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Optimal Timing Control of a Class of Hybrid Autonomous Systems

Xuping XU[1] and Panos J. ANTSAKLIS[2]

[1] Department of Electrical and Computer Engineering, Penn State Erie
Erie, PA 16563, USA

[2] Department of Electrical Engineering, University of Notre Dame
Notre Dame, IN 46556, USA

ABSTRACT: In this paper, optimal timing control problems for a class of hybrid autonomous systems are studied. In particular, we focus on problems in which a prespecified sequence of active subsystems is given and propose an approach to find local optimal switching time instants. The main contributions of the paper are the derivation and computation of the derivatives of the cost with respect to the switching time instants, which then facilitate the application of nonlinear optimization techniques to locate the optimal switching time instants. The approach is then applied to general quadratic problems for hybrid autonomous systems with linear subsystems and state jumps, where it is shown that the special structure of such problems can lead to reduced computational effort. Examples illustrate the results.

AMS(MOS) subject classification: 34H05, 49M37, 65K05, 65K10, 93C10, 93C15

1 INTRODUCTION

A hybrid system is a dynamic system that involves both continuous and discrete event dynamics. The subsystem continuous dynamics are usually described by differential/difference equations and the discrete event dynamics are described by switching laws. Discontinuous jumps of continuous states may occur when the system switches from one subsystem to another. Examples of hybrid systems can be found in chemical processes, automotive systems, and electrical circuit systems.

Recently, optimal control problems for hybrid systems have attracted researchers from science and engineering disciplines. Many theoretical results which extend the classical maximum principle and/or the dynamic programming to such problems have appeared (see, e.g., [3, 4, 14, 16, 17, 18, 20, 23]). On the other hand, many computational methods have been proposed for finding numerical solutions to various subclasses of such problems (see, e.g., [1, 6, 7, 8, 9, 10, 11, 19, 21, 24]). A survey on computational methods can be found in [23].

As indicated in [23], most of the currently available computational methods demand significant amount of computation, or can only find approximations to local optimal solutions, even when a prespecified sequence of active subsystems is given. In this paper, we focus on optimal control problems for a class of hybrid systems in which each subsystem is autonomous (i.e., with no continuous input) and state jumps are present at the switching instants. Such problems arise naturally in the context of multimodal control, logic-based control systems [13], and impulsive control of discrete-continuous systems [12]. Given a prespecified sequence of active subsystems, we develop a very effective approach to find accurate numerical values of local optimal switching time instants. The approach is an extension of the results in [21] to such hybrid systems. Moreover, by taking advantage of autonomous subsystems, the approach only demands reasonable amount of computation and can obtain accurate derivative values as opposed to approximations in [21]. We note that the cost is actually a function of the switching time instants for such problems and use constrained nonlinear optimization techniques to locate the optimal switching time instants. To apply nonlinear optimization techniques, we first need to determine the values of the derivatives of the cost with respect to the switching instants. The main contributions of the paper are the derivation of the expressions of the derivatives and the computation of accurate values of these derivatives. Then the approach is applied to general quadratic problems for hybrid autonomous systems with linear subsystems and state jumps. The computation of the derivatives can further be simplified by utilizing the special structure of such problems.

It is worth noting that most of the available literature results on numerical solutions of hybrid systems optimal control problems are for discrete-time hybrid systems (e.g., [1, 10, 11]), or based on the discretizations of time and/or state spaces (e.g., [9, 19]). However, the discretization approaches may lead to combinatoric explosion and the solutions obtained may not be accurate enough. Unlike these results, the problems we consider in this paper are for continuous-time systems and the approach here is not based on discretization; hence our approach can provide us with accurate values of local minima. The closest literature results to our paper, as far as we are aware of, are [6, 7] which present closed-loop solutions to a special class of problems, i.e., infinite horizon problems for switched linear autonomous systems. However, our approach can deal with finite horizon problems with nonlinear subsystems, and with costs which are not necessarily quadratic, as opposed to infinite horizon problems with linear subsystems and quadratic costs in [6, 7]. In view of the above, we believe our results are new and contribute to the understanding and the solution of optimal control problems of hybrid systems.

The structure of the paper is as follows. In Section 2, we formulate the optimal

control problem and propose an algorithm for solving it. In Section 3, detailed derivations are presented to show how to obtain the expressions of the derivatives. In Section 4, the computation of some parameters for the evaluation of the derivatives is addressed. In Section 5, these results are applied to general quadratic problems for hybrid autonomous systems with linear subsystems and state jumps. Examples are given in Section 6. Section 7 concludes the paper. Our earlier results of the research in this paper can be found in [22, 25].

2 PROBLEM FORMULATION

In this paper, we consider a class of *hybrid autonomous systems* which are defined as follows. The hybrid system consists of autonomous subsystems (i.e., without continuous input)

$$\dot{x} = f_i(x), \quad f_i : \mathbb{R}^n \to \mathbb{R}^n, \quad i \in I = \{1, 2, \cdots, M\}. \tag{1}$$

and whenever the system dynamics switches from subsystem i_k to subsystem i_{k+1}, a discontinuous jump of the state x will occur, which are described by a function

$$x(t_k^+) = \gamma^{i_k, i_{k+1}}(x(t_k^-)) \tag{2}$$

where $x(t_k^+)$ and $x(t_k^-)$ are the righthand limit and lefthand limit of the state x at switching time instant t_k, respectively.

For such a hybrid system, one can control its state trajectory evolution by choosing appropriate switching sequences. Here a *switching sequence* σ in $[t_0, t_f]$ is defined as

$$\sigma = ((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K)), \tag{3}$$

with $0 \le K < \infty$, $t_0 \le t_1 \le \cdots \le t_K \le t_f$, and $i_k \in I$, $k = 0, 1, \cdots, K$. σ indicates that the system switches to subsystem i_k at time instant t_k.

Assumption 1 *In the following, we assume without loss of generality that a prespecified sequence of active subsystem is given as $(1, 2, \cdots, K, K+1)$, i.e., subsystem k is active in $[t_{k-1}, t_k)$ (subsystem $K+1$ in $[t_K, t_f]$).* □

Remark 1 Given any prespecified sequence of active subsystems, we can always make it satisfy Assumption 1 by relabeling the subsystem indices and even expanding the collection of subsystems (i.e., two subsystems may actually refer to the same actual subsystem). Under this assumption, we can simply denote the state jump function at the k-th switching as γ^k. □

We consider the following optimal control problem.

Problem 1 (Optimal Control Problem) *Consider a hybrid autonomous system with state jumps, which consists of subsystems $f_i(x)$, $i \in I$. Assume that a pre-specified sequence of active subsystems $(1, 2, \cdots, K, K+1)$ is given. Find optimal switching instants $t_1, \cdots, t_K (t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f)$ such that the corresponding continuous state trajectory x departs from a given initial state $x(t_0) = x_0$ and the cost*

$$J(t_1, \cdots, t_K) = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x)\, dt + \sum_{k=1}^{K} \psi^k(x(t_k^-)) \tag{4}$$

is minimized. Here t_0, t_f are given. □

Problem 1 is an optimal control problem in Bolza form with terminal cost ψ, running cost $\int_{t_0}^{t_f} L\, dt$, and switching cost ψ^k's. In general, ψ^k depends on the state just prior to switching and the state right after switching, i.e., $\psi^k = \psi^k(x(t_k^-), x(t_k^+))$. However, in the deterministic setting of this paper, $x(t_k^+)$ is related to $x(t_k^-)$ by (2). By substituting (2) into the expression of ψ^k, we can again reduce ψ^k to be only dependent on $x(t_k^-)$.

Assumption 2 *In the sequel, we assume that f_k's are continuously differentiable; ψ, L, ψ^k's, and γ^k's are twice continuously differentiable.* □

Remark 2 Due to the smoothness assumptions for f_i's, L, ψ, ψ^k's, and γ^k's, we can observe that a small disturbance of $(t_1, \cdots, t_K)$ will only cause a small disturbance of the J value. Furthermore, it is not difficult to show that the cost J is a continuously differentiable function in $(t_1, \cdots, t_K)$. □

2.1 An Algorithm

Note that Problem 1 is actually a constrained multivariable optimization problem

$$\begin{array}{c} \min_{\hat{t}} J(\hat{t}) \\ \text{subject to } \hat{t} \in T \end{array} \tag{5}$$

where $T \triangleq \{\hat{t} = (t_1, t_2, \cdots, t_K)^T | t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K \leq t_f\}$. The following algorithm can be adopted to solve such a nonlinear optimization problem.

Algorithm 1

(1). Set the iteration index $j = 0$. Choose an initial $\hat{t}^j$.

(2). Find $J(\hat{t}^j)$, $\frac{\partial J}{\partial \hat{t}}(\hat{t}^j)$ and $\frac{\partial^2 J}{\partial \hat{t}^2}(\hat{t}^j)$.

(3). Use some first-order or second-order feasible direction method (e.g., the gradient projection method or the constrained Newton's method [2]) to update $\hat{t}^j$ to be $\hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$ (here $d\hat{t}^j = -\left(\frac{\partial^2 J}{\partial \hat{t}^2}(\hat{t}^j)\right)^{-1} \left(\frac{\partial J}{\partial \hat{t}}(\hat{t}^j)\right)^T$ and the stepsize α^j can be chosen using, e.g., the Armijo's rule [2]). Set the iteration index $j = j + 1$.

(4). Repeat steps (2), (3) and (4), until a prespecified termination condition is satisfied (e.g. $\|\frac{\partial J}{\partial \hat{t}}(\hat{t}^j)\|_2 < \epsilon$ where ϵ is a given small number). □

In order to apply the above algorithm, one needs to find the values of the derivatives $\frac{\partial J}{\partial \hat{t}}$ and $\frac{\partial^2 J}{\partial \hat{t}^2}$ (step (2)). Let us elaborate more on step (2) in the sequel.

3 DIFFERENTIATIONS OF THE COST FUNCTION

In this section, we propose an approach based on the direct differentiations of the cost function to find the values of the derivatives $\frac{\partial J}{\partial \hat{t}}$ and $\frac{\partial^2 J}{\partial \hat{t}^2}$. This is extends the results in [21].

