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Synthesis of Uniformly Ultimate Boundedness Switching Laws for Discrete-Time Uncertain Switched Linear Systems

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***Abstract*— In this paper, discrete-time switched linear systems affected by both parameter variations and exterior disturbances are considered. The problem of synthesis of switching control laws, which assure that the system state is ultimately bounded within a given compact set containing the origin with an assigned rate of convergence, is investigated. The method is based on set-induced Lyapunov functions. Based on these Lyapunov functions, we compose a global Lyapunov function which guarantees ultimate boundedness for the switched system. The switching laws are characterized by computing conic partitions of the state space.**

I. Introduction

A switched system is a dynamical system that consists of a finite number of subsystems described by differential or difference equations and a logical rule that orchestrates switching between these subsystems. Properties of this type of model have been studied for the past fifty years to consider engineering systems that contain relays and/or hysteresis. Recently, there has been increasing interest in the stability analysis and switching control design of switched systems, see for example [7], [4], [9], [1] and the references cited therein. The motivation for studying such switched systems comes partly from the discovery that there exist large class of nonlinear systems which can be stabilized by switching control schemes, but cannot be stabilized by any smooth state feedback control law. In addition, switched systems and switched multi-controller systems have numerous applications in control of mechanical systems, process control, automotive industry, power systems, aircraft and traffic control, and many other fields. Switched systems with all subsystems described by linear differential or difference equations are called piecewise linear/affine systems or switched linear systems, and have gained the most attention [9], [5], [1]. Recent efforts in switched linear system research typically concentrate on the analysis of the dynamic behaviors, like stability [5], [7], [4], controllability and observability [1], [9] etc., and aim to design controllers with guaranteed stability and performance [9], [5].

In this paper, we will concentrate on robust stabilization problem for the switched linear systems affected by both parameter variations and exterior disturbances. The stability issues of switched systems have been studied extensively in the literature [7], [4], and can be roughly divided into two kinds of problems. One is the stability analysis of switched systems under given switching signals (maybe arbitrary, slow switching etc.), and the other is the synthesis of stabilizing switching signals for a given collection of dynamical systems. The first stability analysis problem is usually dealt with using Lyapunov method, such as common Lyapunov function, multiple Lyapunov functions, see [4], [7] and references therein. Notice that usually (piecewise) quadratic Lyapunov(-like) functions were considered, because of comparable simplicity for calculation by employing LMI techniques. There are less results for the second problem, stabilization switching control for switched systems. Quadratic stabilization for LTI systems was considered in [10], in which it was shown that the existence of a stable convex combination of the subsystem matrices implies the existence of a state-dependent switching rule that stabilizes the switched system along with a quadratic Lyapunov function. There are extensions of [10] to the case of output-dependent switching and discrete-time case [7], [12]. The switching stabilization of second-order LTI systems was considered in [11] via vector field analysis. For robust stabilization of polytopic uncertain switched systems, a quadratic stabilizing switching law was designed for polytopic uncertain switched systems based on LMI techniques in [12].

Because of parameter variations and exterior disturbances considered in this paper, it is only reasonable to stabilize the system within a neighborhood region of the equilibrium, which is the so called practical stabilization or ultimate boundedness control in the literature. In [2], the ultimate boundedness control problem for uncertain discrete-time linear systems was studied based on set-induced Lyapunov functions, and the methods were extended to the continuous-time case in [3]. The problem studied here is *uniformly ultimate boundedness switching control*, that is, to synthesize switching control laws assuring that the system state will be ultimately bounded within a given compact set containing the origin with an assigned rate of convergence. The motivation for considering this problem comes from the following fact. As explained in [6], switching control design methods have become more and more popular. However, switching among these multi-controllers, which are designed with respect to different performance criteria respectively, may leads to undesirable or even unbounded trajectories [4]. Therefore, the stabilizing switching sequences design is not a trivial task and is the central problem in switching control design method. In addition, by switching among multi-controllers, we can achieve better closed-loop performance than a single controller.

This paper is an extension of our group's recent work [6] to uncertain switched systems. In [6], a class of stabilization switching law for switched autonomous linear time-invariant systems is considered. In the present paper, not only time-variant parameter uncertainties in the state matrices but also exterior persistent disturbances are considered in the model. In Section II, a mathematical model for discrete-time switched linear system affected by both parameter variations and exterior disturbances is described, and the ultimate boundedness control problem is formulated. Section III presents the necessary background for set-induced Lyapunov functions. Based on these Lyapunov functions, we compose a global Lyapunov function which guarantees ultimate boundedness of the switched systems. The switching sequences are characterized by computing conic partitions of the state space in Section IV.

