Hybrid Systems,” *CDC*, 2001.
[21] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, “Disturbance attenuation properties of timecontrolled switched systems,” *Journal of the Franklin Institute*, 338, 765779, 2001.