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Robust Stability and Disturbance Attenuation Analysis of a Class of Networked Control Systems

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Abstract— **In this paper, stability and disturbance attenuation issues for a class of Networked Control Systems (NCSs) under uncertain access delay and packet dropout effects are considered. Our aim is to find conditions on the delay and packet dropout rate, under which the system stability and $\mathcal{H}^\infty$ disturbance attenuation properties are preserved to a desired level. The basic idea in this paper is to formulate such Networked Control System as a discrete-time switched system. Then the NCSs' stability and performance problems can be reduced to corresponding problems for the switched systems, which have been studied for decades and for which a number of results are available in the literature. The techniques in this paper are based on recent progress in the discrete-time switched systems and piecewise Lyapunov functions.**

I. INTRODUCTION

By Networked Control Systems (NCSs), we mean feedback control systems where networks, typically digital band-limited serial communication channels, are used for the connections between spatially distributed system components like sensors and actuators to controllers. These channels may be shared by other feedback control loops. In traditional feedback control systems these connections are through point-to-point cables. Compared with the point-to-point cables, the introduction of serial communication networks has many advantages, such as high system testability and resource utilization, as well as low weight, space, power and wiring requirements [9], [14]. These advantages make the networks connecting sensors/actuators to controllers more and more popular in many applications, including traffic control, satellite clusters, mobile robotics etc. Recently modeling, analysis and control of networked control systems with limited communication capability has emerged as a topic of significant interest to control community, see for example [4], [3], [6], [14], [1], [9].

Time delay typically has negative effects on the Networked Control Systems' stability and performance. There are several situations where time delay may arise. First, transmission delay is caused by the limited bit rate of the communication channels. Secondly, the channel in NCSs is usually shared by multiple sources of data, and the channel is usually multiplexed by time-division method. Therefore, there are delays caused by a node waiting to send out a message through a busy channel, which is usually called accessing delay and serves as the main source of delays in NCSs. There are also some delays caused by processing and propagation, which are usually negligible for NCSs. Another interesting problem in NCSs is the packet dropout issue. Because of the uncertainties and noise in the communication channels, there may exist unavoidable errors in the transmitted packet or even loss[1]. If this happens, the corrupted packet is dropped and the receiver (controller or actuator) uses the packet that it received most recently. In addition, packet dropout may occur when one packet, say sampled values from the sensor, reaches the destination later than its successors. In such situation, the old packet is dropped, and its successive packet is used instead. There is another important issue in NCSs, that is the quantization effect. With finite bit-rate constraints, quantization has to be taken into consideration in NCSs. Therefore, quantization and limited bit rate issues have attracted many researchers' attention, see for example [3], [6], [4]. It has been known [3], [6] that an exponential data representation scheme is most desirable under certain conditions.

In this paper, we consider uncertain time delay and packet dropout issues of NCSs in the framework of switched systems. The strength of this approach comes from the solid theoretic results existing in the literature for stability, robust performance etc. for switched systems. By a switched system, we mean a hybrid dynamical system consisting of a finite number of subsystems described by differential or difference equations and a logical rule that orchestrates switching between these subsystems. Properties of this type of model have been studied for the past fifty years to consider engineering systems that contain relays and/or hysteresis. Recently, there has been increasing interest in the stability analysis and switching control design of switched systems (see, e.g., [10], [5], [11], [12] and the references cited therein). Notice that hybrid/switched system research has provided useful results and promising techniques to deal with NCSs with delay and packet dropout effects. For example, in [14], the multiple Lyapunov function method [2] was employed to analyze the stability of NCSs with network-induced delay. In addition, the packet dropout effects in NCSs are closely related to the controller failures studied in the fault tolerance literature. There are some recent work on the controller failure analysis based on switched system techniques [11], [13]. By using a piecewise Lyapunov function, the author in [13] showed that if the controller failures did not occur too frequently or last

[1]Error control coding and/or Automatic Repeat reQuest (ARQ) mechanism may be employed, but the possibility of error occurring still exists.

too long, then global exponential stability and disturbance attenuation property of the system were guaranteed to be preserved.