In this paper, we use the letters $\mathcal{E}, \mathcal{P}, \mathcal{S} \cdots$ to denote sets. $\partial\mathcal{P}$ stands for the boundary of set $\mathcal{P}$, and $int\{\mathcal{P}\}$ its interior. For any real $\lambda \geq 0$, the set $\lambda\mathcal{S}$ is defined as $\{x = \lambda y,\ y \in \mathcal{S}\}$. The term C-set stands for a convex and compact set containing the origin in its interior.

II. Problem Formulation

In this paper, we consider a collection of discrete-time linear systems described by

$$x(t+1) = A_q(w)x(t) + E_q d(t),\ q \in Q = \{1, \cdots, N\} \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $d(t) \in \mathcal{D} \subset \mathbb{R}^r$, $t \in \mathbb{Z}^+$ (the set of nonnegative integers) and state matrices $A_q(w) \in \mathbb{R}^{n\times n}$, $E_q \in \mathbb{R}^{n\times r}$. Assume that $\mathcal{D}$ is a C-set, and that the entries of $A_q(w)$ are continuous function of $w \in \mathcal{W}$, where $\mathcal{W} \subset \mathbb{R}^v$ is an assigned compact set. Note that the origin $x_e = 0$ is an equilibrium for the systems described in (1).

The particular mode q at any given time instant may be selected by a decision-making process, which can be represented by a switching law of the form:

$$q(t) = \delta(x(t)) \quad (2)$$

The discrete mode is determined by the continuous variable state, in fact the partition of the state space. Therefore, we get a class of piecewise constant functions of time $\sigma : \mathbb{Z}^+ \to Q$. Then we can define the following time-varying system as a discrete-time switched linear system

$$x(t+1) = A_{\sigma(t)}(w)x(t) + E_{\sigma(t)}d(t), \quad t \in \mathbb{Z}^+$$

The signal $\sigma(t)$ is called a *switching sequence*.

For this uncertain switched system (1)-(2), we are interested in characterizing the switching law $\delta(\cdot)$ such that the state $x(t)$ asymptotically converges to the equilibrium, $x_e = 0$. Because of the uncertainty and disturbance, we can not drive the state $x(t)$ to the origin exactly, and it is only reasonable to converge into a neighborhood region of the origin. In particular, we introduce the following definition for uniformly ultimate boundedness (UUB).

Definition 1: The uncertain switched system (1)-(2) with the switching law $\delta(\cdot)$ is *Uniformly Ultimately Bounded (UUB)* in the C-set $\mathcal{S}$ iff for every initial condition $x(0) = x_0$, there exists $T(x_0)$, such that for $t \geq T(x_0)$, we have $x(t) \in \mathcal{S}$.

The problem being addressed can be formulated as follows:

Problem: Given the discrete-time uncertain switched linear systems (1)-(2), design switching law $\delta(\cdot)$ to assure that the system state $x(t)$ is uniformly ultimately bounded within a given compact set containing the origin with an assigned rate of convergence.

Our methodology for computing switching sequences that guarantee ultimate boundedness is based on *set-induced Lyapunov functions*, which will be derived in the next section. For systems with linearly constrained uncertainties, it is shown that such a function may be derived by numerically efficient algorithms involving polyhedral sets. Based on these Lyapunov functions, we compose a global Lyapunov function which guarantees uniformly ultimate boundedness of the switched linear system. It is shown that the UUB switching law for switched linear systems is characterized by computing conic partitions of the state space.

III. Set-Induced Lyapunov Functions

In this section, we briefly present some background material necessary for the set-induced Lyapunov functions for uncertain discrete-time linear systems.

Following the notation of [2], we call a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ a *gauge function* if $\Psi(x) \geq 0$, $\Psi(x) = 0 \Leftrightarrow x = 0$; for $\mu > 0$, $\Psi(\mu x) = \mu\Psi(x)$; and $\Psi(x+y) \leq \Psi(x) + \Psi(y)$, $\forall x, y \in \mathbb{R}^n$. A gauge function is convex and it defines a distance of x from the origin which is linear in any direction. If Ψ is a gauge function, we define the closed set (possibly empty) $\bar{N}[\Psi, \xi] = \{x \in \mathbb{R}^n : \Psi(x) \leq \xi\}$. It is easy to show that the set $\bar{N}[\Psi, \xi]$ is a C-set for all $\xi > 0$. On the other hand, any C-set $\mathcal{S}$ induces a gauge function $\Psi_{\mathcal{S}}(x)$ (Known as Minkowski function of $\mathcal{S}$), which is defined as $\Psi(x) \doteq \inf\{\mu > 0 : x \in \mu\mathcal{S}\}$. Therefore a C-set $\mathcal{S}$ can be thought of as the unit ball $\mathcal{S} = \bar{N}[\Psi, 1]$ of a gauge function Ψ and $x \in \mathcal{S} \Leftrightarrow \Psi(x) \leq 1$.