Assume that we have a nominal $\hat{t} = (t_1, \cdots, t_K)^T$ and the corresponding nominal state trajectory $x(t)$. For such nominal values, the cost is

$$J(t_1, \cdots, t_K) = \psi(x(t_f)) + \int_{t_0}^{t_1^-} L(x)\, dt + \int_{t_1^+}^{t_2^-} L(x)\, dt + \cdots + \int_{t_K^+}^{t_f} L(x)\, dt + \sum_{j=1}^{K} \psi^j(x(t_j^-)). \tag{6}$$

Since x_0 and t_0 are given in Problem 1, J will not be a function of them. Next we define the value function at the k-th switching instant to be

$$J^k(x(t_k^+), t_k, \cdots, t_K) \triangleq \psi(x(t_f)) + \int_{t_k^+}^{t_{k+1}^-} L(x)\, dt + \cdots + \int_{t_K^+}^{t_f} L(x)\, dt + \sum_{j=k+1}^{K} \psi^j(x(t_j^-)). \tag{7}$$

Note that, unlike J, J^k for $k = 1, \cdots, K$ will be a function of t_k and of the initial state $x(t_k^+)$ which depends on the trajectory before t_k. Also note that J^K does not have a switching cost and is

$$J^K(x(t_K^+), t_K) \triangleq \psi(x(t_f)) + \int_{t_K^+}^{t_f} L(x)\, dt. \tag{8}$$

The relationship between J^k and J^{k+1} is

$$J^k(x(t_k^+), t_k, \cdots, t_K) = \int_{t_k^+}^{t_{k+1}^-} L(x)\, dt + \psi^{k+1}(x(t_{k+1}^-)) + J^{k+1}(x(t_{k+1}^+), t_{k+1}, \cdots, t_K) \tag{9}$$

for $k = 1, 2, ..., K-1$. In order to make our presentation clear, in the sequel, we denote $\frac{\partial J^k}{\partial x}$ for the function J^k as a row vector J_x^k, $\frac{\partial^2 J^k}{\partial x^2}$ as an $n \times n$ matrix J_{xx}^k and so on.

3.1 Single Switching

Let us first consider the case of a single switching. Given a nominal t_1 and the corresponding nominal state trajectory $x(t)$, we denote by $\hat{x}(t)$ the state trajectory after a variation dt_1 has taken place. In the sequel, we adopt the following notational convention. We write f and f_x with a superscript $1-$ (resp. $1+$) whenever the corresponding active vector field at t_1- (resp. t_1+) is used for evaluation at $x(t_1^-)$ (resp. $x(t_1^+)$). Examples of this convention are $f^{1-} \triangleq f_1(x(t_1^-))$, $f^{1+} \triangleq f_2(x(t_1^+))$, $f_x^{1-} \triangleq \frac{\partial f_1}{\partial x}(x(t_1^-))$, $f_x^{1+} \triangleq \frac{\partial f_2}{\partial x}(x(t_1^+))$. Also, we simply write a function's name with a superscript $1-$ (resp. $1+$) whenever the corresponding function is evaluated at $x(t_1^-)$ (resp. $x(t_1^+)$). Examples are $J^{1+} \triangleq J^1(x(t_1^+), t_1)$, $J_x^{1+} \triangleq \frac{\partial J^1}{\partial x}(x(t_1^+), t_1)$, $L^{1-} \triangleq L(x(t_1^-))$, $L^{1+} \triangleq L(x(t_1^+))$, $L_x^{1-} \triangleq \frac{\partial L}{\partial x}(x(t_1^-))$, $\psi^{1-} \triangleq \psi^1(x(t_1^-)), \cdots$ (be careful to distinguish the values J^{1+}, J_x^{1+}, L^{1-}, $L_x^{1-}, \cdots$ from the functions $J^1(x(t_1^+), t_1)$, $J_x^1(x(t_1^+), t_1)$, $L(x)$, $L_x(x), \cdots$). We also simply denote the lefthand (resp. righthand) limit of $(t_1 + dt_1)$ as $t_1 + dt_1^-$ (resp. $t_1 + dt_1^+$) instead of the longer notation $(t_1 + dt_1)-$ (resp. $(t_1 + dt_1)+$).

Now consider $J(t_1)$ which can be expressed as

$$J(t_1) = \int_{t_0}^{t_1^-} L(x)\, dt + \psi^1(x(t_1^-)) + J^1(x(t_1^+), t_1). \tag{10}$$

For a small variation dt_1 of t_1, we have

$$J(t_1 + dt_1) = \int_{t_0}^{t_1+dt_1^-} L(\hat{x})\, dt + \psi^1(\hat{x}(t_1 + dt_1^-)) + J^1(\hat{x}(t_1 + dt_1^+), t_1 + dt_1). \tag{11}$$

There are three terms in (11). Let us consider the second order Taylor expansion of each term. In the following derivations we denote $dx(t_1^-) \triangleq \hat{x}(t_1 + dt_1^-) - x(t_1^-)$ and $dx(t_1^+) \triangleq \hat{x}(t_1 + dt_1^+) - x(t_1^+)$.

Consider the first term $\int_{t_0}^{t_1+dt_1^-} L(\hat{x})\, dt$ in (11), if $dt_1 \geq 0$, we have

$$\begin{aligned} \int_{t_0}^{t_1+dt_1^-} L(\hat{x})\, dt &= \int_{t_0}^{t_1^-} L(x)\, dt + \int_{t_1^-}^{t_1+dt_1^-} L(\hat{x})\, dt \\ &= \int_{t_0}^{t_1^-} L(x)\, dt + L^{1-} dt_1 + \frac{1}{2} dt_1 L_x^{1-} dx(t_1^-) + \text{H.O.T.} \end{aligned} \tag{12}$$

where H.O.T. stands for Higher Order Terms. Note that in deriving (11), we have used the relationship $\hat{x}(t_1^-) = x(t_1^-)$. If $dt_1 < 0$, we have

$$\begin{aligned} \int_{t_0}^{t_1+dt_1^-} L(\hat{x})\, dt &= \int_{t_0}^{t_1^-} L(x)\, dt + \int_{t_1^-}^{t_1+dt_1^-} L(x)\, dt \\ &= \int_{t_0}^{t_1^-} L(x)\, dt + L^{1-} dt_1 + \frac{1}{2} dt_1 L_x^{1-} dx(t_1^-) + \text{H.O.T.} \end{aligned} \tag{13}$$

which has the same expression as (12) although the derivation is slightly different.

For the second term in (11), we have

$$\begin{aligned} \psi^1(\hat{x}(t_1 + dt_1^-)) &= \psi^1(x(t_1^-) + dx(t_1^-)) \\ &= \psi^{1-} + \psi_x^{1-} dx(t_1^-) + \frac{1}{2}(dx(t_1^-))^T \psi_{xx}^{1-} dx(t_1^-) + \text{H.O.T.} \end{aligned} \tag{14}$$

For the third term in (11), we have the second order expansion

$$\begin{aligned} &J^1(\hat{x}(t_1 + dt_1^+), t_1 + dt_1) = J^1(x(t_1^+) + dx(t_1^+), t_1 + dt_1) \\ &\quad = J^{1+} + J_x^{1+} dx(t_1^+) + J_{t_1}^{1+} dt_1 + \frac{1}{2}(dx(t_1^+))^T J_{xx}^{1+} dx(t_1^+) + \frac{1}{2} J_{t_1 t_1}^{1+} dt_1^2 \\ &\quad\quad + dt_1 J_{t_1 x}^{1+} dx(t_1^+) + \text{H.O.T.} \end{aligned} \tag{15}$$

In order to express (11) into second order expansions with respect to dt_1, we need to find the second order expansions of $dx(t_1^-)$ and $dx(t_1^+)$ in terms of dt_1. First note that

$$dx(t_1^-) \triangleq \hat{x}(t_1 + dt_1^-) - x(t_1^-) = f^{1-} dt_1 + \frac{1}{2} f_x^{1-} f^{1-} dt_1^2 + o(dt_1^2). \tag{16}$$

Note that in (16), $o(dt_1^2)$ refers to a column vector with each element being $o(dt_1^2)$. We will not explicitly mention this later in the paper since it will be clear from the context. Next we have

$$\begin{aligned} dx(t_1^+) &\triangleq \hat{x}(t_1 + dt_1^+) - x(t_1^+) = \gamma^1(\hat{x}(t_1 + dt_1^-)) - \gamma^1(x(t_1^-)) \\ &= \gamma_x^{1-} dx(t_1^-) + \frac{1}{2} \begin{bmatrix} (dx(t_1^-))^T \frac{\partial^2 \gamma_{(1)}^1(x(t_1^-))}{\partial x^2} dx(t_1^-) \\ \vdots \\ (dx(t_1^-))^T \frac{\partial^2 \gamma_{(n)}^1(x(t_1^-))}{\partial x^2} dx(t_1^-) \end{bmatrix} + \text{H.O.T.} \end{aligned} \tag{17}$$

where $\gamma_{(j)}^1$ refers to the j-th element of the vector-valued function γ^1. Note that

$$\begin{bmatrix} (dx(t_1^-))^T \frac{\partial^2 \gamma_{(1)}^1(x(t_1^-))}{\partial x^2} dx(t_1^-) \\ \vdots \\ (dx(t_1^-))^T \frac{\partial^2 \gamma_{(n)}^1(x(t_1^-))}{\partial x^2} dx(t_1^-) \end{bmatrix} = \begin{bmatrix} (f^{1-})^T \frac{\partial^2 \gamma_{(1)}^1(x(t_1^-))}{\partial x^2} \\ \vdots \\ (f^{1-})^T \frac{\partial^2 \gamma_{(n)}^1(x(t_1^-))}{\partial x^2} \end{bmatrix} f^{1-} dt_1^2 + o(dt_1^2). \tag{18}$$

If we define

$$\xi^{1-} \triangleq \begin{bmatrix} (f^{1-})^T \frac{\partial^2 \gamma_{(1)}^1(x(t_1^-))}{\partial x^2} \\ \vdots \\ (f^{1-})^T \frac{\partial^2 \gamma_{(n)}^1(x(t_1^-))}{\partial x^2} \end{bmatrix} \tag{19}$$

and substitute (18) into (17), we obtain

$$dx(t_1^+) = \gamma_x^{1-} f^{1-} dt_1 + \frac{1}{2}(\gamma_x^{1-} f_x^{1-} + \xi^{1-}) f^{1-} dt_1^2 + o(dt_1^2) \tag{20}$$

Substituting (16) and (20) into (12), (14) and (15) and summing them, we obtain

$$
\begin{aligned}
& J(t_1+dt_1) \\
= \; & J(t_1) + L^{1-}dt_1 + \frac{1}{2}dt_1 L_x^{1-}dx(t_1^-) + \psi_x^{1-}dx(t_1^-) + \frac{1}{2}\left(dx(t_1^-)\right)^T \psi_{xx}^{1-}dx(t_1^-) \\
& + J_x^{1+}dx(t_1^+) + J_{t_1}^{1+}dt_1 + \frac{1}{2}\left(dx(t_1^+)\right)^T J_{xx}^{1+}dx(t_1^+) + \frac{1}{2}J_{t_1t_1}^{1+}dt_1^2 \\
& - dt_1 J_{t_1x}^{1+}dx(t_1^+) + \text{H.O.T.} \\
= \; & J(t_1) + (L^{1-} + \psi_x^{1-}f^{1-} + J_x^{1+}\gamma_x^{1-}f^{1-} + J_{t_1}^{1+})dt_1 \\
& + \frac{1}{2}\Big(L_x^{1-}f^{1-} + \psi_x^{1-}f_x^{1-}f^{1-} + (f^{1-})^T\psi_{xx}^{1-}f^{1-} + J_x^{1+}(\gamma_x^{1-}f_x^{1-} + \xi^{1-})f^{1-} \\
& + (f^{1-})^T(\gamma_x^{1-})^T J_{xx}^{1+}\gamma_x^{1-}f^{1-} + J_{t_1t_1}^{1+} + 2J_{t_1x}^{1+}\gamma_x^{1-}f^{1-}\Big)dt_1^2 + o(dt_1^2) \\
\triangleq \; & J(t_1) + J_{t_1}dt_1 + \frac{1}{2}J_{t_1t_1}dt_1^2 + o(dt_1^2)
\end{aligned}
\tag{21}
$$

Now let us consider $J^1(x(t_1^+), t_1)$ which is the value function for the given nominal $x(t_1^+)$ and t_1. The following dynamic programming equation holds for it

$$J_{t_1}^{1+} = -J_x^{1+}f^{1+} - L^{1+} \tag{22}$$

Note that (22) can be derived similarly to the HJB equation. However, the difference between it and the HJB equation is that (22) holds for any trajectory that is not necessarily optimal (for more details see [5]).

