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Practical Stabilization of Integrator Switched Systems

Xuping Xu[1] and Panos J. Antsaklis[2]

[1] Department of Electrical and Computer Engineering
Penn State Erie, Erie, PA 16563, USA. E-mail: Xuping-Xu@psu.edu
[2] Department of Electrical Engineering, University of Notre Dame
Notre Dame, IN 46556, USA. E-mail: antsaklis.1@nd.edu

ABSTRACT: In this paper, practical stabilization problems for integrator switched systems are studied. In such class of switched systems, no subsystem has an equilibrium. However, the system can still exhibit interesting behaviors around a given point under appropriate switching laws. Such behaviors are similar to those of a conventional stable system near an equilibrium. Some practical stability notions are formally introduced to define such behaviors. A necessary and sufficient condition for practical asymptotic stabilizability of such systems is then proved. For practically asymptotically stabilizable systems, a minimum dwell time switching law which can easily be implemented is proposed.

AMS(MOS) subject classification: 34H05, 34D99, 93C15, 93D99

1. Introduction

A switched system is a particular kind of hybrid system that consists of several subsystems and a switching law orchestrating the active subsystem at each time instant. Many results on stability analysis and stabilization of switched systems have been reported (see, e.g., [1,2,7,8] and the references therein). Most of them consider switched systems whose subsystems share a common equilibrium. Methods based on single or multiple Lyapunov functions have been reported for the stability analysis and design of such systems. Methods based on geometric properties of the subsystem vector fields have also been reported [12].

In our recent research, we found that the assumption that all subsystems share a common equilibrium may not hold for all switched systems and may limit the applicability of switched systems stability results. In the case that such an assumption does not hold, i.e., when subsystems have different or no equilibria, under appropriate switching laws a system may still exhibit interesting behaviors. Such behaviors are similar to those of a conventional stable system near an equilibrium. In this paper, we formally introduce some practical stability notions to define such behaviors. Such notions are extensions of the traditional concepts on practical stability [4,5], which are concerned with bringing the system trajectories to be within a given bound.

Similar boundedness behaviors have also been observed by other researchers recently. Lin and Antsaklis in [9] studied the stabilization problem for switched linear systems with uncertainties. Such systems, though still having a common equilibrium, cannot be asymptotically stabilized. Hence ultimate boundedness problem is more reasonable to consider. Zhai and Michel in [13] introduced the notion of practical stability for a class of switched systems. The notion in [13] concerns the boundedness property of the system trajectory with respect to a given bound. Sufficient conditions based on Lyapunov-like functions are proposed. As opposed to [9,13], the notions and results we will propose in this paper concern necessary and sufficient conditions for boundedness property with respect to any bound for a class of switched systems without a common equilibrium.

In this paper, we focus on practical stabilization problems for a simple yet important class of switched systems — integrator switched systems. Many real-world processes including chemical processes [6,10] can be modeled as such systems. After introducing some practical stability notions, we prove a necessary and sufficient condition for the practical asymptotic stabilizability of such systems (Theorem 3.1). Additional feasible ways for checking the condition are also proposed. Moreover, for practically asymptotically stabilizable systems, we propose a minimum dwell time switching law which can easily be implemented to achieve ϵ-practical asymptotic stability. The switching law is applied to a three tank problem in chemical batch process to illustrate its effectiveness.

2. Practical stability notions for switched systems

2.1. Switched systems. A *switched system* is a dynamic system which consists of subsystems

$$\dot{x} = f_i(x), \quad f_i : \mathbf{R}^n \to \mathbf{R}^n, \quad i \in I = \{1, 2, \cdots, M\}, \tag{2.1}$$

and a switching law orchestrating the active subsystem at each time instant. The state trajectory of a switched system is determined by the initial state and the timed sequence of active subsystems. A *switching sequence* in $t \in [t_0, t_f]$ regulates the timed sequence of active subsystems and is defined as follows.

Definition 2.1 (Switching Sequence). A switching sequence σ in $[t_0, t_f]$ is defined as

$$\sigma = ((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K)) \tag{2.2}$$

where $0 \le K < \infty$, $t_0 \le t_1 \le \cdots \le t_K \le t_f$, $i_k \in I$ for $k = 0, 1, \cdots, K$.

We also define $\Sigma_{[t_0,t_f]} = \{$switching sequence σ's in $[t_0, t_f]\}$ and $\Sigma_{[t_0,\infty)} = \{\ \sigma$ defined on $[t_0, \infty)$ satisfying $\sigma_{[t_0,t_f]} \in \Sigma_{[t_0,t_f]}$, $\forall t_f > t_0$, where $\sigma_{[t_0,t_f]}$ is the truncated version of σ in $[t_0, t_f]\}$ □

σ indicates that subsystem i_k is active in $[t_k, t_{k+1})$ (subsystem $i_K \in [t_K, t_f]$) switched system to be well-behaved, we only consider **nonZeno** sequences which at most a finite number of times in any finite time interval $[t_0, t_f]$, though di sequences may have different numbers of switchings. Note that the continuous sta switched system does not exhibit jumps at switching instants.

In this paper, we pay particular attention to switching sequences over the interval $[0, \infty)$. Such switching sequences are usually generated by switching laws d below.

Definition 2.2 (Switching Law over $[0, \infty)$). For switched system (2.1), a switchi S over $[0, \infty)$ is defined to be a mapping $S : \mathbf{R}^n \to \Sigma_{[0,\infty)}$ which specifies a swi sequence $\sigma \in \Sigma_{[0,\infty)}$ for any initial point $x(0)$.

Remark 2.3. S over $[0, \infty)$ is often described by some rules or algorithms, which de how to generate a switching sequence for a given $x(0)$, rather than mathematical for In this paper, we will specify switching laws using such descriptions.

2.2. Some practical stability notions. Many results have appeared on stabilit ysis and stabilization of switched systems. Most of them assume that a common e rium exists for all subsystems. However, this assumption may not be true for all sw systems and may limit the applicability of switched systems. In the case when subs have different or no equilibria, a system may still exhibit interesting behaviors a given point under appropriate switching laws. The behaviors are similar to tho conventional stable system near an equilibrium. The following example illustrate behaviors.