Consider the subsystem of mode q for the discrete-time uncertain switched linear systems (1)-(2) as

$$x(t+1) = A_q(w)x(t) + E_q d(t) \quad (3)$$

for which the UUB in a C-set $\mathcal{S}$ is guaranteed by the existence of a Lyapunov function outside $\mathcal{S}$ [3].

In particular, a Lyapunov function outside $\mathcal{S}$ for the subsystem (3) can be defined as a continuous function $\Psi : \mathbb{R}^n \to \mathbb{R}^+$ such that $\bar{N}[\Psi, \kappa] \subseteq \mathcal{S}$, for some $\kappa > 0$, for which the following conditions hold:

if $x \notin \bar{N}[\Psi, \kappa]$ then there exists $\beta > 0$ such that

$$\Psi(A(w)x + Ed) - \Psi(x) \leq -\beta;$$

if $x \in \bar{N}[\Psi, \kappa]$ then

$$\Psi(A(w)x + Ed) \le \kappa.$$

Lemma 1: [3] If there exists a Lyapunov function outside $\mathcal{S}$ for the system (3), then it is uniformly ultimately bounded (UUB) in $\mathcal{S}$.

In the following, we will assume that for each subsystem (3) there exist a corresponding Lyapunov function Ψ_q, with $\bar{N}[\Psi_q, 1] \subseteq \mathcal{S}$. Under this assumption, we will review the procedure for the construction of such Lyapunov function Ψ_q for each subsystem (3). For notational simplicity, we will drop the subscript q in this subsection.

It can be derived from the definition of the Lyapunov function Ψ that

$$\Psi(x(t)) \le \min\{\lambda^t \Psi(x(0)), 1\}$$

for some λ with $0 < \lambda < 1$. This property motivates the following concept of contractive set.

Definition 2: Given λ, $0 < \lambda < 1$, a set $\mathcal{S}$ is said λ-*contractive* with respect to subsystem (3), if for any $x \in \mathcal{S}$ such that $post_q(x, \mathcal{W}, \mathcal{D}) \subseteq \lambda\mathcal{S}$. Here $post_q(\cdot)$ is defined as

$$post_q(x, \mathcal{W}, \mathcal{D}) = \{x' : x' = A_q(w)x + E_q d, \forall w \in \mathcal{W}, d \in \mathcal{D}\},$$

which represents all the possible next step states of system (3), given current state $x(t)$.

Let $\mathcal{S}$ be an assigned C-set in $\mathbb{R}^n$. We say that a λ-contractive set $\mathcal{P}_m \subseteq \mathcal{S}$ is *maximal* in $\mathcal{S}$ if every λ-contractive set $\mathcal{P}$ contained in $\mathcal{S}$ is also contained in $\mathcal{P}_m$.

Consider the following sequence of sets:

$$\{\mathcal{X}_k\} : \mathcal{X}_0 = \mathcal{S},\ \mathcal{X}_k = pre_q(\lambda\mathcal{X}_{k-1}) \cap \mathcal{S};\ k = 1, 2, \cdots \quad (4)$$

where $pre_q(\mathcal{S})$ is defined as

$$pre_q(\mathcal{S}) = \{x \in \mathbb{R}^n : post_q(x, \mathcal{W}, \mathcal{D}) \subseteq \mathcal{S}\}. \quad (5)$$

Then the maximal λ-contractive set $\mathcal{P}_m \subseteq \mathcal{S}$ is given by $\mathcal{P}_\lambda = \bigcap_{k=0}^{\infty} \mathcal{X}_k$ [2].

Proposition 1: If $\mathcal{P}_\lambda = \bigcap_{k=0}^{\infty} \mathcal{X}_k$ is nonempty, then the system (3) is uniformly ultimately bounded (UUB) in $\mathcal{S}$.

Proof : It can be shown that $\mathcal{P}_\lambda$ is a C-set, when it is nonempty. Let $\psi(x) = \Psi_{\mathcal{P}_\lambda}(x)$ be its Minkowski functional. We have $\psi(x(t+1)) \le \lambda\psi(x(t))$ for all $x(t) \notin int\{\mathcal{P}_\lambda\}$, and $\bar{N}[\psi, 1] \subset \mathcal{S}$. Then ψ is a Lyapunov function outside $\mathcal{S}$ for the system (3). By Lemma 1, the existence of a Lyapunov function outside $\mathcal{S}$ implies the UUB of (3) in $\mathcal{S}$.