By differentiating (22), we obtain

$$J_{t_1x}^{1+} = -(f^{1+})^T J_{xx}^{1+} - J_x^{1+}f_x^{1+} - L_x^{1+} \tag{23}$$

$$J_{t_1t_1}^{1+} = -J_{t_1x}^{1+}f^{1+} = (f^{1+})^T J_{xx}^{1+}f^{1+} + (J_x^{1+}f_x^{1+} + L_x^{1+})f^{1+} \tag{24}$$

Substituting these into (21) we have

$$J_{t_1} = L^{1-} - L^{1+} + \psi_x^{1-}f^{1-} + J_x^{1+}(\gamma_x^{1-}f^{1-} - f^{1+}) \tag{25}$$

$$
\begin{aligned}
J_{t_1t_1} = \; & (L_x^{1-} - L_x^{1+}\gamma_x^{1-})f^{1-} + \psi_x^{1-}f_x^{1-}f^{1-} + (f^{1-})^T\psi_{xx}^{1-}f^{1-} \\
& + J_x^{1+}(\gamma_x^{1-}f_x^{1-} + \xi^{1-} - f_x^{1+}\gamma_x^{1-})f^{1-} - (J_x^{1+}f_x^{1+} + L_x^{1+})(\gamma_x^{1-}f^{1-} - f^{1+}) \\
& + (\gamma_x^{1-}f^{1-} - f^{1+})^T J_{xx}^{1+}(\gamma_x^{1-}f^{1-} - f^{1+})
\end{aligned}
\tag{26}
$$

3.2 Two or More Switchings

In order to construct a second-order optimization algorithm for hybrid systems with two or more switchings, we need more information to derive the derivatives of J with respect to the t_k's. Let us first consider the case of two switchings. Assume that a system switches from subsystem 1 to 2 at t_1 and from subsystem 2 to 3 at t_2

$(t_0 \le t_1 \le t_2 \le t_f)$. The cost then is

$$J(t_1, t_2) = \int_{t_0}^{t_1^-} L(x)\, dt + \psi^1(x(t_1^-)) + J^1(x(t_1^+), t_1, t_2) \tag{27}$$

$$= \int_{t_0}^{t_1^-} L(x)\, dt + \psi^1(x(t_1^-)) + \int_{t_1^+}^{t_2^-} L(x)\, dt + \psi^2(x(t_2^-)) + J^2(x(t_2^+), t_2). \tag{28}$$

Using (27), by holding t_2 fixed, J_{t_1}, $J_{t_1t_1}$ can be derived similarly to that in Section 3.1. On the other hand, if t_1 is held fixed, the first two terms in (28) will not contribute to the coefficients J_{t_2}, $J_{t_2t_2}$. J_{t_2}, $J_{t_2t_2}$ can then be derived using the expansion of the last three terms in (28) with respect to dt_2 similarly to that in Section 3.1. However, we need additional information to derive $J_{t_1t_2}$. Arguments from the calculus of variations will be used in the following to derive it. Let us first define the important notion of incremental change which will be used in the sequel.

Definition 1 (Incremental Change) *Given any variations dt_1 and dt_2, we define $\delta x(t)$, $\min\{t_1^+, t_1 + dt_1^+\} \le t \le \max\{t_2^-, t_2 + dt_2^-\}$ to be the incremental change of the state due to dt_1 and dt_2. In detail, it is defined as follows (see figure 1).*

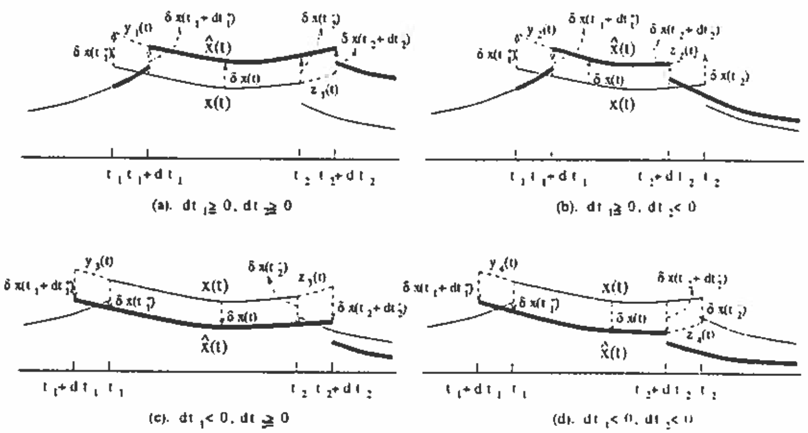

Figure 1: The incremental change $\delta x(t)$ for (a). $dt_1 \ge 0$, $dt_2 \ge 0$; (b). $dt_1 \ge 0$, $dt_2 < 0$; (c). $dt_1 < 0$, $dt_2 \ge 0$; (d). $dt_1 < 0$, $dt_2 < 0$.

Case 1: $dt_1 \ge 0, dt_2 \ge 0$ *(see figure 1(a))*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), & t \in [t_1 + dt_1^+, t_2^-] \\ y_1(t) - x(t), & t \in [t_1^+, t_1 + dt_1^+] \\ \hat{x}(t) - z_1(t), & t \in [t_2^-, t_2 + dt_2^-] \end{cases} \tag{29}$$

where $y_1(t)$ is the solution of

$$\begin{cases} \dot{y}_1(t) = f_2(y_1(t)), \ t \in [t_1^+, t_1 + dt_1^-] \\ y_1(t_1 + dt_1^-) = \hat{x}(t_1 + dt_1^-) \end{cases} \tag{30}$$

and $z_1(t)$ is the solution of

$$\begin{cases} \dot{z}_1(t) = f_2(z_1(t)), \ t \in [t_2^-, t_2 + dt_2^-] \\ z_1(t_2^-) = x(t_2^-). \end{cases} \tag{31}$$

Case 2: *$dt_1 \geq 0, dt_2 < 0$ (see figure 1(b))*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1 + dt_1^+, t_2 + dt_2^-] \\ y_2(t) - x(t), \ t \in [t_1^+, t_1 + dt_1^+] \\ z_2(t) - x(t), \ t \in [t_2 + dt_2^-, t_2^-] \end{cases} \tag{32}$$

where $y_2(t)$ is the solution of

$$\begin{cases} \dot{y}_2(t) = f_2(y_2(t)), \ t \in [t_1^+, t_1 + dt_1^+] \\ y_2(t_1 + dt_1^+) = \hat{x}(t_1 + dt_1^+) \end{cases} \tag{33}$$

and $z_2(t)$ is the solution of

$$\begin{cases} \dot{z}_2(t) = f_2(z_2(t)), \ t \in [t_2 + dt_2^-, t_2^-] \\ z_2(t_2 + dt_2^-) = \hat{x}(t_2 + dt_2^-). \end{cases} \tag{34}$$

Case 3: *$dt_1 < 0, dt_2 \geq 0$ (see figure 1(c))*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1^+, t_2^-] \\ \hat{x}(t) - y_3(t), \ t \in [t_1 + dt_1^+, t_1^+] \\ \hat{x}(t) - z_3(t), \ t \in [t_2^-, t_2 + dt_2^-] \end{cases} \tag{35}$$

where $y_3(t)$ is the solution of

$$\begin{cases} \dot{y}_3(t) = f_2(y_3(t)), \ t \in [t_1 + dt_1^+, t_1^+] \\ y_3(t_1^+) = x(t_1^+) \end{cases} \tag{36}$$

and $z_3(t)$ is the solution of

$$\begin{cases} \dot{z}_3(t) = f_2(z_3(t)), \ t \in [t_2^-, t_2 + dt_2^-] \\ z_3(t_2^-) = x(t_2^-). \end{cases} \tag{37}$$

Case 4: *$dt_1 < 0, dt_2 < 0$ (see figure 1(d))*

In this case, $\delta x(t)$ is defined to be

$$\delta x(t) = \begin{cases} \hat{x}(t) - x(t), \ t \in [t_1^+, t_2 + dt_2^-] \\ \hat{x}(t) - y_4(t), \ t \in [t_1 + dt_1^+, t_1^+] \\ z_4(t) - x(t), \ t \in [t_2 + dt_2^-, t_2^-] \end{cases} \tag{38}$$

where $y_4(t)$ is the solution of

$$\begin{cases} \dot{y}_4(t) = f_2(y_4(t)), \ t \in [t_1 + dt_1^+, t_1^+] \\ y_4(t_1^+) = x(t_1^+) \end{cases} \tag{39}$$

and $z_4(t)$ is the solution of

$$\begin{cases} \dot{z}_4(t) = f_2(z_4(t)), \ t \in [t_2 + dt_2^-, t_2^-] \\ z_4(t_2 + dt_2^-) = \hat{x}(t_2 + dt_2^-). \end{cases} \tag{40}$$

□

Remark 3 Note that $\delta x(t)$ defines the difference between $\hat{x}(t)$ and $x(t)$ in the time interval where subsystem 2 is active. Moreover, by extending the trajectories $\hat{x}$ and x under the dynamics of subsystem 2 to the time interval $[\min\{t_1^+, t_1 + dt_1^+\}, \max\{t_2^-, t_2 + dt_2^-\}]$ in which at least one of $\hat{x}(t)$ and $x(t)$ evolves along subsystem 2, $\delta x(t)$ even defines the difference for this interval. □

In the followings, the expressions of $\delta x(t_2^-)$, $dx(t_2^-)$, and $dx(t_2^+)$ are derived.

Lemma 1 *The expressions of $\delta x(t_2^-)$ and $\delta x(t_2 + dt_2^-)$ are as follows*

$$\begin{aligned} \delta x(t_2^-) &= A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + o(dt_1), \quad (41) \\ \delta x(t_2 + dt_2^-) &= A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + f_x^{2-} A(t_2, t_1)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 dt_2 \\ &\quad + (\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}), \quad (42) \end{aligned}$$

where $A(t_2^-, t_1^+)$ is the state transition matrix for the variational equation

$$\dot{y}(t) = \frac{\partial f_2(x(t))}{\partial x} y(t) \tag{43}$$

for $y(t), t \in [t_1^+, t_2^-]$; in (43), x is the current nominal state.