In this paper, we investigate the robust stability analysis and disturbance attenuation problem for a class of Networked Control Systems (NCSs) under uncertain access delay and packet dropout effects. Our aim is to find conditions concerning the delay and packet dropout rate, under which the system stability and $\mathcal{H}^{\infty}$ disturbance attenuation properties are preserved to a desired level. We first analyze the nature of the uncertain access delay and packet dropout effects on NCSs in Section II. Then in Section III, we model the NCS as a discrete-time switched system. Therefore the NCSs' robust stability and performance problems can be boiled down to the stability analysis and disturbance attenuation problems of switched systems. In Section IV, the robust stability problem for such NCSs with uncertain access delay and packet dropout effects is studied, and disturbance attenuation properties for such NCSs are studied in Section V. The techniques employed in this paper are based on recent progress in the continuous-time and discrete-time switched systems [12], [11]. Finally, concluding remarks are presented.

II. ACCESS DELAY AND PACKET DROPOUT

We assume that the communication scheme has the following properties. For the network link layer, the delays caused by processing and propagation are ignored, and we only consider the access delay which serves as the main source of delays in NCSs. Dependent on data traffic, the communication bus is either busy or idle (available). If available, communication between sender and receiver is instantaneous. Errors may occur during the communication and destroy the packet, and this is considered as a packet dropout.

The model of the NCS discussed in this paper is shown in Figure 1. For simplicity, but without loss of generality, we may combine all the time delay and packet dropout effects into the sensor to controller path and assume that the controller-actuator communicates ideally.

We assume that the plant can be modeled as a continuous-time linear time-invariant system described by

$$\begin{cases} \dot{x}(t) = A^c x(t) + B^c u(t) + E^c d(t) \\ z(t) = C^c x(t) \end{cases}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ is control input, and $z(t) \in \mathbb{R}^p$ is the controlled output. The disturbance input $d(t)$ is contained in $\mathcal{D} \subset \mathbb{R}^r$. $A^c \in \mathbb{R}^{n\times n}$, $B^c \in \mathbb{R}^{n\times m}$ and $E^c \in \mathbb{R}^{n\times r}$ are constant matrices related to the system state, and $C^c \in \mathbb{R}^{p\times n}$ is the output matrix.

For the above NCS, it is assumed that the plant output node (sensor) is clock-driven. In other words, after each clock cycle (sampling time T_s), the output node attempts to send a packet containing the most recent state (output) samples. If the communication bus is idle, then the packet will be transmitted to the controller. Else if the bus is busy, then the output node will wait sometime, say $\varpi < T_s$, and try again. After several attempts or when newer sampled data becomes available, if the transmission still can not be completed, then the packet is discarded, which is also considered as a packet dropout. On the other hand, the controller and actuator are event driven and work in a simpler way. The controller, as a receiver, has a receiving buffer which contains the most recently received data packet from the sensors (the overflow of the buffer may be dealt with as packet dropouts). The controller reads the buffer periodically at a higher frequency than the sampling frequency, say every $\frac{T_s}{N}$ for some integer N large enough. Whenever there is new data in the buffer, then the controller will calculate the new control signal and transmit to the actuator. According to the assumption, the controller-actuator communicates without delay or packet dropout. Upon the new control signal arrival, the actuator will update the output of the Zero-Order-Hold (ZOH) to the new value.

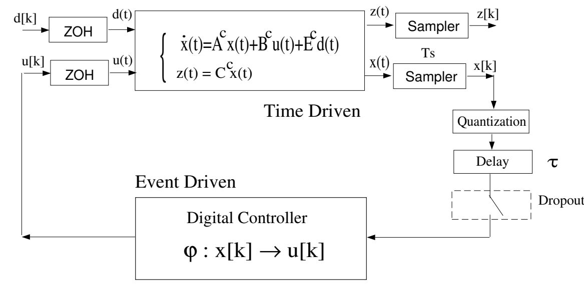

Fig. 1. The Networked Control Systems' model.