□

Lyapunov function ψ is uniquely generated from the target set $\mathcal{S}$ for any fixed λ. Such a function has been named Set-induced Lyapunov Function (SILF) in the literature [3].

A. Linearly Constrained Case

It is known that in practice uncertainties often enter linearly in the system model and they are linearly constrained. To handle this particular but interesting case, we consider the class of polyhedral sets. Such sets have been considered in the literature concerning the control of systems with input and state constraints [2]. Their main advantage is that they are suitable for computation. In the sequel, let us assume polytopic uncertainty in $A_q(w)$. In particular,

$$A_q(w) = \sum_{j=1}^{v} w_j A_q^j,\quad w_j \ge 0,\ \sum_{j=1}^{v} w_j = 1 \quad (6)$$

which provides a classical description of model uncertainty. Notice that the coefficients w_j are unknown and possibly time varying.

For computational efficiency, we assume that $\mathcal{D}$ and $\mathcal{S}$ to be polyhedral C-sets, convex and compact polyhedrons containing the origin, and in addition, $\mathcal{S}$ contains the origin in its interior. A convex polyhedral set $\mathcal{S}$ in $\mathbb{R}^n$ can be represented by a set of linear inequalities

$$\mathcal{S} = \{x \in \mathbb{R}^n : f_i x \le g_i,\ i = 1, \cdots, m\} \quad (7)$$

and for brevity, we denote $\mathcal{S}$ as $\{x : Fx \le g\}$, where $\le$ is with respect to componentwise. The set $\lambda\mathcal{S}$, $\lambda > 0$, is given by $\{x : Fx \le \lambda g\}$. Consider the vector δ whose components are

$$\delta_i = \max_{d \in \mathcal{D}} f_i E_q d \ge 0,\quad i = 1, \cdots, m \quad (8)$$

The vector δ incorporates the effects of the disturbance $d(t)$. For $\lambda > 0$, we have $post_q(x, \mathcal{W}, \mathcal{D}) \subseteq \lambda\mathcal{S}$ iff $FA_q(w)x \le \lambda g - \delta$, for all $w \in \mathcal{W}$, which is equivalent to:

$$FA_q^j x \le \lambda g - \delta,\quad j = 1, \cdots, v \quad (9)$$

The above constraints define a convex polyhedron in the space $\mathbb{R}^n$ which is exactly the set $pre_q(\lambda\mathcal{S})$ by definition. Note that the intersection of finite convex polyhedra produces a convex polyhedron. Therefore, the set $\mathcal{X}_1 = pre_q(\lambda\mathcal{S}) \cap \mathcal{S}$ is a convex polyhedron, which is denoted as $\mathcal{X}_1 = \{x : F^{(1)}x \le g^{(1)}\}$. Following the procedure described in (4), the set $\mathcal{X}_{k+1} = \{x :\ F^{(k+1)}x \le g^{(k+1)}\}$ can be generated inductively as the intersection of $pre_q(\lambda\mathcal{X}_k)$ with $\mathcal{S}$. In view of the convergence of the sequence $\mathcal{X}_k$, $k = 0, 1, \cdots$, we may derive an arbitrarily close external polyhedral approximation of $\mathcal{P}_\lambda$ by $\mathcal{X}_k$ as follows. For every $\lambda^* : \lambda < \lambda^* < 1$, a λ^*-contractive polyhedral C-set $\mathcal{P}_{\lambda^*}$ can be obtained as $\mathcal{P}_{\lambda^*} = \mathcal{X}_k$ for a finite k [2]. Therefore, we can always determine a λ^*-contractive polyhedral C-set $\mathcal{P}_{\lambda^*} \subseteq \mathcal{S}$ in finite number of steps for all λ^*, $\lambda < \lambda^* < 1$, if $\mathcal{S}$ has nonempty λ-contractive subsets. The Minkowski function of a polyhedral C-set $\mathcal{P}$, which can be canonically represented by

$$\mathcal{P} = \{x \in \mathbb{R}^n : f_i x \le 1,\ i = 1, \cdots, m\}, \quad (10)$$

has the following expression

$$\Psi_{\mathcal{P}}(x) = \max_{1 \le i \le m} \{f_i x\}. \quad (11)$$

In this case, the Minkowski function $\Psi_{\mathcal{P}}$ of $\mathcal{P}$ is called as polyhedral Lyapunov function or piecewise-linear Lyapunov function in the literature [8]. In [3], it was shown that if a Lyapunov function exists and solves the uniform ultimate boundedness problem in a certain convex neighborhood of the origin then there exists a polyhedral Lyapunov function that solves the problem in the same neighborhood. In other words, the polyhedral Lyapunov function is universal [3]. Therefore, without loss of generality, we will restrict to polyhedral Lyapunov functions in the sequel. Another advantage of the polyhedral Lyapunov functions is that it can be determined by numerical methods within finite number of iterations under mild assumption. In addition, the polyhedral Lyapunov functions is suitable for control design, which will be explored in the following sections.