Proof: See Appendix. □

In fact, from the proof of Lemma 1 (see Appendix), we can observe that $\delta x(t) = A(t, t_1^+)\delta x(t_1^+) + (\text{H.O.T. in } \delta x(t_1^+)) = A(t, t_1^+)\delta x(t_1^+) + o(dt_1)$ for any $t \in [\min\{t_1^+, t_1 + dt_1^+\}, \max\{t_2^-, t_2 + dt_2^-\}]$. The following important principle can be obtained directly from this observation. We refer to it as *the forward decoupling principle*. It reveals some intrinsic relationship among different switching instants.

The Forward Decoupling Principle:

(a). The value of the incremental change $\delta x(t_1^+)$ at t_1^+ does not depend on dt_2.

(b). The value of the incremental change $\delta x(t_2^-)$ at t_2^- does depend on dt_1. □

The forward decoupling principle tells us that a variation of an earlier switching instant will affect the value of the incremental change at a later switching instant, but not vice versa.

Lemma 2 *The expressions of $dx(t_2^-)$ (i.e., $\hat{x}(t_2+dt_2^-) - x(t_2^-)$) and $dx(t_2^+)$ (i.e., $\hat{x}(t_2+dt_2^+) - x(t_2^+)$) are*

$$\begin{aligned} dx(t_2^-) &= A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2 \\ &\quad + f^{2-}dt_2 + (\textit{terms in } dt_1^2, dt_2^2 \textit{ and H.O.T.}), \qquad (44) \\ dx(t_2^+) &= \gamma_x^{2-}A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1 + (\gamma_x^{2-}f_x^{2-} + \xi^{2-})A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} \\ &\quad - f^{1+})dt_1dt_2 + \gamma_x^{2-}f^{2-}dt_2 + (\textit{terms in } dt_1^2, dt_2^2 \textit{ and H.O.T.}) \qquad (45) \end{aligned}$$

where ξ^{2-} is defined similarly to ξ^{1-} in (19) as

$$\xi^{2-} \triangleq \begin{bmatrix} (f^{2-})^T \frac{\partial^2 \gamma_{(1)}^2 (x(t_2^-))}{\partial x^2} \\ \vdots \\ (f^{2-})^T \frac{\partial^2 \gamma_{(n)}^2 (x(t_2^-))}{\partial x^2} \end{bmatrix} \qquad (46)$$

with $\gamma_{(j)}^2$ referring to the j-th element of the vector-valued function γ^2.

Proof: See Appendix. □

Remark 4 It is very important to point out that in the expressions of $dx(t_2^-)$ and $dx(t_2^+)$, we deliberately express the terms $f_x^{2-}A(t_2^-,t_1^-)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2$ and $(\gamma_x^{2-}f_x^{2-} + \xi^{2-})A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2$ explicitly because they will contribute to the coefficient of dt_1dt_2. □

Now that we have the expressions for $\delta x(t_2^-)$, $\delta x(t_2+dt_2^-)$, $dx(t_2^-)$, and $dx(t_2^+)$, we are ready to derive the coefficient for dt_1dt_2 in the expansion of

$$\begin{aligned} J(t_1+dt_1, t_2+dt_2) &= \int_{t_0}^{t_1+dt_1^-} L(\hat{x}(t))\,dt + \psi^1(\hat{x}(t_1+dt_1^-)) \\ &\quad + \int_{t_1+dt_1^+}^{t_2+dt_2^-} L(\hat{x}(t))\,dt + \psi^2(\hat{x}(t_2+dt_2^-)) + J^2(\hat{x}(t_2+dt_2^+), t_2+dt_2). \qquad (47) \end{aligned}$$

There are five terms in (47). Let us look at each term's Taylor expansion in order to find its contribution to the coefficient of dt_1dt_2.

By using the forward decoupling principle, we can conclude that none of $\delta x(t_1^-)$, $\delta x(t_1^+)$, $dx(t_1^-)$, and $dx(t_1^+)$ will depend on dt_2. Consequently the Taylor expansion of the first two terms will not have terms in dt_2, dt_2^2 and dt_1dt_2. Therefore the first two terms will not contribute to the coefficient of dt_1dt_2.

For the third term in (47), we have the following Lemma.

Lemma 3 *The contribution of $\int_{t_1+dt_1^+}^{t_2+dt_2^-} L(\hat{x})\,dt$ to the coefficient of dt_1dt_2 is*

$$L_x^{2-}A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+}). \qquad (48)$$

Proof: See Appendix. □

The fourth term in (47) can be expanded as

$$\begin{aligned} \psi^2(\hat{x}(t_2+dt_2^-)) &= \psi^2(x(t_2^-) + dx(t_2^-)) \\ &= \psi^{2-} + \psi_x^{2-}dx(t_2^-) + \frac{1}{2}(dx(t_2^-))^T \psi_{xx}^{2-} dx(t_2^-) + \text{H.O.T.} \qquad (49) \end{aligned}$$

Therefore the contribution to the coefficient of dt_1dt_2 by the fourth term is

$$(\psi_x^{2-}f_x^{2-} + (f^{2-})^T\psi_{xx}^{2-})A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+}). \qquad (50)$$

For the fifth term in (47), similar to the single switching case, we can obtain its Taylor expansion as

$$\begin{aligned} J^2(\hat{x}(t_2+dt_2^+), t_2+dt_2) &= J^{2+} + J_x^{2+}dx(t_2^+) + J_{t_2}^{2+}dt_2 + \frac{1}{2}(dx(t_2^+))^T J_{xx}^{2+} dx(t_2^+) \\ &\quad + \frac{1}{2}J_{t_2t_2}^{2+}dt_2^2 + dt_2 J_{t_2x}^{2+}dx(t_2^+) + \text{H.O.T.} \qquad (51) \end{aligned}$$

In (51), the terms that will possibly contribute to the coefficient of dt_1dt_2 are those containing $dx(t_2^+)$. They are

$$J_x^{2+}dx(t_2^+),\ \frac{1}{2}(dx(t_2^+))^T J_{xx}^{2+}dx(t_2^+),\ dt_2J_{t_2x}^{2+}dx(t_2^+). \qquad (52)$$

Substituting the expression of $dx(t_2^+)$ into (52) and summing them, we obtain the contribution of the fifth term to the coefficient of dt_1dt_2 as

$$(J_x^{2+}(\gamma_x^{2-}f_x^{2-} + \xi^{2-}) + (f^{2-})^T(\gamma_x^{2-})^T J_{xx}^{2+}\gamma_x^{2-} + J_{t_2x}^{2+}\gamma_x^{2-})A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+}). \qquad (53)$$

Summing (48), (50), and (53) and also substituting into the sum the expression of $J_{t_2x}^{2+}$ which can be obtained similarly to the expression of $J_{t_1x}^{1+}$ in (23), we conclude that the coefficient of dt_1dt_2 (i.e., $J_{t_1t_2}$ in the expansion of $J(t_1+dt_1, t_2+dt_2)$ is

$$\begin{aligned} J_{t_1t_2} &= (L_x^{2-} + \psi_x^{2-}f_x^{2-} + (f^{2-})^T\psi_{xx}^{2-} + J_x^{2+}(\gamma_x^{2-}f_x^{2-} + \xi^{2-}) + (f^{2-})^T(\gamma_x^{2-})^T J_{xx}^{2+}\gamma_x^{2-} \\ &\quad + J_{t_2x}^{2+}\gamma_x^{2-})A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+}) \\ &= (L_x^{2-} - L_x^{2+}\gamma_x^{2-} + \psi_x^{2-}f_x^{2-} + (f^{2-})^T\psi_{xx}^{2-} + J_x^{2+}(\gamma_x^{2-}f_x^{2-} + \xi^{2-} - f_x^{2+}\gamma_x^{2-}) \\ &\quad + (\gamma_x^{2-}f^{2-} - f^{2+})^T J_{xx}^{2+}\gamma_x^{2-})A(t_2^-,t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+}). \qquad (54) \end{aligned}$$

Remark 5 The above results still holds even when $t_1 = t_2$ (we can consider $t_2 > t_1$ first and then let $t_2 \to t_1$ to prove this). □

The above result can also be similarly extended to the case of K switchings to relate $dx(t_l^-)$, $dx(t_l^-)$ to dt_l and dt_k ($k < l$). The expression for $J_{t_k t_l}$ can similarly be obtained. We summarize and extend the results obtained in this section by the following theorem.

Theorem 1 *The cost J in Problem 1 satisfies*

$$\begin{aligned} & J(t_1 + dt_1, t_2 + dt_2, \cdots, t_K + dt_K) \\ = \; & J(t_1, t_2, \cdots, t_K) + \sum_{k=1}^{K} J_{t_k} dt_k + \frac{1}{2}\sum_{k=1}^{K} J_{t_k t_k} dt_k^2 + \sum_{1 \le k < l \le K} J_{t_k t_l} dt_k dt_l \\ & + (\textit{higher order terms}) \end{aligned} \tag{55}$$

where

$$J_{t_k} = L^{k-} - L^{k+} + \psi_x^{k-} f^{k-} + J_x^{k+}(\gamma_x^{k-} f^{k-} - f^{k+}), \tag{56}$$

$$\begin{aligned} J_{t_k t_k} = \; & (L_x^{k-} - L_x^{k+}\gamma_x^{k-}) f^{k-} + \psi_x^{k-} f_x^{k-} f^{k-} + (f^{k-})^T \psi_{xx}^{k-} f^{k-} \\ & + J_x^{k+}(\gamma_x^{k-} f_x^{k-} + \xi^{k-} - f_x^{k+}\gamma_x^{k-}) f^{k-} - (J_x^{k+} f_x^{k+} + L_x^{k+})(\gamma_x^{k-} f^{k-} - f^{k+}) \\ & + (\gamma_x^{k-} f^{k-} - f^{k+})^T J_{xx}^{k+} (\gamma_x^{k-} f^{k-} - f^{k+}), \end{aligned} \tag{57}$$

for any $k = 1, \cdots, K$, and

$$\begin{aligned} J_{t_k t_l} = \; & (L_x^{l-} - L_x^{l+}\gamma_x^{l-} + \psi_x^{l-} f_x^{l-} + (f^{l-})^T \psi_{xx}^{l-} + J_x^{l+}(\gamma_x^{l-} f_x^{l-} + \xi^{l-} - f_x^{l+}\gamma_x^{l-}) \\ & + (\gamma_x^{l-} f^{l-} - f^{l+})^T J_{xx}^{l+}\gamma_x^{l-}) H(t_l^-, t_k^+)(\gamma_x^{k-} f^{k-} - f^{k+}), \end{aligned} \tag{58}$$

for any $1 \le k < l \le K$. Here $H(t_l^-, t_k^+)$ is the state transition matrix under state jumps

$$H(t_l^-, t_k^+) = A(t_l^-, t_{l-1}^+)\gamma_x^{(l-1)-} A(t_{l-1}^-, t_{l-2}^+) \bullet \cdots \bullet \gamma_x^{(k+1)-} A(t_{k+1}^-, t_k^+) \tag{59}$$

where $A(t_{j+1}^-, t_j^+)$, $k \le j \le l-1$ is the state transition matrix for the time interval $[t_j^+, t_{j+1}^-]$ for the variational equation

$$\dot{y}(t) = \frac{\partial f_{j+1}(x(t))}{\partial x} y(t). \tag{60}$$

Also here

$$\xi^{k-} \triangleq \begin{bmatrix} (f^{k-})^T \frac{\partial \gamma_{(1)x}^k (x(t_k^-))}{\partial x} \\ \vdots \\ (f^{k-})^T \frac{\partial \gamma_{(n)x}^k (x(t_k^-))}{\partial x} \end{bmatrix}, \quad k = 1, \cdots, K, \tag{61}$$

with $\gamma_{(j)}^k$ referring to the j-th element of the vector-valued function γ^k. □