Example 2.4. Consider a switched system consisting of: subsystem 1: $\dot{x} = [-3,$ subsystem 2: $\dot{x} = [-2.5, -3]^T$; subsystem 3: $\dot{x} = [3, -2.5]^T$; subsystem 4: $\dot{x} = [2.$ If we apply the switching law which makes subsystem 1 active in quadrant I, sub 2 in quadrant II, subsystem 3 in quadrant III, and subsystem 4 in quadrant IV, th system will exhibit "convergent behaviors" around the origin. Figure 1 shows a trajectory starting from $x_0 = [2, 1]^T$ under this law in a finite time duration.

In Example 2.4, under the given switching rule the origin exhibits behaviors to those of an asymptotically stable system. However, as the trajectory becomes and closer to the origin, the system needs to switch faster and faster. This viola nonZenoness requirement for valid switching sequences. In practice, a lower bound time between switchings will usually be imposed that prevents Zenoness. Such bound is called the ***minimum dwell time*** [3] and its value may be different for d application objectives. If we incorporate a minimum dwell time into the switching Example 2.4, system trajectories starting from any point in $\mathbf{R}^2$ will be attracted the origin and eventually oscillate near the origin within certain bound.

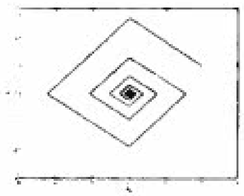

FIGURE 1. A sample trajectory starting from $x_0 = [2, 1]^T$ for Example 2.4.

The concept of bringing the system trajectories to be within a given bound is quite useful in practice. For example, in temperature control systems, we are more interested in keeping the temperature within certain bounds, rather than in stabilizing the system asymptotically to a set-point. In fact, such concept has been formally termed practical stability in [4,5] for ordinary differential equations. In the following, we adapt and expand some practical stability notions to switched systems and formally define the notion of practical asymptotic stabilizability. We will use $\|\cdot\|$ to denote the 2-norm of a vector. Without loss of generality, we only discuss the case of the origin and let the initial time be $t_0 = 0$.

Definition 2.5 (ϵ-Practical Stability). Assume a switching law S over $[0, \infty)$ is given for the switched system (2.1). Given $\epsilon > 0$, the switched system is said to be ϵ-practically stable under S if there exists $\delta = \delta(\epsilon) > 0$ such that $x(t) < \epsilon$, $\forall t > 0$, whenever $x(0)$ satisfies $\|x(0)\| < \delta$. □

Definition 2.6 (ϵ-Attractivity). Assume a switching law S over $[0, \infty)$ is given for the switched system (2.1). Given $\epsilon > 0$, the origin is said to be ϵ-attractive if for any $x(0) \in \mathbf{R}^n$, there exists a $T = T(x(0)) \geq 0$ such that $\|x(t)\| < \epsilon$ for any $t \geq T$. □

Remark 2.7. Note that ϵ-practical stability is related to the boundedness behavior of the system trajectory and ϵ-attractivity is related to the convergent behavior. However, ϵ-attractivity does not imply ϵ-practical stability, because it is possible that for any $\delta < \epsilon$, a trajectory exists that starts at $x(0)$ with $\|x(0)\| < \delta$ and violates $\|x(t)\| < \epsilon$ for some time and finally settles down with $\|x(t)\| < \epsilon$. This still satisfies ϵ-attractivity; however, ϵ-practical stability is not satisfied. □

Definition 2.8 (ϵ-Practical Asymptotic Stability). Assume a switching law S over $[0, \infty)$ is given for the switched system (2.1). Given $\epsilon > 0$, the switched system is said to be ϵ-practically asymptotically stable under S if it is ϵ-practically stable and the origin is ϵ-attractive. □

Definition 2.9 (Practical Asymptotic Stabilizability). The switched system (2.1) to be practically asymptotically stabilizable if for any $\epsilon > 0$, there exists a switch $S = S(\epsilon)$ such that the system is ϵ-practically asymptotically stable under S.

Remark 2.10. In the definition of practical asymptotic stabilizability, the ϵ varied as opposed to the fixed ϵ in the previous several definitions. Hence a pra asymptotically stabilizable system has the property that, for any given bound, switching law can be constructed which brings the system trajectory into the bou keeps it within the bound.

3. Practical stabilization results for integrator switched systems

In the sequel, we focus on a special class of switched systems — integrator sy systems, which consist of subsystems

$$\dot{x} = a_i, \; i \in I = \{1, 2, \cdots, M\} \tag{3.1}$$

where $a_i \in \mathbf{R}^n$ ($a_i \neq 0$), $i \in I$ are constant vectors and $x \in \mathbf{R}^n$ is the con state. Such systems receives particular attention due to the followings. First can model many real world processes, such as chemical batch processes [6,10]. S the simple structure of such systems makes possible rigorous analysis which le nice theoretical and practical results. Third, the complete exploration of such s is the first step toward the study of practical stability properties of general no switched systems, because a nonlinear switched system may be approximated loc an integrator switched systems around a given point when this point is not an equil for any subsystem.

3.1. Conditions for practical asymptotic stabilizability.

In the following, w help of some convex analysis notions and results (see [11]), we prove some necessa sufficient conditions for practical asymptotic stabilizability of system (3.1). Th 3.1 provides a necessary and sufficient condition for practical asymptotic stabiliz Lemma 3.2 provides a feasible way of verifying the condition in Theorem 3.1. Theorem 3.4 and three corollaries are proposed which illustrate some implications necessary and sufficient condition and emphasize more on systems with $n+1$ subs in $\mathbf{R}^n$. In order to make the main results stand out, we have put all the proofs in Ap A.

Theorem 3.1 (Necessary and Sufficient Condition). *An integrator switched syste in $\mathbf{R}^n$ is practically asymptotically stabilizable if and only if $C = \mathbf{R}^n$, where C is the cone $C = \{\sum_{i=1}^{M} \lambda_i a_i | \lambda_1 \geq 0, \cdots, \lambda_M \geq 0\}$.*

Proof: See Appendix A.

In order to apply Theorem 3.1, we need to verify the validity of the condition $C = \mathbf{R}^n$. Exhaustively checking whether $x \in C$ for any $x \in \mathbf{R}^n$ is not an option due to the infinite number of x to check. The following lemma provides a necessary and sufficient condition which is equivalent to $C = \mathbf{R}^n$ and computational feasible to verify.