IV. Ultimate Boundedness Switching Law

It is known that the stability (or UUB) of all the subsystems (3) can not guarantee the stability (or UUB) of the switched system (1)-(2). Such a switched system might become unboundedness for certain switching sequences [7], [4]. Therefore, it is important to characterize switching sequences that result in ultimately bounded trajectories. In this section, we will present an approach to design the ultimately bounded switching law for the uncertain switched system (1)-(2). This method is based on set induced Lyapunov functions derived in the previous section.

Recall the problem we concerned is to synthesize switching law $\delta(\cdot)$ so as to assure that the system state $x(t)$ is uniformly ultimately bounded within a given compact set containing the origin, say a polyhedral C-set $\mathcal{T}$, with an assigned rate of convergence, say $0 < \lambda < 1$. In the sequel, we assume that each individual subsystem admits a λ_q-contractive polyhedral C-set ($\lambda_q \leq \lambda$), which is described by

$$\mathcal{P}_q = \{x \in \mathbb{R}^n : F_q x \leq \bar{1}\} \subseteq \mathcal{T} \tag{12}$$

where $F^q \in \mathbb{R}^{m_q \times n}$ and $\bar{1} = [1, \cdots, 1]^T \in \mathbb{R}^{m_q}$. Such $\mathcal{P}_q$ can be generated by the procedure described in (4). We denote the rows of the matrix F^q by $f_i^q \in \mathbb{R}^{1\times n}$, $i = 1, \cdots, m_q$. By Equation (11), the Lyapunov function induced by the polyhedral C-set $\mathcal{P}_q$ can be described by $\psi_q(x) = \max_{1\leq i\leq m_q}\{f_i^q x\}$.

First, we briefly describe the necessary notations from convex analysis in order to construct the conic partition. Given a polyhedral C-set $\mathcal{P}$, let $vert\{\mathcal{P}\} = \{v_1, v_2, \cdots, v_N\}$ stands for the vertices of a polytope $\mathcal{P}$, while $face\{\mathcal{P}\} = \{F_1, F_2, \cdots, F_M\}$ denotes its faces. The hyperplane that corresponds to the k-th face F_k is defined by

$$H_k = \{x \in \mathbb{R}^n : f_k x = 1\} \tag{13}$$

where $f_k \in \mathbb{R}^{1\times n}$ is the corresponding gradient vector of face F_k. The set of vertices of F_k can be found as $vert(F_k) = vert(\mathcal{P}) \cap F_k$. Finally, we denote the cone generated by the vertices of F_k by $cone(F_k) = \{x \in \mathbb{R}^n : \sum_i \alpha_i v_{k_i},\ \alpha_i \geq 0,\ v_{k_i} \in vert(F_k)\}$. The $cone(F_k)$ has the property that $\forall x \in cone(F_k)$, $\psi(x) = f_k x$.

Next we will characterize a conic partition of the state space based on these polyhedral Lyapunov functions $\psi_q(x)$. Consider any pair of subsystems with modes q_1 and q_2, with $q_1 \neq q_2 \in Q$, we want to compute the region

$$\Omega_{q_1}^{q_2} = \{x \in R^n : \psi_{q_1}(x) \leq \psi_{q_2}(x)\} \tag{14}$$

For this purpose, we first consider a pair of faces $F_{i_1}^{q_1}$ and $F_{i_2}^{q_2}$ of the polyhedral C-sets $\mathcal{P}_{q_1}$ and $\mathcal{P}_{q_2}$ respectively and consider

$$C_{q_1,i_1}^{q_2,i_2} = cone(F_{i_1}^{q_1}) \cap cone(F_{i_2}^{q_2}) \tag{15}$$

The set $C_{q_1,i_1}^{q_2,i_2}$ is either empty or a polyhedral cone. If $C_{q_1,i_1}^{q_2,i_2} \neq \emptyset$, then all the state $x \in C_{q_1,i_1}^{q_2,i_2}$ has the property that, $\psi_{q_1}(x) = f_{i_1}^{q_1} x$ and $\psi_{q_2}(x) = f_{i_2}^{q_2} x$. Next, we intersect the set $C_{q_1,i_1}^{q_2,i_2}$ with the half-space defined by

$$HF_{q_1,i_1}^{q_2,i_2} = \{x \in R^n : (f_{i_1}^{q_1} - f_{i_2}^{q_2})x \leq 0\} \tag{16}$$

and get the set $\Omega_{q_1,i_1}^{q_2,i_2} = C_{q_1,i_1}^{q_2,i_2} \cap HF_{q_1,i_1}^{q_2,i_2}$. The reason for specifying the region $\Omega_{q_1,i_1}^{q_2,i_2}$ can be clarified by the following lemma [6].