Based on the above assumptions and discussion, the time delay and packet dropout pattern can be shown in Figure 2. In this figure, The small bullet, •, stands for the packet being transmitted successfully from the sensor to the controller's receiving buffer, maybe with some delay, and being read by the controller, at some time $t = kT_s + \kappa\frac{T_s}{N}$, and the new control signal is updated in the actuator instantly. The actuator will hold this new value until next update control signal comes. The symbol, ◦, denotes the packet being dropped, due to error, bus busy, conflict or buffer overflow etc.

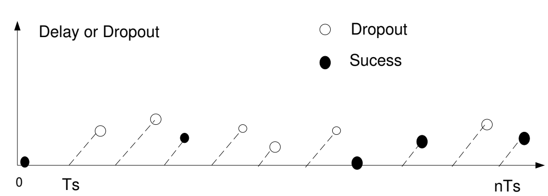

Fig. 2. The illustration of uncertain time delay and packet dropout of Networked Control Systems.

III. MODELS FOR NETWORKED CONTROL SYSTEMS

In this section, we will consider the sampled-data model of the plant. Because we do not assume the synchronization between the sampler and the digital controller, the control signal is no longer of constant value within a sampling period. Therefore the control signal within a sampling period has to be divided into subintervals corresponding to the controller's reading buffer period, $T = \frac{T_s}{N}$. Within each subinterval, the control signal is constant under the assumptions of the previous section. Hence the continuous-time plant may be discretized into the following sampled-data systems:

$$x[k+1] = Ax[k] + \underbrace{[B\ B\ \cdots B]}_{N} \begin{bmatrix} u^1[k] \\ u^2[k] \\ \vdots \\ u^N[k] \end{bmatrix} + Ed[k] \quad (1)$$

where $A = e^{A^c T_s}$, $B = \int_0^{\frac{T_s}{N}} e^{A^c \eta} B^c d\eta$ and $E = \int_0^{T_s} e^{A^c \eta} E^c d\eta$. Note that for linear time-invariant plant and constant-periodic sampling, the matrices A, B and E are constant.

During each sampling period, several different cases may arise, which can be listed as follows.

1) No delay, $\tau = 0$, $u^1[k] = u^2[k] = \cdots = u^N[k] = u[k]$, then the state transition equation (1) for this case can be written as follows.

$$\begin{aligned} x[k+1] &= Ax[k] + [B\ B\ \cdots B] \begin{bmatrix} u[k] \\ u[k] \\ \vdots \\ u[k] \end{bmatrix} \\ &\quad + Ed[k] \\ &= Ax[k] + N \cdot Bu[k] + Ed[k] \end{aligned}$$

2) Delay $\tau = \kappa \times \frac{T_s}{N}$, where $\kappa = 1, 2, \cdots, D_{max}$. For this case $u^1[k] = u^2[k] = \cdots = u^\kappa[k] = u[k-1]$, $u^{\kappa+1}[k] = u^{\kappa+2}[k] = \cdots = u^N[k] = u[k]$, and Equation (1) can be written as:

$$\begin{aligned} x[k+1] &= Ax[k] + [B\ B\ \cdots B] \begin{bmatrix} u[k-1] \\ \vdots \\ u[k-1] \\ u[k] \\ \vdots \\ u[k] \end{bmatrix} + Ed[k] \\ &= Ax[k] + \kappa \cdot Bu[k-1] + (N-\kappa) \cdot Bu[k] + Ed[k] \end{aligned}$$

Let us assume that the controller uses just the time-invariant linear feedback control law, $u[k] = Kx[k]$, which may be obtained as the solution of a LQR problem without considering the network induced effects. Then, we may plug in the $u[k] = Kx[k]$ and get

$$\begin{aligned} & x[k+1] \\ &= Ax[k] + \kappa BKx[k-1] + (N-\kappa)BKx[k] + Ed[k] \\ &= [A + (N-\kappa)BK]x[k] + \kappa BKx[k-1] + Ed[k] \end{aligned}$$

If we let $\hat{x}[k] = \begin{bmatrix} x[k-1] \\ x[k] \end{bmatrix}$, then the above equations can be written as:

$$\begin{aligned} \hat{x}[k+1] &= \begin{bmatrix} x[k] \\ x[k+1] \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \\ \kappa BK & A + (N-\kappa)BK \end{bmatrix} \begin{bmatrix} x[k-1] \\ x[k] \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ E \end{bmatrix} d[k] \end{aligned}$$

where $\kappa = 0, 1, 2, \cdots, D_{max}$. Note that $\kappa = 0$ implies $\tau = 0$, which corresponds to the previous "no delay" case. And the controlled output $z[k]$ is given by

$$z[k] = \begin{bmatrix} 0 & C \end{bmatrix} \hat{x}[k]$$

where $C = C^c$.