Lemma 3.2. *$C = \mathbf{R}^n$ if and only if there exists a subset $\{a_{i_1}, \cdots, a_{i_l}\}$ of $\{a_1, \cdots, a_M\}$ which satisfies the following conditions:*

(a). $\mathrm{span}\{a_{i_1}, \cdots, a_{i_l}\} = \mathbf{R}^n$ *and*
(b). *there exist $\lambda_j > 0$, $j = 1, \cdots, l$, such that $\sum_{j=1}^{l} \lambda_j a_{i_j} = 0$.*

Proof: See Appendix A. □

Remark 3.3. Lemma 3.2 provides us with a feasible way of checking whether $C = \mathbf{R}^n$ or not. By exhaustively checking all possible subsets of $\{a_1, \cdots, a_M\}$ for the validity of conditions (a) and (b), we can determine whether a given system is practically asymptotically stabilizable or not. Because there are at most 2^M subsets, the computation can finish in finite time and therefore is feasible. □

Furthermore, from Lemma 3.2, we can immediately conclude that the number of subsystems in a practically asymptotically stabilizable system should be on less than $n+1$. The following theorem confirms it.

Theorem 3.4.

(a). *If an integrator switched system (3.1) in $\mathbf{R}^n$ is practically asymptotically stabilizable, then there are at least $n+1$ subsystems.*
(b). *Moreover, for every n, there exists an integrator switched system consisting of $n+1$ subsystems which is practically asymptotically stabilizable.*

Proof: See Appendix A. □

The case of $n+1$ subsystems that form a practically asymptotically stabilizable system is important, because in many practically stabilizable systems such $n+1$ subsystems do exist. The following three corollaries related to such case can be inferred from the above theorems and lemma.

Corollary 3.5. *An integrator switched system (3.1) in $\mathbf{R}^n$ consisting of $n+1$ subsystems with vector fields $a_1, \cdots, a_{n+1}$ is practically asymptotically stabilizable if and only if $\mathrm{span}\{a_1, \cdots, a_{n+1}\} = \mathbf{R}^n$ and there exist $\lambda_i > 0$, $i = 1, \cdots, n+1$ such that $\sum_{i=1}^{n+1} \lambda_i a_i = 0$.*

Proof: See Appendix A. □

Corollary 3.6. *An integrator switched system (3.1) in $\mathbf{R}^n$ with $n+1$ subsyste vector fields $a_1, \cdots, a_{n+1}$ is practically asymptotically stabilizable if and only if vectors in the set $\{a_1, \cdots, a_{n+1}\}$ are linearly independent and there exist $\lambda_i >$ $1, \cdots, n+1$ such that $\sum_{i=1}^{n+1} \lambda_i a_i = 0$.*

Proof: See Appendix A.

In many cases, even though a system has many subsystems, we can find subsystems which can be used for determining the practical asymptotic stabilizat the system. The following corollary provides a sufficient condition for doing so.

Corollary 3.7 (A Sufficient Condition). *An integrator switched system (3.1) in M ($M \geq n+1$) subsystems is practically asymptotically stabilizable if there exists of $n+1$ subsystems which, if regarded as a switched system with $n+1$ subsyst practically asymptotically stabilizable.*

Proof: See Appendix A.

3.2. **A minimum dwell time switching law.** Now we construct an easy-to-imp switching law that makes the system ϵ-practically asymptotically stable if the sy determined to be so by the conditions in Section 3.1. Note that in the proof of t part of Theorem 3.1 (see Appendix A), a valid switching law is constructed. How is not easy to implement in practice due to the need to solve the convex combina each x (see equation (A3) in Appendix A).

Here we focus on practically asymptotically stabilizable integrator switched s in $\mathbf{R}^n$ with $n+1$ subsystems and propose a valid minimum dwell time switching l mentioned in Section 3.1, the case of $n+1$ subsystems is important because in man such $n+1$ subsystems do exist. Hence the switching law proposed here can actu applied to many switched systems with more than $n+1$ subsystems. Let us first ill the idea of our switching law by the following example.

Example 3.8. Consider a switched system in $\mathbf{R}^2$ consisting of three subsystem system 1: $\dot{x} = a_1 = [1, 0.5]^T$; subsystem 2: $\dot{x} = a_2 = [-1, 1.5]^T$; subsys $\dot{x} = a_3 = [-0.5, -1]^T$ (see figure 2(a)). By Corollary 3.6, the system is prac asymptotically stabilizable.

Denote by C_1 the convex cone generated by the vectors $-a_2, -a_3$; by C_2 the cone generated by $-a_1, -a_3$; by C_3 the convex cone generated by $-a_1, -a_2$ (see 2(b)). Note that C_1, C_2, and C_3 have mutually disjoint interiors and $C_1 \cup C_2 \cup$ $\mathbf{R}^2$. For a given $\epsilon > 0$, we now propose a valid switching law to achieve ϵ- prac asymptotic stability as follows.

A minimum dwell time switching law: Let subsystem 2 be active in $\mathrm{Int}(C_1)$, s tem 3 be active in $\mathrm{Int}(C_2)$, subsystem 1 be active in $\mathrm{Int}(C_3)$. When the state is

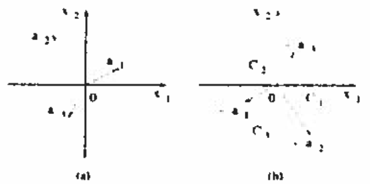

FIGURE 2. Example 3.8: (a) Vector fields a_1, a_2, a_3. (b) Convex cones C_1, C_2, C_3.

common boundary of any two convex cones, choose the active subsystem to be the one corresponding to the convex cone that the trajectory has the potential to enter next, if the system still evolves according to the current active subsystem (e.g., if x evolves in C_1 following subsystem 2 and intersects the ray in the same direction as $-a_3$, then subsystem 3 will become active). Moreover, in order to eliminate the Zenoness phenomenon near the origin, besides the above rules, we also impose a minimum dwell time τ (i.e., the minimum time duration that any subsystem must be active before the system can switch again).