Lemma 2: For every $x \in \Omega_{q_1,i_1}^{q_2,i_2}$, we have that $\psi_{q_1}(x) \leq \psi_{q_2}(x)$.

Proof : By definition, $\Omega_{q_1,i_1}^{q_2,i_2} = C_{q_1,i_1}^{q_2,i_2} \cap HF_{q_1,i_1}^{q_2,i_2}$, where $C_{q_1,i_1}^{q_2,i_2} = cone(F_{i_1}^{q_1}) \cap cone(F_{i_2}^{q_2})$. The $cone(F_{i_1}^{q_1})$ and $cone(F_{i_2}^{q_2})$ have the property that $\forall x \in cone(F_{i_1}^{q_1})$, $\psi_{q_1}(x) = f_{i_1}^{q_1} x$, and $\forall x \in cone(F_{i_2}^{q_2})$, $\psi_{q_2}(x) = f_{i_2}^{q_2} x$. Note that $\forall x \in HF_{q_1,i_1}^{q_2,i_2}$, $f_{i_1}^{q_1}(x) \leq f_{i_2}^{q_2}(x)$. Therefore, for all $x \in \Omega_{q_1,i_1}^{q_2,i_2}$, we have that $\psi_{q_1}(x) \leq \psi_{q_2}(x)$.

□

The illustration of the conic region $\Omega_{q_1,i_1}^{q_2,i_2}$ is shown in Figure 1. From this figure, an important observation can be made, that is the hyperplane $H_{(q_1,i_1)}^{(q_2,i_2)} = \{x \in R^n : (f_{i_2}^{q_2} - f_{i_1}^{q_1})x = 0\}$ goes through the origin and the intersections of the faces $F_{i_1}^{q_1}$ and $F_{i_2}^{q_2}$. This is simply comes from the fact that $\psi_{q_1}(0) = \psi_{q_2}(0) = 0$, and for $x \in F_{i_1}^{q_1} \bigcap F_{i_2}^{q_2} \Rightarrow \psi_{q_1}(x) = \psi_{q_2}(x) = 1$. We will show later that this observation dramatically simplify the design procedure for conic partition based switching law.

Based on the above lemma, we have

$$\Omega_{q_1}^{q_2} = \bigcup_{i_1,i_2} \Omega_{q_1,i_1}^{q_2,i_2} \tag{17}$$

where i_1 and i_2 go through all the faces' index of $\mathcal{P}_{q_1}$ and $\mathcal{P}_{q_2}$ respectively. And the following corollary holds.

Corollary 1: For every $x \in \Omega_{q_1}^{q_2}$, we have that $\psi_{q_1}(x) \leq \psi_{q_2}(x)$.

Because $\Omega_{q_1,i_1}^{q_2,i_2}$ is an intersection of a polyhedral cone with a half-space, so it is either an empty set or a polyhedral cone. Hence $\Omega_{q_1}^{q_2}$ is finite union of polyhedral cones. And it is obvious that for $x \notin \Omega_{q_1}^{q_2}$, we have that $\psi_{q_1}(x) > \psi_{q_2}(x)$. Therefore, $\Omega_{q_1}^{q_2} \bigcup \Omega_{q_2}^{q_1} = \mathbb{R}^n$.

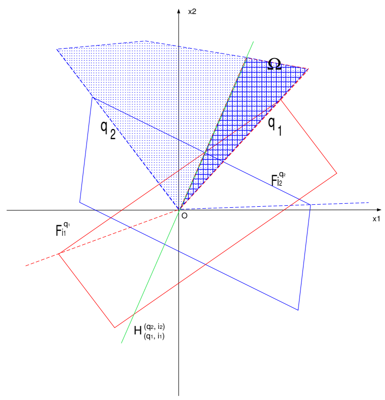

Fig. 1. The conic region of Ω.

Finally define

$$\Omega_q = \bigcap_{q_i \in Q,\ q_i \neq q} \Omega_q^{q_i},$$

which has the property as follows.