3) If packet dropout happens, due to corrupted packet or delay $\tau > D_{max} \times \frac{T_s}{N}$, then the actuator will implement the previous control signal, i.e. $u^1[k] = u^2[k] = \cdots = u^N[k] = u[k-1]$, then the state transition equation (1) for this case can be written as follows.

$$\begin{aligned} x[k+1] &= Ax[k] + [B\ B\ \cdots B] \begin{bmatrix} u[k-1] \\ u[k-1] \\ \vdots \\ u[k-1] \end{bmatrix} + Ed[k] \\ &= Ax[k] + N \cdot Bu[k-1] + Ed[k] \end{aligned}$$

Using the same variable transformation as in the above case, we get

$$\begin{aligned} \hat{x}[k+1] &= \begin{bmatrix} x[k] \\ x[k+1] \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \\ NBK & A \end{bmatrix} \begin{bmatrix} x[k-1] \\ x[k] \end{bmatrix} + \begin{bmatrix} 0 \\ E \end{bmatrix} d[k] \end{aligned}$$

The controlled output $z[k]$ is given by

$$z[k] = \begin{bmatrix} 0 & C \end{bmatrix} \hat{x}[k]$$

where $C = C^c$.

In the next section, we will formulate the above NCSs as a class of discrete-time switched systems.

IV. STABILITY ANALYSIS

Motivated by the above analysis of NCSs, we introduce a family of discrete-time linear systems described by the following difference equations.

$$x[k+1] = A_q x[k] + E_q d[k], \quad k \in \mathbb{Z}^+ \quad (2)$$

where $x[k] \in \mathbb{R}^n$ is the state variable, and the disturbance input $d[k]$ is contained in $\mathcal{D} \subset \mathbb{R}^r$. $A_q \in \mathbb{R}^{n \times n}$ and $E_q \in$

$\mathbb{R}^{n\times r}$ are constant matrices indexed by $q \in Q$, where the finite set $Q = \{q_1, q_2, \cdots, q_n\}$ is called the set of *modes*.

Combine the family of discrete-time uncertain linear systems (2) with a class of piecewise constant functions of time $\sigma : \mathbb{Z}^+ \to Q$. Then we can define the following linear time-varying system as a discrete-time switched linear system

$$x[k+1] = A_{\sigma[k]}x[k] + E_{\sigma[k]}d[k], \quad k \in \mathbb{Z}^+ \tag{3}$$

The signal $\sigma[k]$ is called a *switching signal*.

Associated with the switched system (3), a controlled output $z[k]$ is considered.

$$z[k] = C_{\sigma[k]}x[k]$$

where $C_{\sigma[k]} \in \mathbb{R}^{p\times n}$ and $z[k] \in \mathbb{R}^p$.

For the NCS we considered in this paper, we may formulate it as a switched system with $D_{max} + 2$ different modes, which can be expressed as follows.

$$\left\{ \begin{array}{rcl} \hat{x}[k+1] & = & A_\kappa \hat{x}[k] + E_\kappa d[k] \\ z[k] & = & C_\kappa \hat{x}[k] \end{array} \right. \tag{4}$$

where $A_\kappa = \begin{bmatrix} 0 & I \\ \kappa BK & A + (N-\kappa)BK \end{bmatrix}$, $E_\kappa = \begin{bmatrix} 0 \\ E \end{bmatrix}$ and $C_\kappa = \begin{bmatrix} 0 & C \end{bmatrix}$ for $\kappa = 0, 1, 2, \cdots D_{max}, N$. And the set of modes Q is given by $Q = \{0, 1, 2, \cdots D_{max}, N\}$. Note that $\kappa = 0$ implies $\tau = 0$, which corresponds to the "no delay" case, while $\kappa = N$ corresponds to the "packet dropout" case.