Remark 6 In general in the interval $[t_k^+, t_l^-]$, there will be discontinuous jumps and they must be taken into consideration when we consider the incremental change $\delta(x)$ in this interval, hence $H(t_l^-, t_k^+)$ appears in (58) (instead of $A(t_l^-, t_k^+)$) if we follow the similar derivations as in the two switchings case. In the special case when $l = k + 1$, $H(t_l^-, t_k^+)$ is reduced to $A(t_{k+1}^-, t_k^+)$. □

4 COMPUTATION OF $H(t_l^-, t_k^+)$, J_x^{k+}, AND J_{xx}^{k-}

It should be pointed out that in Theorem 1, in order to compute J_{t_k}, $J_{t_k t_k}$ and $J_{t_k t_l}$, we need to know the values of $H(t_l^-, t_k^+)$, J_x^{k+} and J_{xx}^{k-}. However, given nominal $\hat{t}$ and x, these values are not readily available. In general, numerical methods need to be used to compute their values. An efficient numerical method based on solving additional initial value ordinary differential equations (ODEs) with jumps is developed in this section.

First note that if $l = k + 1$ then $H(t_l^-, t_k^+)$ is equal to $A(t_{k+1}^-, t_k^+)$, which is the state transition matrix for

$$\dot{y}(t) = \frac{\partial f_{k+1}(x(t))}{\partial x} y(t). \tag{62}$$

To find its value, we can first find the solution $y^{(1)}(t), \cdots, y^{(n)}(t)$ corresponding to initial conditions

$$y^{(1)}(t_k^+) = e_1, \; \cdots, \; y^{(n)}(t_k^+) = e_n \tag{63}$$

respectively, where e_j is the unit column vector with all 0's except that the j-th element being 1, $j = 1, 2, \cdots, n$. From linear systems theory, $A(t_{k+1}^-, t_k^+)$, is equal to the square matrix whose j-th column is $y^{(j)}(t_{k+1}^-)$, i.e., in this case

$$H(t_l^-, t_k^+) = A(t_{k+1}^-, t_k^+) = [y^{(1)}(t_{k+1}^-), \cdots, y^{(n)}(t_{k+1}^-)]. \tag{64}$$

Now if $l > k$, a similar method can be adopted to compute $H(t_l^-, t_k^+)$. Instead of solving initial value ODEs for $y^{(j)}$'s, $y^{(j)}(t)$'s are now obtained by solve the following ODEs with jumps with initial conditions (63).

$$\begin{cases} \dot{y}(t) = \frac{\partial f_{j+1}(x(t))}{\partial x} y(t), \text{ for } t_j^+ \le t \le t_{j+1}^-, \\ y(t_j^+) = \gamma_x^{j-} y(t_j^-), k < j < l. \end{cases} \tag{65}$$

We then have

$$H(t_l^-, t_k^+) = [y^{(1)}(t_l^-), \cdots, y^{(n)}(t_l^-)]. \tag{66}$$

To obtain the value of J_x^k, note that

$$J^k(x(t_k^+), t_k, \cdots, t_K) = \psi(x(t_f)) + \sum_{j=k}^{K} \int_{t_j^+}^{t_{j+1}^-} L(x(t))\, dt + \sum_{j=k+1}^{K} \psi^j(x(t_j^-)). \tag{67}$$

Note that, for simplicity of notation, we regard t_f as t_{K+1}^- in (67).

If $x(t_k^+)$ has a variation $\delta x(t_k^+)$, then

$$
\begin{aligned}
& J^k(x(t_k^+) + \delta x(t_k^+), t_k, \cdots, t_K) \\
= \; & \psi(x(t_f) + H(t_f, t_k^+)\delta x(t_k^+) + \text{H.O.T.}) + \sum_{j=k}^{K} \int_{t_j^+}^{t_{j+1}^-} L(x(t) + H(t, t_k^+)\delta x(t_k^+) \\
& + \text{H.O.T}) \, dt + \sum_{j=k+1}^{K} \psi^j(x(t_j^-) + H(t_j^-, t_k^+)\delta x(t_k^+) + \text{H.O.T.}) \\
= \; & J^k(x(t_k^+), t_k, \cdots, t_K) + \Big(\psi_x(x(t_f))H(t_f, t_k^+) + \sum_{j=k}^{K} \int_{t_j^+}^{t_{j+1}^-} L_x(x(t))H(t, t_k^+) \, dt \\
& + \sum_{j=k+1}^{K} \psi_x^j(x(t_j^-))H(t_j^-, t_k^+)\Big)\delta x(t_k^+) + \text{H.O.T.}
\end{aligned}
\tag{68}
$$

Hence

$$
J_x^{k+} = \psi_x(x(t_f))H(t_f, t_k^+) + \sum_{j=k}^{K} \int_{t_j^+}^{t_{j+1}^-} L_x(x(t))H(t, t_k^+) \, dt + \sum_{j=k+1}^{K} \psi_x^j(x(t_j^-))H(t_j^-, t_k^+).
\tag{69}
$$

If we apply the similar procedure by varying $x(t_k^+)$ as in (68) to $J_x^k(x(t_k^+), t_k, \cdots, t_K)$, we can obtain

$$
\begin{aligned}
J_{xx}^{k+} = \; & H^T(t_f, t_k^+)\psi_{xx}(x(t_f))H(t_f, t_k^+) + \sum_{j=k}^{K} \int_{t_j^+}^{t_{j+1}^-} H^T(t, t_k^+)L_{xx}(x(t))H(t, t_k^+) \, dt \\
& + \sum_{j=k+1}^{K} H^T(t_j^-, t_k^+)\psi_{xx}^j(x(t_j^-))H(t_j^-, t_k^+).
\end{aligned}
\tag{70}
$$

From the above discussions, we find that $H(t_l^-, t_k^+)$ can be obtained by solving ODEs with jumps (65) along with initial conditions (63). $H(t_f, t_k^+)$ can be obtained in the same fashion. J_x^{k+} and J_{xx}^{k+} are in the forms (69) and (70) which can be easily rewritten as ODEs with jumps. By solving the following initial value ODEs with jumps from t_k^+ to t_f (along with the hybrid system ODEs with jumps which provides us with the state trajectory)

$$
\begin{cases}
\dot{H}(t, t_k^+) = \frac{\partial f_{j+1}(x(t))}{\partial x} H(t, t_k^+), \; t_j^+ \le t \le t_{j+1}^-, \\
H(t_j^+, t_k^+) = \gamma_x^{j-} H(t_j^-, t_k^+),
\end{cases}
\tag{71}
$$

$$
\begin{cases}
\dot{\eta}_1 = L_x(x(t))H(t, t_k^+), \; t_j^+ \le t \le t_{j+1}^-, \\
\eta_1(t_j^+) = \eta_1(t_j^-) + \psi_x^j(x(t_j^-))H(t_j^-, t_k^+),
\end{cases}
\tag{72}
$$

$$
\begin{cases}
\dot{\eta}_2 = H^T(t, t_k^+)L_{xx}(x(t))H(t, t_k^+), \; t_j^+ \le t \le t_{j+1}^-, \\
\eta_2(t_j^+) = \eta_2(t_j^-) + H^T(t_j^-, t_k^+)\psi_{xx}^j(x(t_j^-))H(t_j^-, t_k^+),
\end{cases}
\tag{73}
$$

along with initial conditions (63) and

$$
\begin{aligned}
\eta_1(t_k) &= 0_{1\times n}, & (74)\\
\eta_2(t_k) &= 0_{n\times n}, & (75)
\end{aligned}
$$

we can find the value of $H(t_f, t_k^+)$ and can obtain the values of J_x^{k+}, J_{xx}^{k+} from

$$
\begin{aligned}
J_x^{k+} &= \psi_x(x(t_f))H(t_f, t_k^+) + \eta_1(t_f), & (76)\\
J_{xx}^{k+} &= H^T(t_f, t_k^+)\psi_{xx}(x(t_f))H(t_f, t_k^+) + \eta_2(t_f). & (77)
\end{aligned}
$$

Remark 7 (Computational Cost) All other terms in (56)-(58) except for $H(t_l^-, t_k^+)$, J_x^{k+} and J_{xx}^{k+} are readily available once the nominal trajectory $x(t)$ is known. Therefore the main computational cost for J_{t_k}, $J_{t_k t_k}$, J_{t_k, t_l} occurs in the computation of $H(t_l^-, t_k^+)$, J_x^{k+} and J_{xx}^{k+}. The above method we propose reduces the computation of $H(t_l^-, t_k^+)$ to solving initial value ODEs with jumps (65) for any $k < l$ and the computation of J_x^{k+} and J_{xx}^{k+} to solving initial value ODEs with jumps (71)-(73) for $k = 1, 2, \cdots, K$. Hence we altogether need to solve $\frac{(K-1)K}{2} + K = \frac{K(K+1)}{2}$ sets of initial value ODEs with jumps. With today's powerful ODE solvers (e.g., ode45 function in MATLAB), these equations can be solved efficiently and accurately. For our purpose of efficient optimization of open-loop solutions of optimal switching instants, such computation suffices. Moreover, for general quadratic problems for switched autonomous linear systems which we will elaborate on in the next section, the computational costs of these values can be reduced greatly. □

5 GENERAL QUADRATIC PROBLEMS FOR HYBRID AUTONOMOUS SYSTEMS WITH LINEAR SUBSYSTEMS AND STATE JUMPS

In this section, we apply the approach developed in Section 3 to a special class of problems, namely, general quadratic problems for hybrid autonomous systems with linear subsystems and linear state jumps. In particular, we show that due to the special structure of the problem, the computation of $H(t_l^-, t_k^+)$, J_x^{k+} and J_{xx}^{k+} can further be simplified.