Lemma 3: For every $x \in \Omega_q$, we have that $\psi_q(x) \leq \psi_{q_i}(x)$, $\forall q_i \in Q$ and $q_i \neq q$.

Proof : For every $x \in \Omega_q = \bigcap_{q_i \in Q,\ q_i \neq q} \Omega_q^{q_i}$, then $x \in \Omega_q^{q_i}$ for all $q_i \in Q$ and $q_i \neq q$. Therefore, $\psi_q(x) \leq \psi_{q_i}(x)$, $\forall q_i \in Q$ because of Corollary 1.

$\square$

Some observations about Ω_q are important for the following design procedure. First, in the region of Ω_q, $q \in \arg\min_{q \in Q} \psi_q(x)$. Secondly, Ω_q is finite union of polyhedral cones. Finally, $\bigcup_{q \in Q} \Omega_q = \mathbb{R}^n$, so Ω_q, $q \in Q$, serves as a conic partition of the state space.

Define the following switching law based on the conic partition of the state space Ω_q, $q \in Q$.

$$x \in \Omega_q \Rightarrow \delta(\cdot) = q \tag{18}$$

For the case $x \in \Omega_q \cap \Omega_{q'}$, one simply remains the previous mode q.

It can be shown that the switching law defined as above can guarantee the UUB for the uncertain switched system (1)-(2).

Theorem 1: Consider the class of switching law defined in (18), the uncertain switched system (1)-(2) is UUB.

Proof : Define the function $V(x) = \min_{q \in Q} \psi_q(x)$. In the following, we will prove that such $V(x)$ is a Lyapunov function for the switched system (1)-(2)with the above switching law. First, it is straightforward to verify that $V(x)$ is positive definite, $V(x) = 0$ iff $x = 0$ etc. The key point is to show that $V(x)$ decreases along all the trajectories of the switched systems under above switching law. First, for the case of $x \notin int(\bigcup_{q \in Q}(\mathcal{P}_q))$. Assume that at time t, $x(t) \in \Omega_q$ and current mode $q(t) = q$. If no switching occur, i.e. $x(t+1) \in \Omega_q$, then $V(x(t)) = \min_{q \in Q} \psi_q(x(t)) = \psi_q(x(t))$ and $V(x(t+1)) = \psi_q(x(t+1)) \leq \lambda_q \psi_q(x(t)) \leq \lambda V(x(t))$. Else, if switching occur at time t, say $x(t+1) \in \Omega_{q'}$, then $V(x(t+1)) = \min_{q \in Q} \psi_q(x(t+1)) = \psi_{q'}(x(t+1)) \leq \psi_q(x(t+1)) \leq \lambda_q \psi_q(x(t)) \leq \lambda V(x(t))$. Therefore, for $x \notin int(\bigcup_{q \in Q}(\mathcal{P}_q))$, we have $V(x(t+1)) \leq \lambda V(x(t))$.

Similarly, it can be shown that for $x \in int(\bigcup_{q \in Q}(\mathcal{P}_q))$, we have $V(x(t+1)) \leq \lambda$. Therefore, by definition, the uncertain switched system (1)-(2) is UUB with convergence index λ with the class of switching law defined by $\delta(\cdot) = q$ for $x \in \Omega_q$.

$\square$

A. Simplified Design Procedure

As it has been pointed out that some geometric characteristics can be used to simplify the determination of the conic partition Ω_q. In the following, we will describe the simplified design procedure through an example.

Consider a second order three mode discrete-time switched system, and assume that the target region is given as a polyhedral C-set $\mathcal{T}$, and the assigned rate of convergence is $0 < \lambda < 1$. Assume that each individual subsystem admits a λ_{q_i}-contractive polyhedral C-set $\mathcal{P}_{q_i}$, $\lambda_{q_i} \leq \lambda$ for $i = 1, 2, 3$. Such $\mathcal{P}_{q_i}$ can be generated by the procedure described in (4). In Figure 2, the two dimensional case $\mathcal{P}_{q_i}$, for $i = 1, 2, 3$, is plotted.

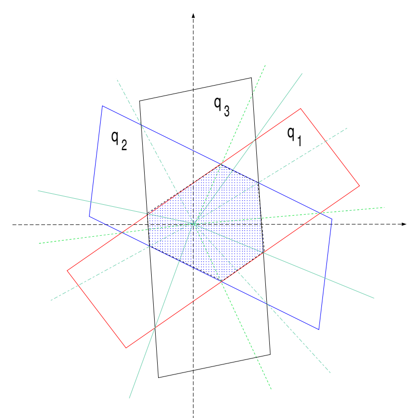

Fig. 2. The λ_{q_i}-contractive polyhedral C-set $\mathcal{P}_{q_i}$, $\lambda_{q_i} \leq \lambda$ for $i = 1, 2, 3$.