The choice of a minimum dwell time τ: In general, the smaller the τ is, the smaller the ϵ can be, so that the system can be made ϵ-practically asymptotically stable. As $\tau \to 0$, we find that ϵ can also go to 0. However when $\tau = 0$, Zenoness problem will occur, therefore τ cannot be infinitesimally small either. For this example in $\mathbf{R}^2$, some geometric observations suggest that we can choose a τ satisfying the following inequality

$$\tau \leq \min\left\{\frac{1}{\|a_1\|}\left(\epsilon - \frac{\delta}{\sin\theta_{12}}\right), \frac{1}{\|a_2\|}\left(\epsilon - \frac{\delta}{\sin\theta_{23}}\right), \frac{1}{\|a_3\|}\left(\epsilon - \frac{\delta}{\sin\theta_{31}}\right), \frac{\delta}{\|a_1\|}, \frac{\delta}{\|a_2\|}, \frac{\delta}{\|a_3\|}\right\}, \tag{3.2}$$

where θ_{12} is the angle extended by $-a_1$ and $-a_2$ ($0 < \theta_{12} < \pi$). Similar definitions apply for θ_{23} and θ_{31}. The δ in (3.2) corresponds to the δ in Definition 2.5 and can be chosen to be a value that satisfies

$$\delta < \min\{\epsilon \sin\theta_{12}, \epsilon \sin\theta_{23}, \epsilon \sin\theta_{31}\}. \tag{3.3}$$

The details of the derivation of (3.2) and (3.3) are given in Appendix B.

Equipped with the switching law and (3.2), we return to our example. We choose $\epsilon = 0.3$, and $\delta = 0.1$ which satisfies (3.3) for this example, it can then be determined from (3.2) that $\tau \leq 0.0555$ will lead to a valid switching law that makes the system ϵ-practically asymptotically stable. Figure 3 shows a trajectory starting from $[1, 1]^T$ with $\tau = 0.05$. Figure 4 shows $x_1(t)$ and $x_2(t)$. Note that when time becomes large, the maximum deviation from 0 is -0.1165 for x_1, and 0.075 for x_2. So the state is actually within a ball with radius 0.1386 which is smaller than 0.3. The requirement is therefore satisfied. □

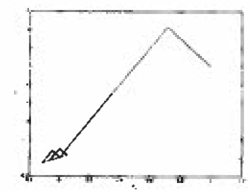

FIGURE 3. Example 3.8: A trajectory starting from $[1, 1]^T$.

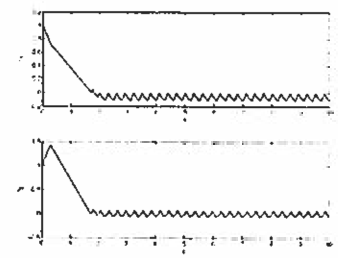

FIGURE 4. Example 3.8: $x_1(t)$ and $x_2(t)$.

The switching law proposed in Example 3.8 can be extended to the case of prac asymptotically stabilizable systems in $\mathbf{R}^n$ with $n+1$ subsystems. Denote by C_k the c cone generated by the vectors $-a_1, \cdots, -a_{k-1}, -a_{k+1}, \cdots, -a_{n+1}$ (if $k = n+1$ then subsystem 1 as subsystem $k+1$) for all $1 \leq k \leq n+1$. It can be shown that $C_1, \cdots$ have mutually disjoint interiors and $C_1 \cup \cdots \cup C_{n+1} = \mathbf{R}^n$.

A minimum dwell time switching law: Let subsystem $k+1$ be active whenev state is in $\mathrm{Int}(C_k)$, $1 \leq k \leq n+1$. When the state is on the common boundary of c cones, choose the active subsystem to be the one corresponding to the convex con the trajectory has the potential to enter next, if the system still evolves accord the current active subsystem. Moreover, in order to eliminate the Zenoness pheno near the origin, besides the above rules, we also impose a minimum dwell time τ.

The choice of a minimum dwell time τ: For systems in $\mathbf{R}^n$, geometric observ as those in Example 3.8 are currently still under our research, because direct exte of results $\mathbf{R}^2$ into higher dimensional space are not readily available. However, w that given any $\epsilon > 0$, the above switching law will behave as ϵ-practically asympto stable if we choose τ small enough. In practice, we usually specify an ϵ and then t the value of τ and test the resulting trajectory until the ϵ-practically asymptotic st is achieved.

Remark 3.9. The switching law proposed above enjoys the benefit of easy implementation, as opposed to the theoretically sound valid Switching Law C in the proof of Theorem 3.1. Given ϵ, we rigorously derive bounds for τ and δ for $\mathbf{R}^2$ case (see Appendix B). Similar bounds for $\mathbf{R}^n, n \geq 3$ are not readily available and are still under our investigation. However, we point out that this does not hinder the usefulness of the above proposed switching law. In many practical problems, by adjusting τ several times, we can achieve ϵ-practically asymptotic stability (see Example 4.1 in Section 4). □

4. A three tank example

Now we apply the practical stabilization results developed in Section 3 to a chemical batch process example.

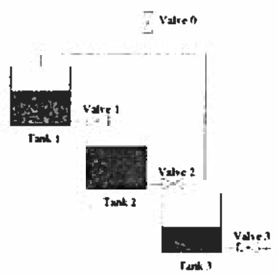

FIGURE 5. Example 4.1: The three tanks system.

Example 4.1. (A Three Tank Example) Consider the three tanks system in figure 5. All three tanks are identical and all flows cause the tank-levels to rise or decrease by 0.1 unit/sec. There are four allowable operating modes: Mode 1: Valve 0 is on, Valves 1,2,3 are off, the corresponding dynamics $\dot{x} = a_1 = [0.1, 0.1, 0.1]^T$; Mode 2: Valve 1 on, Valves 0,2,3 off, the dynamics $\dot{x} = a_2 = [-0.1, 0.1, 0]^T$; Mode 3: Valve 2 on, Valves 0,1,3 off, the dynamics $\dot{x} = a_3 = [0, -0.1, 0.1]^T$; Mode 4: Valve 3 on, Valves 0,1,2 off, the dynamics $\dot{x} = a_4 = [0, 0, -0.1]^T$. We want to develop a switching law such that the water levels in the tanks are driven toward the desired value $[80, 50, 70]^T$ and each tank level is then kept within $[-2, +2]$ range around the desired level.