Problem 2 *Consider a hybrid autonomous system with linear subsystems $\dot{x} = A_i x, i \in I$. Assume a prespecified sequence of active subsystems $(1, 2, \cdots, K, K+1)$ is given. Also assume that when the system switches from subsystem k to $k+1$ $(k = 1, \cdots, K)$, there is a discontinuous jump of the continuous state which has the linear relationship*

$$
x(t_k^+) = \gamma^k(x(t_k^-)) = \Theta_k x(t_k^-) + \Gamma_k
\tag{78}
$$

where Θ_k, Γ_k are matrices of appropriate dimensions. Find optimal switching instants $t_1, \cdots, t_K$ ($t_0 \le t_1 \le \cdots \le t_K \le t_f$) such that the cost in general quadratic form

$$J(t_1, \cdots, t_K) = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x)\, dt + \sum_{k=1}^{K} \psi^k(x(t_k^-)) \tag{79}$$

where

$$\psi(x(t_f)) = \frac{1}{2}(x(t_f))^T Q_f x(t_f) + M_f x(t_f) + W_f, \tag{80}$$

$$L(x) = \frac{1}{2}(x(t))^T Q x(t) + M x(t) + W, \tag{81}$$

$$\psi^k(x(t_k^-)) = \frac{1}{2}(x(t_k^-))^T Q_k x(t_k^-) + M_k x(t_k^-) + W_k, \tag{82}$$

is minimized. Here t_0, t_f and $x(t_0) = x_0$ are given; Q_f, M_f, W_f, Q, M, W are matrices of appropriate dimensions with $Q_f \ge 0$, $Q \ge 0$. Q_k, M_k, W_k, ($k = 1, \cdots, K$), are matrices of appropriate dimensions which form the quadratic terms for the switching costs from subsystem k to $k+1$ and $Q_k \ge 0$. □

In view of the special structure of Problem 2, we can readily observe that

$$A(t_{k+1}^-, t_k^+) = e^{A_{k+1}(t_{k+1} - t_k)} \tag{83}$$

for any $k = 1, \cdots, K$. Moreover,

$$\begin{aligned} H(t_l^-, t_k^+) &= e^{A_l(t_l - t_{l-1})} \gamma_x^{(l-1)-} e^{A_{l-1}(t_{l-1} - t_{l-2})} \bullet \cdots \bullet \gamma_x^{(k+1)-} e^{A_{k+1}(t_{k+1} - t_k)} \\ &= e^{A_l(t_l - t_{l-1})} \Theta_{l-1} e^{A_{l-1}(t_{l-1} - t_{l-2})} \bullet \cdots \bullet \Theta_{k+1} e^{A_{k+1}(t_{k+1} - t_k)}. \end{aligned} \tag{84}$$

The computation of J_x^{k+} and J_{xx}^{k+} is discussed next. Assume a nominal $\hat{t}$ is given. If for any $x \in \mathbb{R}^n$ and any $t \in [t_0, t_f]$ we denote by $\bar{J}(x, t)$ the cost incurred if the system starts from the state x at time instant t and evolves according to the portion of the switching sequence generated by $\hat{t}$ in $[t, t_f]$. In other words,

$$\bar{J}(x, t) = \psi(x(t_f)) + \int_t^{t_f} L(x(\tau))\, d\tau + \sum_{k \text{ with } t_k \in [t, t_f]} \psi^k(x(t_k^-)) \tag{85}$$

where $x(t) = x$. Dynamic programming approach similar to (22) can be applied to $\bar{J}(x, t)$ to obtain

$$\bar{J}(x, t) = \frac{1}{2} x^T P(t) x + S(t) x + T(t) \tag{86}$$

where $P(t) = P^T(t)$ and $P(t)$, $S(t)$, $T(t)$ obey the following differential equations

with jumps

$$\begin{cases} -\dot{P} = PA_{j+1} + A_{j+1}^T P + Q, & t_j^+ \le t \le t_{j+1}^-, \\ P(t_j^-) = \Theta_j^T P(t_j^+) \Theta_j + Q_j, \end{cases} \tag{87}$$

$$\begin{cases} -\dot{S} = SA_{j+1} + M, & t_j^+ \le t \le t_{j+1}^-, \\ S(t_j^-) = \Gamma_j^T P(t_j^+) \Theta_j + S(t_j^+) \Theta_j + M_j, \end{cases} \tag{88}$$

$$\begin{cases} -\dot{T} = W, & t_j^+ \le t \le t_{j+1}^-, \\ T(t_j^-) = \frac{1}{2} \Gamma_j^T P(t_j^+) \Gamma_j + S(t_j^+) \Gamma_j + T(t_j^+) + W_j, \end{cases} \tag{89}$$

along with initial conditions

$$P(t_f) = Q_f, \tag{90}$$

$$S(t_f) = M_f, \tag{91}$$

$$T(t_f) = W_f. \tag{92}$$

From the definitions of the functions $\bar{J}$ and J^k, if $\hat{t}$ is fixed, we have

$$J^k(x(t_k^+), t_k, \cdots, t_K) = \bar{J}(x(t_k^+), t_k^+), \tag{93}$$

$$J_x^k(x(t_k^-), t_k, \cdots, t_K) = \bar{J}_x(x(t_k^-), t_k^-), \tag{94}$$

$$J_{xx}^k(x(t_k^-), t_k, \cdots, t_K) = \bar{J}_{xx}(x(t_k^-), t_k^-). \tag{95}$$

Therefore the values of J_x^{k+} and J_{xx}^{k+} can be obtained as

$$J_x^{k+} = \bar{J}_x(x(t_k^+), t_k^+) = (x(t_k^+))^T P(t_k^+) + S(t_k^+), \tag{96}$$

$$J_{xx}^{k+} = \bar{J}_{xx}(x(t_k^+), t_k^+) = P(t_k^+). \tag{97}$$

Remark 8 (Computational Cost) The computation of $H(t_l^-, t_k^+)$'s using (84) is straightforward and do not resort to an ODE solver. The computation of J_x^{k+} and J_{xx}^{k+} using (96) and (97) relies on the values of $P(t_k^+)$'s and $S(t_k^+)$'s which are easy to obtain by solving the initial value ODEs with jumps (87)-(92) backward in time only once. Therefore, the computational cost for Problem 2 is greatly reduced as opposed to the general case in Section 4. □

6 EXAMPLES

In this section, we present two examples to illustrate the effectiveness of the approach developed in this paper. The examples are computed using MATLAB implementation of our approach. Our approach and implementation can solve these examples very efficiently.

Example 1 Consider a hybrid autonomous system consisting of

$$\text{subsystem 1: } \begin{cases} \dot{x}_1 = x_1 + 0.5\sin x_2 \\ \dot{x}_2 = -0.5\cos x_1 - x_2 \end{cases} \tag{98}$$

$$\text{subsystem 2: } \begin{cases} \dot{x}_1 = 0.3\sin x_1 + 0.5x_2 \\ \dot{x}_2 = -0.5x_1 + 0.3\cos x_2 \end{cases} \tag{99}$$

$$\text{subsystem 3: } \begin{cases} \dot{x}_1 = -x_1 - 0.5\cos x_2 \\ \dot{x}_2 = 0.5\sin x_1 + x_2 \end{cases} \tag{100}$$

Assume that $t_0 = 0$, $t_f = 3$ and the system switches at $t = t_1$ from subsystem 1 to 2 and at $t = t_2$ from subsystem 2 to 3 ($0 \le t_1 \le t_2 \le 3$). Also assume that the system has the state jump

$$\begin{cases} x_1(t_1^+) = x_1(t_1^-) + 0.2 \\ x_2(t_1^+) = x_2(t_1^-) + 0.2 \end{cases} \tag{101}$$

when switching from subsystem 1 to 2 and

$$\begin{cases} x_1(t_2^+) = x_1(t_2^-) + 0.2 \\ x_2(t_2^+) = x_2(t_2^-) - 0.2 \end{cases} \tag{102}$$

when switching from subsystem 2 to 3. We want to find optimal switching instants t_1, t_2 such that the cost

$$J = \frac{1}{2}x_1^2(3) + \frac{1}{2}x_2^2(3) + \frac{1}{2}\int_0^3 \left(x_1^2(t) + x_2^2(t)\right) dt + \sum_{k=1}^{2}\left(\frac{1}{2}x_1^2(t_k^-) + \frac{1}{2}x_2^2(t_k^-)\right) \tag{103}$$

is minimized. Here $x_1(0) = 1$ and $x_2(0) = 3$.

For this problem, we choose initial nominal $t_1 = 1$, $t_2 = 1.5$. We derive the derivatives of J using the result in Theorem 1. The computation of $H(t_2^-, t_1^+)$, J_x^{1+}, J_{xx}^{1+}, J_x^{2+}, and J_{xx}^{2+} is based on results in Section 4. By using Algorithm 1 with the constrained Newton's method, after 8 iterations we find that the optimal switching instants are $t_1 = 0.4847$, $t_2 = 1.9273$ and the corresponding optimal cost is 18.8310. The computation takes 4.01 seconds of CPU time when it is performed using Matlab 6.1 on an AMD Athlon4 900MHz PC with 256MB of RAM. The corresponding state trajectory is shown in figure 2. Figure 3 shows the plot of the cost function for different $0 \le t_1 \le t_2 \le 3$. By comparing the J value for different t_1 and t_2, we verify that the solution we obtain is the global optimal (although it is difficult to tell from the cost surface, our computation shows us so). □

Example 2 (Switched Autonomous System with Time Delay) Consider a switched autonomous system consisting of

$$\text{subsystem 1: } \dot{x} = A_1 x = \begin{bmatrix} -0.1 & 0.5 \\ -5 & -0.1 \end{bmatrix} x, \tag{104}$$

$$\text{subsystem 2: } \dot{x} = A_2 x = \begin{bmatrix} -0.1 & 5 \\ -0.5 & -0.1 \end{bmatrix} x, \tag{105}$$

$$\text{subsystem 3: } \dot{x} = A_3 x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x. \tag{106}$$

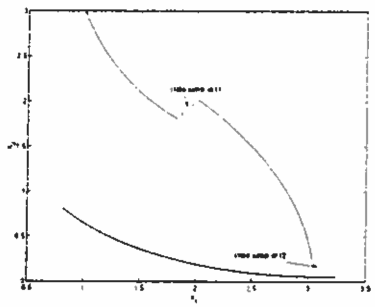

Figure 2: The state trajectory for Example 1.

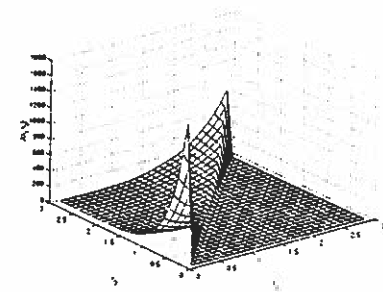

Figure 3: The cost for Example 1 for different (t_1, t_2)'s.