It is reasonable to assume that, for the cases of no delay ($\kappa = 0$) or small delay ($\kappa \le \kappa_0$), the corresponding state matrix A_κ's are Schur stable, while, for the cases of large delay ($\kappa > \kappa_0$) or packet dropout ($\kappa = N$), the A_κ's are not Schur stable. Therefore, in this paper it is assumed that the first r, corresponding to κ_0, of all the $D_{max} + 2$ matrices in $\{A_\kappa\}$ are Schur stable, while the rest matrices are not Schur stable, where $r \le D_{max} + 2$ and $\kappa \in Q = \{0, 1, 2, \cdots, D_{max}, N\}$. In the sequel, for simplicity of notation, we will index the switched NCS model with i, for $i \in Q = \{0, 1, 2, \cdots, D_{max}, N\}$. In this section, we set $d[k] = 0$ in (4) for the purpose to study its stability.

It is known that for Schur stable systems $x[k+1] = A_i x[k]$, there always exist positive scalars $\lambda_1 < 1$ and h_i's, $i \le r$ such that $\|A_i^k\| \le h_i \lambda_1^k$ for any $k \ge 1$[2]. Note that for any Schur unstable system $x[k+1] = A_i x[k]$ ($i > r$), there always exist a constant $0 < \sigma < 1$ making the system $x[k+1] = \sigma A_i x[k]$ Schur stable. Hence we may assume that there exist positive scalars $\lambda_2 \ge 1$ and h_i's, $i > r$ such that $\|A_i^k\| \le h_i \lambda_2^k$ for any $k \ge 1$. Therefore, we get

$$\|A_i^k\| \le \left\{ \begin{array}{ll} h_i\lambda_1^k & i \le r \\ h_i\lambda_2^k & i > r \end{array} \right. \tag{5}$$

Following [12], we introduce the notations as below. Denote $h = \max_i\{h_i\}$. For any switching signal $\sigma(k)$ and any $k_2 > k_1 > 0$, let $N_\sigma(k_1, k_2)$ denote the number of switchings of $\sigma(k)$ on the interval $[k_1, k_2)$. Let $K_i(k_1, k_2)$ denote the total period that the i-th subsystem is activated during $[k_1, k_2)$. Define $K^-(k_1, k_2) = \sum_{i\le r,\ i\in Q} K_i(k_1, k_2)$, which stands for the total activation period of the Schur stable subsystems. On the other hand, $K^+(k_1, k_2) = \sum_{i>r,\ i\in Q} K_i(k_1, k_2)$ denotes the total activation period of the Schur unstable subsystems. We have $K^-(k_1, k_2) + K^+(k_1, k_2) = k_2 - k_1$.

For given $N_0 \ge 0$, τ_a, let $\mathcal{S}_a(\tau_a)$ denote the set of all switching signals satisfying

$$N_\sigma(0, k) \le N_0 + \frac{k}{\tau_a} \tag{6}$$

where the constant τ_a is called the *average dwell time* and N_0 the *chatter bound*. The idea is that there may exist consecutive switching separated by less than τ_a, but the average time interval between consecutive switchings is not less than τ_a. Note that the concept of average dwell time between subsystems was originally proposed for continuous-time switched systems in [8]. With these assumptions and notations, we may apply the techniques and results developed in [12] to the NCSs and get the following theorem for globally exponential stability. The proof of the theorem is not difficult by using the technique of Theorem 3 in [12], and thus is omitted here.

Theorem 1: For any given $\lambda \in (\lambda_1, 1)$, the NCS (4) is globally exponentially stable with stability degree λ if there exists a finite constant τ_a^* and $\lambda^* \in (\lambda_1, \lambda)$ such that the $K^+(0, k)$ and $N_\sigma(0, k)$ satisfy the following two conditions

1) $\inf_{k>0} \frac{K^-(0,k)}{K^+(0,k)} \ge \frac{\ln\lambda_2 - \ln\lambda^*}{\ln\lambda^* - \ln\lambda_1}$ holds for some scalar $\lambda^* \in (\lambda_1, \lambda)$;
2) The average dwell time is not smaller than τ_a^*, i.e. $N_\sigma(0, k) \le N_0 + \frac{k}{\tau_a^*}$, where $\tau_a^* = \frac{\ln h}{\ln\lambda - \ln\lambda^*}$, and N_0 may be specified arbitrarily.