By Corollary 3.6, this system with 4 subsystems in $\mathbf{R}^3$ is practically asymptotically stabilizable. For this problem, we can choose $\epsilon = 2$ and apply the switching law proposed in Section 3.2 to make the system ϵ-practically asymptotically stable around the point $[80, 50, 70]^T$ (although the point is not the origin, but with state shift, the stabilization result can be applied). We choose $\tau = 5$ sec. Figure 6 shows the three tank levels starting

from $[90, 45, 75]^T$. Note that when time becomes large, the maximum deviatic the desired point are 0.4999 for x_1, 0.9999 for x_2, and 1.4999 for x_3. They sat requirements.

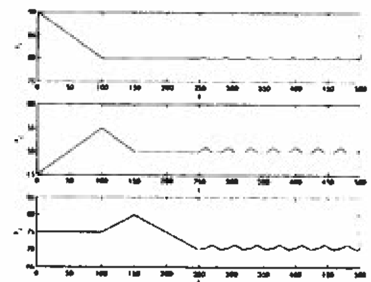

FIGURE 6. Example 4.1: The three tank levels starting from $[90, 45, 75]$

5. Conclusion

This paper reports some results for practical stabilization problems of integrator s systems. Some practical stability notions were introduced, and a necessary and s condition for practical asymptotic stabilizability of integrator switched systems w proved. Moreover, an easy-to-implement minimum dwell time switching law f tically asymptotically stabilizable systems in $\mathbf{R}^n$ consisting of $n+1$ subsyste proposed to achieve ϵ-practical asymptotic stability. The research in this paper i step toward the studies of general nonlinear subsystems. Future research inclu extensions of the results to the studies of local behaviors of switched systems wi integrator subsystems, and the estimation of bound for minimum dwell time for in $\mathbf{R}^n$.

6. Acknowledgement

The partial support of the National Science Foundation (NSF CCR01 13131) is gr acknowledged.

Appendix A. Some proofs for section 3

Proof of Theorem 3.1: "Only if" part: Assume that system (3.1) is globally pra stabilizable, but $C \neq \mathbf{R}^n$. Then there exists a vector $b \neq 0$ and $b \notin C$. It must that $0 = -b + b \notin -b + C$.

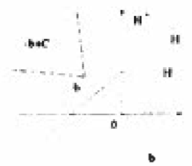

FIGURE 7. There exists a hyperplane H which strictly separates $-b+C$ and $\{0\}$.

Note that $-b+C$ is a translation of the set C. So $-b+C$ is a closed convex set because C is so. Since 0 is not in $-b+C$, there exists a hyperplane H which strictly separates $-b+C$ and $\{0\}$ (see figure 7). It can be seen that under any switching sequence, the trajectory starting from $x(0)=-b$ must be within the set $-b+C$. Therefore the trajectory cannot enter the open halfspace $\text{Int}(H^-)$ where 0 is in. Hence $\exists\epsilon>0$ such that $B(0,\epsilon)\triangleq\{x\in\mathbf{R}^n|\|x-0\|<\epsilon\}$ is all in the open halfspace $\text{Int}(H^-)$. Consequently, the trajectory starting from $x(0)=-b$ cannot enter $B(0,\epsilon)$ under any valid switching law. This leads to a contradiction because 0 should be ϵ-attractive under some switching control law due to the practical asymptotic stabilizability assumption.

"If" part: We prove this by constructing a valid switching law $\mathcal{S}=\mathcal{S}(\epsilon)$ that renders the system ϵ-practically stable and the origin ϵ-attractive, given any $\epsilon>0$.

Given an $\epsilon>0$, let us first construct a switching law for the ϵ-practical stability. In order to do so, we first claim that we can find a switching law such that $\exists G>0$ and for any initial point in $B[0,1]\triangleq\{x\in\mathbf{R}^n|\|x-0\|\le 1\}$, the trajectory satisfies $\|x(t)\|<G$ for any $t\ge 0$. This switching law is constructed as follows. First let us consider the unit vectors $e_1,\cdots,e_n$ in $\mathbf{R}^n$ and their negatives $-e_1,\cdots,-e_n$. We denote them as $\hat{e}_1=e_1$, $\cdots$, $\hat{e}_n=e_n$, $\hat{e}_{n+1}=-e_1,\cdots,\hat{e}_{2n}=-e_n$. Since $C=\mathbf{R}^n$, they have the representations

$$\text{(A1)}\qquad \hat{e}_1=\sum_{i=1}^{M}\lambda_{1,i}a_i,\cdots,\hat{e}_{2n}=\sum_{i=1}^{M}\lambda_{2n,i}a_i,$$

with $\lambda_{k,i}\ge 0$. Furthermore, we note that every vector $x\in B[0,1]$ can be represented as $x=\sum_{k=1}^{n}\alpha_k e_k$ where $\alpha_k\in\mathbf{R}$, $\sum_{k=1}^{n}\alpha_k^2\le 1$. By using the $\hat{e}_k$'s, x can be represented as

$$\text{(A2)}\qquad x=\sum_{k=1}^{2n}\beta_k\hat{e}_k.$$

where $\beta_k=\begin{cases}\alpha_k, & \text{if } \alpha_k\le 0\\ 0, & \text{if } \alpha_k>0\end{cases}$ for $1\le k\le n$ and $\beta_k=\begin{cases}-\alpha_{k-n}, & \text{if } \alpha_{k-n}>0\\ 0, & \text{if } \alpha_{k-n}\le 0\end{cases}$ for $n+1\le k\le 2n$. Note that every $\beta_k\le 0$ and $\sum_{k=1}^{2n}\beta_k^2=\sum_{k=1}^{n}\alpha_k^2\le 1$.

Substituting (A1) into (A2), we can write x as

$$\text{(A3)}\qquad x=\sum_{k=1}^{2n}\beta_k\hat{e}_k=\sum_{k=1}^{2n}\beta_k(\sum_{i=1}^{M}\lambda_{k,i}a_i)=\sum_{i=1}^{M}(\sum_{k=1}^{2n}\beta_k\lambda_{k,i})a_i=\sum_{i=1}^{M}\gamma_i a_i$$

where $\gamma_i=\sum_{k=1}^{2n}\beta_k\lambda_{k,i}\le 0$ for any $1\le i\le M$.