Assume that the system has a time delay $T = 0.1$ sec for each switching to actually take effect. Also assume that $t_0 = 0$, $t_f = 3$ and the system switches at $t = t_1$ from subsystem 1 to 2 and at $t = t_2$ from subsystem 2 to 3 (here we require that $0 \le t_1 < t_1 + T \le t_2 < t_2 + T \le 3$). We want to find optimal t_1, t_2 such that the cost

$$J = \frac{1}{2}\int_0^3 x^T(t)x(t)\, dt \tag{107}$$

is minimized. Here $x_1(0) = 0$ and $x_2(0) = -3$.

To find optimal t_1 and t_2, we first translate the original problem into a problem for a hybrid autonomous system and then use the approach in this paper to solve it. We translate the problem as follows. Consider a hybrid autonomous system with subsystems (104)-(106). Regard the original system dynamics in the interval $[t_1, t_1 + T]$ as a state jump at t_1

$$x(t_1^+) = e^{A_1 T}x(t_1^-) = \begin{bmatrix} 0.9777 & 0.0493 \\ -0.4930 & 0.9777 \end{bmatrix} x(t_1^-). \tag{108}$$

In shrinking the dynamics in $[t_1, t_1 + T]$ into a state jump, instead of regarding subsystem 2 as active in $[t_1 - T, t_2]$ in the original system, we can now regard subsystem 2 as active in $[t_1, t_2 - T]$ in the translated system.

Similarly, we now regard the original system dynamics in the interval $[t_2, t_2 + T]$ as a state jump at $t_2 - T$

$$x(t_2 - T^+) = e^{A_2 T} x(t_2 - T^-) = \begin{bmatrix} 0.9777 & 0.4930 \\ -0.0493 & 0.9777 \end{bmatrix} x(t_2 - T^-). \tag{109}$$

In shrinking the dynamics in $[t_2, t_2 + T]$ into a state jump at $t_2 - T$, instead of regarding subsystem 3 as active in $[t_2 + T, 3]$ in the original system, we now regard subsystem 3 as active in $[t_2 - T, 3 - 2T]$ in the translated system.

Once we have translated the original system dynamics into a hybrid autonomous system in $[0, 3 - 2T]$ as above, we can translate the original cost function into the following one for the translated system

$$\begin{aligned} J &= \frac{1}{2}\int_0^{3-2T} x^T(t)x(t)\,dt + \frac{1}{2}x^T(t_1^-)\Big(\int_0^T e^{A_1^T t}e^{A_1 t}\,dt\Big)x(t_1^-) \\ &\quad + \frac{1}{2}x^T(t_2 - T^-)\Big(\int_0^T e^{A_2^T t}e^{A_2 t}\,dt\Big)x(t_2 - T^-) \\ &= \frac{1}{2}\int_0^{2.8} x^T(t)x(t)\,dt + \frac{1}{2}x^T(t_1^-)\begin{bmatrix} 0.1064 & -0.0220 \\ -0.0220 & 0.0983 \end{bmatrix} x(t_1^-) \\ &\quad + \frac{1}{2}x^T(t_2 - T^-)\begin{bmatrix} 0.0983 & 0.0220 \\ 0.0220 & 0.1064 \end{bmatrix} x(t_2 - T^-). \end{aligned} \tag{110}$$

The problem now becomes finding t_1 and t_2 so that the J in (110) is minimized; it is a general quadratic problem for hybrid autonomous system with linear subsystems and state jumps.

For this problem, we choose initial nominal $t_1 = 0.5$ and $t_2 = 1.6$. We derive the derivatives of J using the result in Theorem 1. The computation of $H(t_2^-, t_1^+)$, J_x^{1+}, J_{xx}^{1+}, J_x^{2+}, and J_{xx}^{2+} is based on results in Section 5. By using Algorithm 1 with the constrained Newton's method, after 13 iterations we find that the optimal switching instants are $t_1 = 0.9423$ and $t_2 = 1.5168$, and the corresponding optimal cost is 2.4461. The computation takes 9.78 seconds of CPU time when it is performed using Matlab 6.1 on an AMD Athlon4 900MHz PC with 256MB of RAM. We then translate the result back to the original switched system. The corresponding state trajectory is shown in figure 4. Figure 5 shows the plot of the cost function for different (t_1, t_2)'s. Note from figure 5 that there are several local minima for this problem. By comparing the J value for different t_1 and t_2, we verify that the one local minimum we obtain here is actually the global optimal (although it is difficult to tell from the cost surface, our computation shows us so). □

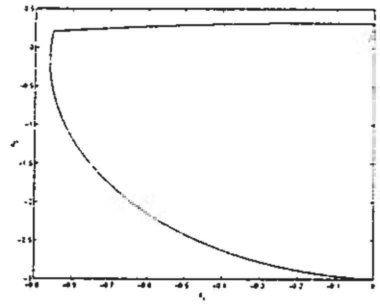

Figure 4: The state trajectory for Example 2.

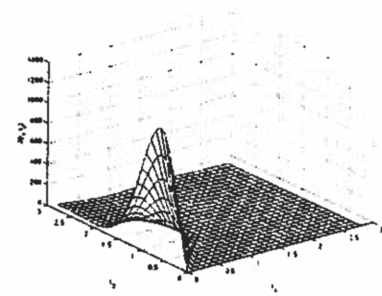

Figure 5: The cost for Example 2 for different (t_1, t_2)'s.

7 CONCLUSION

In this paper, we proposed an approach for solving optimal timing control problems for a class of hybrid autonomous systems given prespecified sequences of active subsystems. In particular, we derive the derivatives of the cost with respect to the switching time instants and use nonlinear optimization techniques to locate the optimal switching time instants. It is also shown in the paper that the computational burden can be eased in the case of general quadratic problems for hybrid autonomous systems with linear subsystems and state jumps. The approach developed in the paper has been implemented using MATLAB. The software we developed can solve the optimal control problems studied in this paper very efficiently. Future research topics include the incorporation of intelligent optimization methods for finding global optima and the development of methods for finding optimal switching sequences when the sequence of active subsystems is not prespecified.

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APPENDIX: SOME PROOFS FOR SECTION 3.2

PROOF OF LEMMA 1: Although the results in the Lemma hold for all cases in the definition of $\delta x(t)$, we need to discuss each case in order to show the validity of them.

Case 1: $dt_1 \geq 0,\ dt_2 \geq 0$

$$\begin{aligned}
\delta x(t_1 + dt_1^+) &= \hat{x}(t_1 + dt_1^+) - x(t_1 + dt_1^+) \\
&= \gamma^1\Big(x(t_1^-) + \int_{t_1^-}^{t_1+dt_1^-} f_1(\hat{x}(t))\ dt\Big) - \Big(\gamma^1(x(t_1^-)) + \int_{t_1^+}^{t_1+dt_1^+} f_2(x(t))\ dt\Big) \\
&= \gamma^1\Big(x(t_1^-) + f_1(x(t_1^-))dt_1 + o(dt_1)\Big) - \Big(\gamma^1(x(t_1^-)) + f_2(x(t_1^+))dt_1 + o(dt_1)\Big) \\
&= (\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + o(dt_1).
\end{aligned} \tag{A1}$$

We then conclude from the property of the variational equation that

$$\begin{aligned}
\delta x(t_2^-) &= A(t_2^-, t_1 + dt_1^+)\delta x(t_1 + dt_1^+) + (\text{H.O.T. in } \delta x(t_1 + dt_1^+)) \\
&= \big(A(t_2^-, t_1^+) + A_{t_1}(t_2^-, t_1^+)dt_1 + o(dt_1)\big)\big((\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + o(dt_1)\big) + o(dt_1) \\
&= A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + o(dt_1).
\end{aligned} \tag{A2}$$

$$\begin{aligned}
\delta x(t_2 + dt_2^-) &= \hat{x}(t_2 + dt_2^-) - z_1(t_2 + dt_2^-) \\
&= \Big(\hat{x}(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2(\hat{x}(t))\ dt\Big) - \Big(z_1(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2(z_1(t))\ dt\Big) \\
&= \delta x(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} \Big(f_2(\hat{x}(t)) - f_2(z_1(t))\Big)\ dt \\
&= \delta x(t_2^-) + \Big(f_2(\hat{x}(t_2^-)) - f_2(z_1(t_2^-))\Big)dt_2 + o(dt_2) \\
&= \delta x(t_2^-) + f_x^{2-}\delta x(t_2^-)dt_2 + o(dt_2) \\
&= A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + f_x^{2-} A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 dt_2 \\
&\quad + (\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}).
\end{aligned} \tag{A3}$$

Case 2: $dt_1 \geq 0,\ dt_2 < 0$

The arguments for proving (A1) in Case 1 can be applied in this case to show its validity. In this case,

$$
\begin{aligned}
\delta x(t_2 + dt_2^-) &= z_2(t_2 + dt_2^-) - x(t_2 + dt_2^-) \\
&= \left(z_2(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(z_2(t)\big)\ dt\right) - \left(x(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(x(t)\big)\ dt\right) \\
&= \delta x(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} \Big(f_2\big(z_2(t)\big) - f_2\big(x(t)\big)\Big)\ dt \\
&= \delta x(t_2^-) + \Big(f_2\big(z_2(t_2^-)\big) - f_2\big(x(t_2^-)\big)\Big)dt_2 + o(dt_2) \\
&= \delta x(t_2^-) + f_x^{2-}\delta x(t_2^-)dt_2 + o(dt_2) \\
&= A(t_2^-, t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2^-, t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2 \\
&\quad +(\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}).
\end{aligned} \tag{A4}
$$

Case 3: $dt_1 < 0,\ dt_2 \geq 0$

In this case, we have

$$
\begin{aligned}
\delta x(t_1^+) &= \hat{x}(t_1^+) - x(t_1^+) \\
&= \left(\gamma^1\big(x(t_1 + dt_1^-)\big) + \int_{t_1+dt_1^+}^{t_1^+} f_2\big(\hat{x}(t)\big)\ dt\right) - \gamma^1\left(x(t_1 + dt_1^-) + \int_{t_1+dt_1^-}^{t_1^-} f_1\big(x(t)\big)\ dt\right) \\
&= \Big(\gamma^1\big(x(t_1 + dt_1^-)\big) + f_2\big(\hat{x}(t_1 + dt_1^+)\big)(-dt_1) + o(dt_1)\Big) - \gamma^1\Big(x(t_1 + dt_1^-) \\
&\quad + f_1\big(x(t_1^-)\big)(-dt_1) + o(dt_1)\Big) \\
&= -f_2\Big(\gamma^1\big(x(t_1 + dt_1^-)\big)\Big)dt_1 + \gamma_x^1\big(x(t_1 + dt_1^-)\big)f^{1-}dt_1 + o(dt_1) \\
&= -f_2\Big(\gamma^1\big(x(t_1^-)\big) + O(dt_1)\Big)dt_1 + \gamma_x^1\big(x(t_1^-) + O(dt_1)\big)f^{1-}dt_1 + o(dt_1) \\
&= \Big(\gamma_x^1\big(x(t_1^-)\big)f^{1-} - f_2\Big(\gamma_1\big(x(t_1^-)\big)\Big)\Big)dt_1 + o(dt_1) \\
&= (\gamma_x^{1-}f^{1-} - f^{1+})dt_1 + o(dt_1).
\end{aligned} \tag{A5}
$$