Remark 1: The first condition implies that if we expect the entire system to have decay rate λ, we should restrict the total number of lost packets and large delay packets in the sense that on average $K^+(0, k)$ has an upper-bound, $K^+(0, k) \le \frac{\ln\lambda^* - \ln\lambda_1}{\ln\lambda_2 - \ln\lambda_1}k$.

Remark 2: The main point of the second condition can be described as follows. Although the first condition may be satisfied, which means that on average the packet lost is limited and the total number of large delayed packet is bounded, in the worst case the packet dropout and large access delay happen in a burst fashion. For such worst case, the NCSs may fail to achieve the decay rate. The second condition restricts the frequency of the packet dropout and large delayed packet, and to make sure the above worst case can not happen.

Remark 3: The above theorem says that the NCSs' stability, with most of the packets arriving in a timely fashion, does not degenerate seriously, which is reasonable.

[2]The vector/matrix norm considered here is the l^2 norm and its induced matrix norm.

V. DISTURBANCE ATTENUATION ANALYSIS

In this section, we will study the disturbance attenuation property for the NCSs (4). Note that the $\mathcal{L}_2$ gain property of discrete-time switched systems was studied in [12] under the assumption that all subsystems were Schur stable. In this section, we will extend the $\mathcal{L}_2$ gain property of discrete-time switched system to the case that not all subsystems are Schur stable. The techniques used in this section are similar to those in [11] for continuous-time switched systems.

Following the assumptions in [12], the initial state is assumed to be the origin, $x[0] = 0$. And we assume that the Schur stable subsystems achieve an $\mathcal{L}_2$ gain smaller than γ_0. It is known that there exist a positive scalar $\lambda_- < 1$ and a set of positive definite matrices P_i, for $i \leq r$ and $i \in Q$, such that

$$A_i^T P_i A_i - \lambda_-^2 P_i + C_i^T C_i + A_i^T P_i E_i(\gamma_0^2 I - E_i^T P_i E_i)^{-1} E_i^T P_i A_i < 0$$

holds [7]. Observing that for Schur unstable subsystems, there always exist a constant $0 < \sigma < 1$, such that the subsystems $(\sigma A_i, E_i, \sigma C_i)$ can achieve the $\mathcal{L}_2$ gain level γ_0. Therefore, we assume that for Schur unstable subsystems there exist a positive scalar $\lambda_+ \geq 1$ and a set of positive definite matrices P_i, for $i > r$ and $i \in Q$, such that

$$A_i^T P_i A_i - \lambda_+^2 P_i + C_i^T C_i + A_i^T P_i E_i(\gamma_0^2 I - E_i^T P_i E_i)^{-1} E_i^T P_i A_i < 0$$

Using the solution P_i's, we define the following *piecewise Lyapunov function* candidate

$$V(k) = V_{\sigma[k]}(x) = x^T[k] P_{\sigma[k]} x[k] \tag{7}$$

for the switched system, where $P_{\sigma[k]}$ is switched among the solution P_i's in accordance with the piecewise constant switching signal $\sigma[k]$. It can be shown as in [12] that there always exist constant scalars $\alpha_1, \alpha_2 > 0$, for example, $\alpha_1 = \inf_{i\in Q} \lambda_m(P_i)$, $\alpha_2 = \sup_{i\in Q} \lambda_M(P_i)$, such that

$$\alpha_1 \|x\|^2 \leq V_i(x) \leq \alpha_2 \|x\|^2, \ \forall x \in \mathbb{R}^n, \ \forall i \in Q \tag{8}$$

Here $\lambda_M(P_i)$ and $\lambda_m(P_i)$ denotes the largest and smallest eigenvalue of P_i respectively. There exist a constant scalar $\mu \geq 1$ such that

$$V_i(x) \leq \mu V_j(x), \ \forall x \in \mathbb{R}^n, \ \forall i, j \in Q \tag{9}$$

A conservative choice is $\mu = \sup_{k,l\in Q} \frac{\lambda_M(P_k)}{\lambda_m(P_l)}$.