Based on (A3), we can construct the following switching law:

Switching Law A (for $x(0)\in B[0,1]$):

(1). Assume that the system trajectory starts from $x(0)\in B[0,1]$ at time t current state $x_{current}=x(0)$ and the current time $t_{current}=0$.

(2). Obtain the expression for the current state $x_{current}=\sum_{i=1}^{M}\gamma_i a_i$. First subsystem 1 and stay for time $|\gamma_1|$, then switch to subsystem 2 and stay fo and so on. In other words, we obtain a switching sequence $((t_{current},1),(t_{current}+|\gamma_1|,2),(t_{current}+|\gamma_1|+|\gamma_2|,3),\cdots,(t_{current}+|\gamma_1|+\cdots+|\gamma_{M-1}|,M))$ $t_{current}$ to $t_{current}+\sum_{i=1}^{M}|\gamma_i|$.

(3). At time $t_{current}+\sum_{i=1}^{M}|\gamma_i|$, the trajectory reaches 0, then we can let the sy at subsystem M for time $\frac{1}{\|a_M\|}$ until it intersects the unit sphere.

(4). Update $x_{current}$ to be the intersecting point and $t_{current}$ to be the time intersection. Repeat steps (2) and (3).

Using Switching Law A, we obtain nonZeno switching sequences for initial $x(0)$ The switching sequences are valid because $\frac{1}{\|a_M\|}>0$ and every repetition of ste (3) will require at most M switchings in a time duration $\frac{1}{\|a_M\|}+\sum_{i=1}^{M}|\gamma_i|$.

Switching Law A also generates bounded trajectories for initial $x(0)\in B[$ that a trajectory starting from $x(0)$ will take time $\sum_{i=1}^{M}|\gamma_i|$ to reach the orig (2). For any $0\le t\le\sum_{i=1}^{M}|\gamma_i|$ during this time period, we must have

$$\begin{aligned}\|x(t)\| &\le \|x(0)\|+\sum_{i=1}^{M}|\gamma_i|\cdot\max_{1\le i\le M}(\|a_i\|)\\ &\le 1+\sum_{i=1}^{M}\left((\sum_{k=1}^{2n}\beta_k^2)^{\frac{1}{2}}(\sum_{k=1}^{2n}\lambda_{k,i}^2)^{\frac{1}{2}}\right)\cdot\max_{1\le i\le M}(\|a_i\|)\\ &\le 1+M\max_{1\le i\le M}(\sum_{k=1}^{2n}\lambda_{k,i}^2)^{\frac{1}{2}}\cdot\max_{1\le i\le M}(\|a_i\|)\end{aligned}$$

Define $G\triangleq 2+M\max_{1\le i\le M}(\sum_{k=1}^{2n}\lambda_{k,i}^2)^{\frac{1}{2}}\cdot\max_{1\le i\le M}(\|a_i\|)$, we will have $\|x(t$ any $t\ge 0$. It follows that for any $\epsilon>0$, if we choose $\delta=\frac{\epsilon}{G}$, we can design a law similar to Switching Law A except for the scaling to the points starting We call such a switching law **Switching Law B**. Under Switching Law B, stability of the system can be achieved.

Next let us consider the ϵ-attractiveness of the origin. Starting from any initial point $x(0) \in \mathbf{R}^n$, since $C = \mathbf{R}^n$, we have $-x(0) = \sum_{i=1}^{M} \lambda_i a_i$, $\lambda_i \geq 0$. Hence $x(0) = \sum_{i=1}^{M}(-\lambda_i)a_i$. We modify Switching Law B as follows. If $x(0) \in B[0,\delta]$, apply Switching Law B for $x(0) \in B[0,\delta]$ as mentioned above. If $x(0) \notin B[0,\delta]$, we can first choose the switching sequence as

$$((0,1),(\lambda_1,2),(\lambda_1+\lambda_2,3),\cdots,(\lambda_1+\cdots+\lambda_{M-1},M))$$

until the trajectory reaches 0 at time $\sum_{i=1}^{M} \lambda_i$ and then follow Switching Law B. Hence the trajectory will always be inside the ball $B(0,\epsilon)$ after $T = \sum_{i=1}^{M} \lambda_i$. Such a modification provides us with a new switching law. We call it **Switching Law C**.

Clearly from the above constructions, if we choose $S = S(\epsilon)$ to be Switching Law C, S renders the system ϵ-practical stable and the origin ϵ-attractive. □

Proof of Lemma 3.2: "If" part: Assume that a subset $\{a_{i_1},\cdots,a_{i_l}\}$ exists such that conditions (a) and (b) are satisfied. Without loss of generality, assume it is $\{a_1,\cdots,a_l\}$.

For any $x \in \mathbf{R}^n$, from (a), there exist α_j's, $j = 1,\cdots,l$ such that

$$x = \sum_{j=1}^{l} \alpha_j a_j. \tag{A4}$$

From (b), we have

$$\sum_{j=1}^{l} \lambda_j a_j = 0,\ \lambda_j > 0,\ j = 1,\cdots,l. \tag{A5}$$

Multiplying (A5) by a constant $c > 0$ and then adding to (A4), we obtain

$$x = \sum_{j=1}^{l} (\alpha_j + c\lambda_j)a_j. \tag{A6}$$

Since $\lambda_j > 0$, we can have $\alpha_j + c\lambda_j \geq 0$ for all $1 \leq j \leq l$ if we choose c to be large enough. Now define

$$\hat{\lambda}_i = \begin{cases} \alpha_i + c\lambda_i, & i = 1,\cdots,l, \\ 0, & i = l+1,\cdots,M. \end{cases} \tag{A7}$$

x can then be expressed as $\sum_{i=1}^{M} \hat{\lambda}_i a_i$. Consequently, this shows $x \in C$ for any $x \in \mathbf{R}^n$.

"Only if" part: Consider the unit vectors $\hat{e}_1,\cdots,\hat{e}_{2n}$ defined in the proof of the 'If' part of Theorem 3.1 and their representations (A1). Define A_1 to be the set of all a_i's for which the corresponding $\lambda_{1,i} > 0$ in the expression $\hat{e}_1 = \sum_{i=1}^{M} \lambda_{1,i} a_i$. Similarly define A_2 to be the set of all a_i's for which the corresponding $\lambda_{2,i} > 0$ in the expression $\hat{e}_2 = \sum_{i=1}^{M} \lambda_{2,i} a_i$, and so on. In this way, we can define the subsets $A_1, A_2,\cdots,A_{2n}$ corresponding to the expressions of $\hat{e}_1, \hat{e}_2,\cdots,\hat{e}_{2n}$. Now if we define the subset $A = \cup_{k=1}^{2n} A_k$, we claim that A satisfies conditions (a) and (b). The reasons are as follows.