Next, in order to calculate the region $\Omega_{q_1}^{q_2}$, we simply draw some radii which star from the origin and go through the intersection points of faces of $\mathcal{P}_{q_1}$ and $\mathcal{P}_{q_2}$. These radii partition the state space into finite union of conic regions. Notice that on these radii, $\psi_{q_1}(x) = \psi_{q_2}(x)$, and that within each conic region partitioned by these radii either $\psi_{q_1}(x) \geq \psi_{q_2}(x)$ or $\psi_{q_1}(x) \leq \psi_{q_2}(x)$ holds. Therefore, $\Omega_{q_1}^{q_2}$ is just the union of some of these conic regions. To determine whether one of these polyhedral cones is contained in $\Omega_{q_1}^{q_2}$, one simply check whether there exists one point in this cone which is on the edge of $\mathcal{P}_{q_1}$ but not contained in $int(\mathcal{P}_{q_2})$. If such points exist in the cone, then include this cone into the region $\Omega_{q_1}^{q_2}$ (from the geometric interpretation of Minkowski function). The region $\Omega_{q_1}^{q_2}$ is just the union of such cones.

Similarly, we get $\Omega_{q_1}^{q_3}$. And the region $\Omega_{q_1} = \Omega_{q_1}^{q_2} \bigcap \Omega_{q_1}^{q_3}$, which is illustrated in the leftmost plot in Figure 3. The middle plot of Figure 3 illustrates the region Ω_{q_2}, while Ω_{q_3} is the rightmost plot of Figure 3. And the conic partition of the state space is plotted in Figure 4. From this conic partition, the UUB switching law, $\delta(\cdot) = q_i$ for $x \in \Omega_{q_i}$, can be easily implemented.

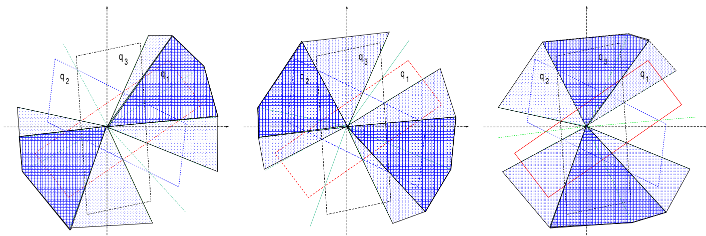

Fig. 3. Determine the region of Ω_q as finite union of polyhedral cones.

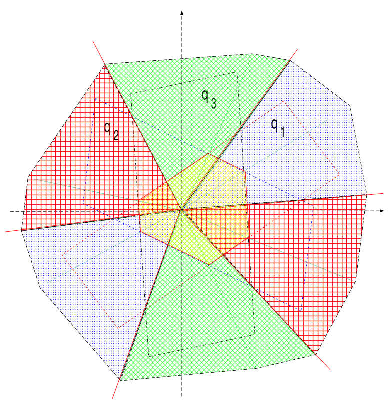

Fig. 4. Conic partition based switching law.

In [6], $\Omega_q^{q'}$ was obtained based on the computation of $\Omega_{q,i}^{q',j}$ of all possible pairs of faces, F_i^q and $F_j^{q'}$, of $\mathcal{P}_q$ and $\mathcal{P}_{q'}$ respectively. Therefore, it may be computationally expensive to calculate $\Omega_q^{q'}$. In the present paper, a simplified method is developed to obtain the conic partition $\Omega_q^{q'}$ by employing geometric characteristics of $\mathcal{P}_q$ and $\mathcal{P}_{q'}$ as explained above. In addition, the stabilization switching sequences in [6] is based on partition $\Omega_q^{q'}$, which leads to possibly nondeterministic switching law. However, in this paper the UUB switching law is based on the conic partition Ω_q of the state space, and switching is deterministic.

V. CONCLUDING REMARKS

In this paper, discrete-time switched linear systems affected by both parameter variation and exteriors disturbance were considered. The problem of switching control law synthesis, assuring that the system state is ultimately bounded within a given compact set containing the origin with an assigned rate of convergence, was investigated. Given an uncertain switched linear system, a systematic method for computing switching control laws that guaranteed ultimately boundedness was proposed. The method was based on set-induced Lyapunov functions. For systems with linearly constrained uncertainties, it was shown that such a function could be derived by numerically efficient algorithms involving polyhedral sets. Based on these set-induced Lyapunov functions, a procedure to construct UUB switching control laws based on the conic partition of the state space was presented.

VI. ACKNOWLEDGMENTS

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