Following the steps in [12], for each $V_i(x) = x^T[k] P_i x[k]$ along the solutions of the corresponding subsystem, we may obtain that

$$\begin{aligned} & V_i(x[k+1]) - V_i(x[k]) \\ \leq & \left\{ \begin{array}{l} -(1-\lambda_-^2) V_i(x[k]) - z^T[k]z[k] + \gamma_0 d^T[k]d[k] \\ -(1-\lambda_+^2) V_i(x[k]) - z^T[k]z[k] + \gamma_0 d^T[k]d[k] \end{array} \right. \end{aligned}$$

For a piecewise constant switching signal $\sigma[k]$ and any given integer $k > 0$, we let $k_1 < \cdots < k_i$ $(i \geq 1)$ denote the switching points of $\sigma[k]$ over the interval $[0, k)$. Then, using the above difference inequalities, we obtain

$$V(k) \leq \left\{ \begin{array}{l} \lambda_-^{2(k-k_i)} V(k_i) - \sum_{j=k_i}^{k-1} \lambda_-^{2(k-1-j)} \Gamma(j) \\ \lambda_+^{2(k-k_i)} V(k_i) - \sum_{j=k_i}^{k-1} \lambda_+^{2(k-1-j)} \Gamma(j) \end{array} \right.$$

where $\Gamma(j) = z^T[j]z[j] - \gamma_0^2 d^T[j]d[j]$. Since $V(k_i) \leq \mu V(k_i^-)$ holds on every switching point k_i, we obtain by induction that

$$\begin{aligned} V(k) \ \leq \ & \mu^{N_\sigma(0,k)} \lambda_-^{2K^-(0,k)} \lambda_+^{2K^+(0,k)} V(0) \\ & - \sum_{j=0}^{k-1} \mu^{N_\sigma(j,k-1)} \lambda_-^{2K^-(j,k-1)} \lambda_+^{2K^+(j,k-1)} \Gamma(j) \\ = \ & - \sum_{j=0}^{k-1} \mu^{N_\sigma(j,k-1)} \lambda_-^{2K^-(j,k-1)} \lambda_+^{2K^+(j,k-1)} \Gamma(j) \end{aligned}$$

The last equality is because of the zero initial state assumption $x[0] = 0$.

We assume that on any interval $[k_1, k_2)$ the total activation periods of the unstable subsystems satisfies $K^+(k_1, k_2) \leq \frac{\ln \lambda^* - \ln \lambda_-}{\ln \lambda_+ - \ln \lambda_-}(k_2 - k_1)$, or equivalently

$$\frac{K^-(k_1, k_2)}{K^+(k_1, k_2)} \geq \frac{\ln \lambda_+ - \ln \lambda^*}{\ln \lambda^* - \ln \lambda_-} \tag{10}$$

holds for some scalar $\lambda^* \in (\lambda_-, 1)$ and $\forall k_2 > k_1 \geq 0$. Then we get

$$\begin{aligned} & \lambda_-^{2K^-(j,k-1)} \lambda_+^{2K^+(j,k-1)} \\ \leq \ & (\lambda^*)^{2K^-(j,k-1) + 2K^+(j,k-1)} = (\lambda^*)^{2(k-1-j)} \end{aligned}$$

Therefore, we get

$$V(k) \leq - \sum_{j=0}^{k-1} \mu^{N_\sigma(j,k-1)} (\lambda^*)^{2(k-1-j)} \Gamma(j) \tag{11}$$

When $\mu = 1$, we get from $V(k) \geq 0$ and (11) that

$$\sum_{j=0}^{k-1} \mu^{N_\sigma(j,k-1)} (\lambda^*)^{2(k-1-j)} \Gamma(j) \leq 0 \tag{12}$$