For (a), since any $x \in \mathbf{R}^n$ can be represented as a linear combination of $\hat{e}$ and every $\hat{e}_k$ can be represented as a linear combination of the vectors in A, we that x can be represented as a linear combination of the vectors in A. Hence true.

For (b), assume that $A = \{a_{i_1},\cdots,a_{i_l}\}$. Now consider $\sum_{k=1}^{2n} \hat{e}_k$, by substi expressions $\hat{e}_1 = \sum_{i=1}^{M} \lambda_{1,i} a_i,\cdots,\hat{e}_{2n} = \sum_{i=1}^{M} \lambda_{2n,i} a_i$ into it, we conclude that $\sum_{j=1}^{l} \hat{\lambda}_j a_{i_j}$. Note $\hat{\lambda}_j > 0$, $j = 1,\cdots,l$, because for each a_{i_j} there must be at $\hat{e}_k$ in the expression of which $\lambda_{k,i_j} > 0$. On the other hand, we note that $\sum_{k=1}^{n} e_k - \sum_{k=1}^{n} e_k = 0$. Therefore, (b) holds true.

Proof of Theorem 3.4: (a). If system (3.1) is practically asymptotically st then by Theorem 3.1, we have $C = \mathbf{R}^n$. By Lemma 3.2, there exists a subset $\{a_{i_1}$ of $\{a_1,\cdots,a_M\}$ whose span is $\mathbf{R}^n$, and therefore $l \geq n$. However, if $l = n$ solution to $\sum_{j=1}^{l} \lambda_j a_{i_j} = 0$ is $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$ since a_{i_j}'s must l independent in this case. This is a contradiction to condition (b) in Lemma $l > n$ and consequently we conclude that $M \geq l \geq n+1$.

(b). We can construct an integrator switched system consisting of $n+1$ s which is practically asymptotically stabilizable. Assume that $a_1,\cdots,a_n$ a independent vectors in $\mathbf{R}^n$, let

$$a_{n+1} = -\sum_{i=1}^{n} \lambda_i a_i,\ \lambda_i > 0. \tag{A8}$$

It is then not difficult to see that the convex cone formed by $\{a_1, a_2,\cdots,a_n,$ $C = \{\sum_{i=1}^{n+1} \lambda_i a_i | \lambda_i \geq 0\}$ satisfies $C = \mathbf{R}^n$.

Proof of Corollary 3.5: From the proof of part (a) of Theorem 3.4, we con the only subset satisfying the conditions of Lemma 3.2 is the set $\{a_1,\cdots,a_n$ case.

Proof of Corollary 3.6: It is not difficult to see that the condition in Corol equivalent to the condition in Corollary 3.5.

Proof of Corollary 3.7: Assume without loss of generality that the $\{a_1,\cdots,a_{n+1}\}$, then we only need to switch among these $n+1$ subsystem to practically asymptotically stabilize the system. The practical asymptotic st ity of the system consisting of these $n+1$ subsystems can therefore lead to th asymptotic stabilizability of the original system.

Appendix B. Some geometric observations for choosing τ in exam

Given an $\epsilon > 0$, we can choose τ based on the following reasonings.

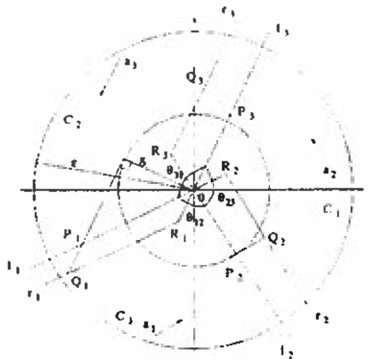

FIGURE 8. Geometric observations for choosing τ for Example 3.8.

First of all, let us consider ϵ-practical stability. From Definition 2.5, we need to have a ball $B[0,\delta]$ such that any trajectory starting in this ball remains in $B(0,\epsilon)$. Figure 8 depicts the two balls. l_i's are the rays corresponding to $-a_i$'s. Assume subsystem 3 is active and $x \in B[0,\delta] \cap C_2$. Also assume the points P_1, Q_1 are on the line tangent to $B[0,\delta]$ and parallel to l_3. P_1 is on l_1, and $|P_1Q_1| = \|a_3\|\tau$. Moreover, assume the points 0, P_1, Q_1, and R_1 form a parallelogram. Now we note that for any point in $B[0,\delta] \cap C_2$, when subsystem 3 is active and the system follows the minimum dwell time switching law proposed in Example 3.8, the trajectory will either intersect l_1 and switch to subsystem 1 immediately (when the time elapsed is no less than τ), or it will enter $0P_1Q_1R_1$ and then switch to subsystem 1 (when time elapsed is equal to τ). The importance of $0P_1Q_1R_1$ lies in the fact that for trajectories starting from $B[0,\delta] \cap C_2$ and following subsystem 3, all trajectories will switch to subsystem 1 in $0P_1Q_1R_1$. As long as the line segment $0R_1$ is in $B[0,\delta]$, by following subsystem 1, the trajectory will eventually intersect OR_1 and hence be in $B[0,\delta] \cap C_3$. Similar arguments can be applied to show that the trajectories starting in $B[0,\delta] \cap C_3$ will switch in the parallelogram $0P_2Q_2R_2$ and then enter into $B[0,\delta] \cap C_1$; and the trajectories starting in $B[0,\delta] \cap C_1$ will switch in the parallelogram $0P_3Q_3R_3$ and then enter into $B[0,\delta] \cap C_2$. Now in order to achieve ϵ-practical stability, a sufficient condition is to require that the farthest point Q_1 of the parallelogram $0P_1Q_1R_1$ be inside $B(0,\epsilon)$. A sufficient condition for this is $|0P_1| + |P_1Q_1| \le \epsilon$ which can also be written as