In the derivations of the third to the last equations in (A5), we use the relationship

$$
x(t_1 + dt_1^-) = x(t_1^-) + f^{1-}dt_1 + o(dt_1) = x(t_1^-) + O(dt_1), \tag{A6}
$$

and the Taylor expression of f_2. Therefore, we have

$$
\begin{aligned}
\delta x(t_2^-) &= A(t_2^-, t_1^+)\delta x(t_1^+) + \big(\text{H.O.T. in } \delta x(t_1^+)\big) \\
&= A(t_2^-, t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1 + o(dt_1)
\end{aligned} \tag{A7}
$$

$$
\begin{aligned}
\delta x(t_2 + dt_2^-) &= \hat{x}(t_2 + dt_2^-) - z_3(t_2 + dt_2^-) \\
&= \left(\hat{x}(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(\hat{x}(t)\big)\ dt\right) - \left(z_3(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(z_3(t)\big)\ dt\right) \\
&= \delta x(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} \Big(f_2\big(\hat{x}(t)\big) - f_2\big(x_3(t)\big)\Big)\ dt \\
&= \delta x(t_2^-) + \Big(f_2\big(\hat{x}(t_2^-)\big) - f_2\big(z_3(t_2^-)\big)\Big)dt_2 + o(dt_2) \\
&= \delta x(t_2^-) + f_x^{2-}\delta x(t_2^-)dt_2 + o(dt_2) \\
&= A(t_2^-, t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2^-, t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2 \\
&\quad +(\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}).
\end{aligned} \tag{A8}
$$

Case 4: $dt_1 < 0,\ dt_2 < 0$

The arguments for proving (A7) in Case 3 can be applied in this case to show its validity. In this case, we have

$$
\begin{aligned}
\delta x(t_2 + dt_2^-) &= z_4(t_2 + dt_2^-) - x(t_2 + dt_2^-) \\
&= \left(z_4(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(z_4(t)\big)\ dt\right) - \left(x(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} f_2\big(x(t)\big)\ dt\right) \\
&= \delta x(t_2^-) + \int_{t_2^-}^{t_2+dt_2^-} \Big(f_2\big(z_4(t)\big) - f_2\big(x(t)\big)\Big)\ dt \\
&= \delta x(t_2^-) + \Big(f_2\big(z_4(t_2^-)\big) - f_2\big(x(t_2^-)\big)\Big)dt_2 + o(dt_2) \\
&= \delta x(t_2^-) + f_x^{2-}\delta x(t_2^-)dt_2 + o(dt_2) \\
&= A(t_2^-, t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1 + f_x^{2-}A(t_2^-, t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2 \\
&\quad +(\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}).
\end{aligned} \tag{A9}
$$

□

PROOF OF LEMMA 2: (44) follows directly from the fact that

$$
dx(t_2^-) = \delta x(t_2 + dt_2^-) + f_2\big(x(t_2^-)\big)dt_2 + o(dt_2). \tag{A10}
$$

To prove (45), we note that

$$
\begin{aligned}
dx(t_2^+) &= \gamma^2\big(\hat{x}(t_2 + dt_2^-)\big) - \gamma^2\big(x(t_2^-)\big) \\
&= \gamma^2\big(x(t_2^-) + dx(t_2^-)\big) - \gamma^2\big(x(t_2^-)\big) \\
&= \gamma_x^{2-}dx(t_2^-) + \frac{1}{2}\begin{bmatrix} \big(dx(t_2^-)\big)^T \frac{\partial^2\gamma^2_{(1)}\left(x(t_2^-)\right)}{\partial x^2} dx(t_2^-) \\ \vdots \\ \big(dx(t_2^-)\big)^T \frac{\partial^2\gamma^2_{(n)}\left(x(t_2^-)\right)}{\partial x^2} dx(t_2^-) \end{bmatrix} + \big(\text{H.O.T. in } dx(t_2^-)\big).
\end{aligned} \tag{A11}
$$

Now since

$$
\begin{aligned}
&\frac{1}{2}\begin{bmatrix} \big(dx(t_2^-)\big)^T \frac{\partial^2\gamma^2_{(1)}\left(x(t_2^-)\right)}{\partial x^2} dx(t_2^-) \\ \vdots \\ \big(dx(t_2^-)\big)^T \frac{\partial^2\gamma^2_{(n)}\left(x(t_2^-)\right)}{\partial x^2} dx(t_2^-) \end{bmatrix} \\
&= \begin{bmatrix} (f^{2-})^T \frac{\partial^2\gamma^2_{(1)}\left(x(t_2^-)\right)}{\partial x^2} \\ \vdots \\ (f^{2-})^T \frac{\partial^2\gamma^2_{(n)}\left(x(t_2^-)\right)}{\partial x^2} \end{bmatrix} A(t_2^-, t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2 + (\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}) \\
&= \xi^{2-}A(t_2^-, t_1^+)(\gamma_x^{1-}f^{1-} - f^{1+})dt_1dt_2 + (\text{terms in } dt_1^2, dt_2^2 \text{ and H.O.T.}),
\end{aligned} \tag{A12}
$$

we can substitute (A12) into (A11) to obtain (45). □

PROOF OF LEMMA 3: We first note that

$$
\int_{t_1+dt_1^+}^{t_2+dt_2^-} L(\hat{x})\ dt = \begin{cases} \int_{\max\{t_1^+, t_1+dt_1^+\}}^{t_2+dt_2^-} L(\hat{x})\ dt, & \text{if } dt_1 \geq 0, \\ \int_{t_1+dt_1^+}^{\max\{t_1^+, t_1+dt_1^+\}} L(\hat{x})\ dt + \int_{\max\{t_1^+, t_1+dt_1^+\}}^{t_2+dt_2^-} L(\hat{x})\ dt, & \text{if } dt_1 < 0. \end{cases} \tag{A13}
$$

In the light of the forward decoupling principle, the term $\int_{t_1+dt_1^+}^{\max\{t_1^+, t_1+dt_1^+\}} L(\hat{x})\ dt$ in the case of $dt_1 < 0$ will not depend on dt_2; therefore, it will not contribute to the coefficient of dt_1dt_2. So we conclude that no matter $dt_1 \geq 0$ or $dt_1 < 0$, we only need to consider the term $\int_{\max\{t_1^-, t_1+dt_1^-\}}^{t_2-dt_2^-} L(\hat{x})\, dt$. For this term, we discuss as follows.

Case 1: $dt_2 \geq 0$

In this case, we have

$$\int_{\max\{t_1^-, t_1+dt_1^+\}}^{t_2+dt_2^-} L(\hat{x})\, dt = \int_{\max\{t_1^+, t_1+dt_1^+\}}^{t_2^-} L(x+\delta x)\, dt + \int_{t_2^-}^{t_2+dt_2^-} L(\hat{x})\, dt. \tag{A14}$$

The first term in (A14) will not be contributing due to the reason that

$$\delta x(t) = A(t, t_1^-)(\gamma_x^{1-} f^{1-} - f^{1+})dt_1 + o(dt_1), \tag{A15}$$

for $t \in [\max\{t_1^-, t_1 + dt_1^+\}, t_2^-]$ and therefore they do not depend on dt_2.

The second term is shown to be

$$\begin{aligned} \int_{t_2^-}^{t_2+dt_2^-} L(\hat{x})\, dt &= L(\hat{x}(t_2^-))dt_2 + o(dt_2) \\ &= L^{2-}dt_2 + L_x^{2-}\delta x(t_2^-)dt_2 + \left(\text{terms in } (\delta x(t_2^-))^2 dt_2, dt_2^2 \text{ and H.O.T.}\right). \end{aligned} \tag{A16}$$

By substituting the expression of $\delta x(t_2^-)$ into (A16), we obtain the coefficient of $dt_1 dt_2$ contributed by this term as

$$L_x^{2-} A(t_2^-, t_1^+)(\gamma_x^{1-} f^{1-} - f^{1+}). \tag{A17}$$

Case 2: $dt_2 < 0$

In this case, since $x(t) + \delta x(t) = \hat{x}(t)$ for $t \in [\max\{t_1^+, t_1 + dt_1^+\}, t_2 + dt_2^-]$, we have

$$\begin{aligned} \int_{\max\{t_1^+, t_1+dt_1^+\}}^{t_2+dt_2^-} L(\hat{x})\, dt &= \int_{\max\{t_1^-, t_1+dt_1^-\}}^{t_2+dt_2^-} L(x+\delta x)\, dt \\ &= \int_{\max\{t_1^+, t_1+dt_1^+\}}^{t_2^-} L(x+\delta x)\, dt + \int_{t_2^-}^{t_2+dt_2^-} L(x+\delta x)\, dt. \end{aligned} \tag{A18}$$

Similar to Case 1, the first term in (A18) will not be contributing. The second term is shown to be

$$\begin{aligned} \int_{t_2^-}^{t_2+dt_2^-} L(x+\delta x)\, dt &= L(x(t_2^-) + \delta x(t_2^-))dt_2 + o(dt_2) \\ &= L^{2-}dt_2 + L_x^{2-}\delta x(t_2^-)dt_2 + \left(\text{terms in } (\delta x(t_2^-))^2 dt_2, dt_2^2 \text{ and H.O.T.}\right). \end{aligned} \tag{A19}$$

Therefore, by substituting the expression of $\delta x(t_2^-)$ into (A19), we obtain the same coefficient (A17). □

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Hybrid Automata Model of Manufacturing Systems and its Optimal Control Subject to Logical Constraints *

[†]Fabio Balduzzi, [‡]Ratnesh Kumar

[†] Dip. di Automatica ed Informatica
Politecnico di Torino, 10129 Torino, Italy
balduzzi@polito.it

[‡] Dept. of Electrical and Computer Engineering
Iowa State University, Ames, IA 50014-3060
rkumar@iastate.edu

Abstract

In this paper we present a hybrid system formulation for the modeling and control of automated manufacturing systems, combining both discrete and continuous dynamics in a single hybrid automaton representation. We derive a hierarchical two-level production control policy based on hybrid automata models, capable of providing closed-loop behaviors which both satisfy logic-based and optimize performance specifications. We show that the hybrid automaton model of an elementary service is Zeno and linear. We propose a non-Zeno regularization, and also an outer approximation for it; the latter is non-Zeno, and also an initialized linear hybrid automaton and hence decidable. Our main contribution is the integration of fluid approximation techniques within a hybrid automata model framework aimed at performance optimization subject to logical constraints. By introducing the notions of macro-states and macro-events, the logical constraints are satisfied through a upper level supervisory controller which achieves a desired macro-events trajectory set, represented by a generated language, upon which the optimization is performed by a lower level controller.

Keywords: Manufacturing systems, supervisory control, optimization, production control policy, fluid approximation.

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