We sum (12) from $k = 1$ to $k = +\infty$ to obtain

$$\begin{aligned} & \sum_{k=1}^{+\infty} \left(\sum_{j=0}^{k-1} \mu^{N_\sigma(j,k-1)} (\lambda^*)^{2(k-1-j)} \Gamma(j) \right) \\ = \ & \sum_{j=1}^{+\infty} \Gamma(j) \left(\sum_{k=j+1}^{+\infty} \mu^{N_\sigma(j,k-1)} (\lambda^*)^{2(k-1-j)} \right) \\ = \ & (1 - (\lambda^*)^2)^{-1} \sum_{j=1}^{+\infty} \Gamma(j) \leq 0 \end{aligned}$$

which means

$$\sum_{j=0}^{+\infty} z^T[j]z[j] \leq \gamma_0^2 \sum_{j=0}^{+\infty} d^T[j]d[j] \tag{13}$$

Therefore, $\mathcal{L}_2$ gain γ_0 is achieved for the switched system, namely the NCS (4).

For the case $\mu > 1$, we multiply both sides of (11) by $\mu^{-N_\sigma(0,k-1)}$ to get

$$\begin{aligned} & \sum_{j=0}^{k-1} \mu^{-N_\sigma(0,j)} (\lambda^*)^{2(k-1-j)} z^T[j]z[j] \\ \leq \ & \gamma_0^2 \sum_{j=0}^{k-1} \mu^{-N_\sigma(0,j)} (\lambda^*)^{2(k-1-j)} d^T[j]d[j] \end{aligned} \tag{14}$$

Now, we choose a positive scalar λ larger than 1 to consider the following average dwell time condition: for any positive integer $j > 0$,

$$N_\sigma(0, j) \leq \frac{j}{\tau_a^*}, \quad \tau_a^* = \frac{\ln \mu}{2 \ln \lambda} \tag{15}$$

Therefore $\mu^{-N_\sigma(0,j)} > \lambda^{-2j}$ holds for any $j > 0$, where $\lambda = \mu^{(2\tau_a^*)^{-1}}$. Then, from (14) we obtain

$$\sum_{j=0}^{k-1} \lambda^{-2j} (\lambda^*)^{2(k-1-j)} z^T[j] z[j] \leq \gamma_0^2 \sum_{j=0}^{k-1} (\lambda^*)^{2(k-1-j)} d^T[j] d[j]$$

Similarly, we sum both sides of the above inequality from $k = 1$ to $k = +\infty$ to get

$$(1 - (\lambda^*)^2)^{-1} \sum_{j=1}^{+\infty} \lambda^{-2j} z^T[j] z[j] \leq (1 - (\lambda^*)^2)^{-1} \gamma_0^2 \sum_{j=1}^{+\infty} d^T[j] d[j]$$

and thus

$$\sum_{j=1}^{+\infty} \lambda^{-2j} z^T[j] z[j] \leq \gamma_0^2 \sum_{j=1}^{+\infty} d^T[j] d[j] \tag{16}$$

holds for any $d[k] \in \mathcal{L}_2[0, +\infty)$. Following the notation in [12], we say that a *weighted* $\mathcal{L}_2$ *gain* γ_0 is achieved. In summary, we prove the following theorem.

Theorem 2: The NCS (4) achieves a weighted $\mathcal{L}_2$ gain γ_0 if the $K^+(k_1, k_2)$ satisfies (10) and $N_\sigma(0, k)$ satisfy the condition of (15).

Remark 4: Similarly, the condition (10) restricts the number of the packet dropout and large delayed packet, while the condition (15) restricts the happening frequency of them. Both of the conditions are given in the sense of average over time.

VI. CONCLUDING REMARKS

In this paper, we modeled a class of NCSs under uncertain access delay and packet dropout as discrete-time linear switched systems. The stability and disturbance attenuation issues for such NCSs were studied in the framework of switched systems. The strength of this approach comes from the solid theoretic results existing in the literature of switched/hybrid systems. It was shown that the exponential stability and disturbance attenuation level might be preserved for the NCSs under certain bounds on the amount and rate of the dropped and large delayed packets. Although we only consider state feedback control law here, the techniques and results developed here can be easily extended to the case of static output feedback control law. It should be pointed out that the conditions concerning the delay and packet dropout rate for the preservation of the NCSs' stability and $\mathcal{H}^\infty$ disturbance attenuation properties were based on Lyapunov theory. Therefore, the conditions are sufficient only and maybe conservative for some cases.

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