(B1) $$\|a_3\|\tau + \frac{\delta}{\sin\theta_{31}} \le \epsilon.$$

where θ_{31} is the angle extended by l_3 and l_1 ($0 < \theta_{31} < \pi$). Also note from discussion, we require that $0R_1$ be in $B[0,\delta]$, which is equivalent to

(B2) $$\|a_3\|\tau \le \delta.$$

Similarly, we can obtain the inequalities

(B3) $$\|a_1\|\tau + \frac{\delta}{\sin\theta_{12}} \le \epsilon,\ \|a_1\|\tau \le \delta,\ \|a_2\|\tau + \frac{\delta}{\sin\theta_{23}} \le \epsilon,\ \|a_2\|\tau \le \delta.$$

Combining all the inequalities in (B1)-(B3), we find that if we choose

(B4) $$\tau \le \min\left\{\frac{1}{\|a_1\|}(\epsilon - \frac{\delta}{\sin\theta_{12}}),\ \frac{1}{\|a_2\|}(\epsilon - \frac{\delta}{\sin\theta_{23}}),\ \frac{1}{\|a_3\|}(\epsilon - \frac{\delta}{\sin\theta_{31}}),\ \frac{\delta}{\|a_1\|},\ \frac{\delta}{\|a_2}\right.$$

then the switching law with τ satisfying (B4) will lead to ϵ-practical stability (B4) corresponds to the δ in Definition 2.5 and can be chosen by the designe it must satisfy the following condition

(B5) $$\delta < \min\{\epsilon\sin\theta_{12}, \epsilon\sin\theta_{23}, \epsilon\sin\theta_{31}\},$$

so that the τ can take positive value in the inequalities $\|a_3\|\tau + \frac{\delta}{\sin\theta_{31}} \le \epsilon$, $\|a_1\|$ ϵ, and $\|a_2\|\tau + \frac{\delta}{\sin\theta_{23}} \le \epsilon$. Besides the constraint (B5), we can freely choose δ different bounds for τ.

We claim that the switching law in Example 3.8 with τ satisfying (B4) al ϵ-attractiveness. This is because any trajectory starting in C_2 following subsy enter into the band formed by l_1, $0R_1$, and the ray r_1 which emits from R_1 an direction of R_1Q_1 (see figure 8) and then switch to subsystem 1. Therefore switching from subsystem 3 to 1, all trajectories starting in C_2 can then int which is in $B[0,\delta]$. Then by the above arguments for ϵ-practical stability, the will always be in $B(0,\epsilon)$.

References

[1] M.S. Branicky. Multiple Lyapunov functions and other analysis too switched and hybrid systems. *IEEE Transactions on Automatic C* 43(4):475–482, April 1998.

[2] R.A. DeCarlo, M.S. Branicky, S. Pettersson, and B. Lennartson. Perspe and results on the stability and stabilizability of hybrid systems. *Proce of the IEEE*, 88(7):1069–1082.

[3] J.P. Hespanha. *Logic-Based Switching Algorithms in Control*. PhD thesis University, 1998.

[4] V. Lakshmikantham, S. Leela, and A.A. Martynyuk. *Practical Stability o linear Systems*. World Scientific, 1990.

[5] J.P. LaSalle and S. Lefschetz. *Stability by Lyapunov's Direct Method wi plications*. Academic Press, New York, 1961.

[6] B. Lennartson, M. Tittus, B. Egardt, and S. Pettersson. Hybrid systems in process control. *IEEE Control Systems Magazine*, 16(5):45–56, October 1996.

[7] D. Liberzon and A.S. Morse. Basic problems in stability and design of switched systems. *IEEE Control Systems Magazine*, 19(5):59–70, October 1999.

[8] D. Liberzon. *Switching in Systems and Control*. Birkhäuser, 2003.

[9] H. Lin and P.J. Antsaklis. Uniformly ultimate boundedness control for uncertain switched linear systems. ISIS Technical Report, ISIS-2003-004, University of Notre Dame, August 2003, available at: `http://www.nd.edu/~isis/tech.html`

[10] M. Tittus and B. Egardt. Control design for integrator hybrid systems. *IEEE Transactions on Automatic Control*, 43(4):491–500, April 1998.

[11] R.T. Rockafellar. *Convex Analysis*. Princeton University Press, 1970.

[12] X. Xu and P.J. Antsaklis. Stabilization of second order LTI switched systems. *International Journal of Control*, 73(14):1261–1279, 2000.

[13] G. Zhai and A.N. Michel. On practical stability of switched systems. *International Journal of Hybrid Systems*, 2(1–2):141–154, 2002.

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Delay–Dependent Static Output Feedbacl Stabilization for Singular Linear Systems

E. K. Boukas[1]

[1]Mechanical Engineering Department, École Polytechnique de Montréal P.O. Box 6079, Station Centre-ville, Montréal (Québec) Canada, H3C 3A7

ABSTRACT: This paper deals with the class of continuous-time singular linear : tems with time delay in the state vector. Delay-independent and delay depenc sufficient conditions on static output feedback stabilization are developed. A de algorithm for a memoryless static output feedback controllers which guarantee that closed-loop dynamics will be regular, impulse free and stable is proposed in term the solutions to linear matrix inequalities (LMIs). Two numerical examples are gi to show the effectiveness of the developed results.

Keywords. Singular systems, Continuous-time linear systems, Linear matrix equality, Stability, Stabilizability, Static output feedback controller.

1. Introduction

The class of singular continuous-time linear systems is an important class (that has attracted a lot researchers from mathematics and control communities systems are also referred to as descriptor systems, implicit systems, generali space systems, differential-algebraic systems or semi-state systems [4, 9]. It v in many studies that the class of singular systems is more appropriate to de behavior of some practical systems in different fields ranging from chemical pr robotics (see [4] and some references therein). Many problems for this class (either in the continuous-time and discrete-time have been tackled and interesti have been reported in the literature. Among these contributions we quot [13, 19, 17, 5, 14, 15, 16, 12, 7, 8, 10, 11, 3], and the references therein.

Some practical systems that can be modelled by the class of singular sys we are considering here may have time-delay in their dynamics which may be of instability and performance degradation of such systems (see [2]). Therel attention should be paid to these class of systems. To the best of our know class of continuous-time singular systems with time delays has not yet been fi tigated. Particularly delay-dependent sufficient conditions for stabilization are not existing in the literature.

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