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Controller Failure Time Analysis for Symmetric $\mathcal{H}_\infty$ Control Systems

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***Abstract*— In this paper, we consider a controller failure time analysis problem for a class of symmetric linear time-invariant (LTI) systems controlled by a pre-designed symmetric static output feedback. We assume that the controller fails from time to time due to physical or purposeful reason, and analyze stability and $\mathcal{H}_\infty$ disturbance attenuation properties for the entire system. Our objective is to find conditions concerning controller failure time, under which the system's stability and $\mathcal{H}_\infty$ disturbance attenuation properties are preserved to a desired level. For both stability and $\mathcal{H}_\infty$ disturbance attenuation analysis, we show that if the unavailability rate of the controller is smaller than a specified constant, then global exponential stability of the entire system and a reasonable $\mathcal{H}_\infty$ disturbance attenuation level is achieved. The key point is to establish a common quadratic Lyapunov-like function for the case where the controller works and the case where the controller fails.**

I. Introduction & Problem Description

In this paper, we consider some quantitative properties for linear time-invariant (LTI) control systems with controller failures. The motivation of studying such a problem stems from the fact that controller failures always exist in real control systems due to various environmental factors. For example, in a feedback control system which is composed of a system and a feedback controller, controller failures occur when the signals are not transmitted perfectly on the route between the system and the controller, or when the controller itself is not available sometimes due to some known or unknown reason. Another important motivation concerning controller failures is that we can think about "failure" in a positive way: "suspension". That is, in the situation that economical issue or system life consideration is concerned, we desire to suspend the controller purposefully from time to time.

For feedback control systems, the problem of possessing integrity was considered in [1], where it was proposed to design a state feedback controller so that the closed-loop system remains stable even when some part of the controller fails. In [2], similar control systems were dealt with using the name of asynchronous dynamical systems (ADS), and two real systems, the control over asynchronous network and the parallelized algorithm, were discussed. In that context, a Lyapunov-based approach was proposed to construct the controller so that the system has desired properties. Ref. [3] stated similar control problems in the framework of networked control systems (NCS), where informations (reference input, plant output, control input, etc.) are exchanged through a network among control system components (sensors, controller, actuators, etc.), and thus packet dropouts occuring inevitably due to unreliable transmission paths lead to controller failures. Certainly, we can think of package dropouts positively in the sense that we expect to use limited rate of data and information to control our system. The control problems in that case also fall in the framework of feedback control system with controller failures.

In our recent works, we have considered several analysis problems for feedback control systems with occassional controller failures. First, we considered in [4] a controller failure time analysis problem for exponential stability of LTI continuous-time systems with a pre-designed stabilizing state feedback. By using a piecewise Lyapunov function, we showed that if the unavailability rate of the controller is smaller than a specified constant and the average time interval between controller failures (ATBCF) is large enough, then global exponential stability of the system is guaranteed. In [5], the result of [4] was extended to LTI discrete-time systems. Furthermore, we extended the consideration to $\mathcal{L}_2$ gain analysis for LTI continuous-time feedback control systems with controller failures in [6].

Recently, we extended the results in [4], [6] to dynamical output feedback case in [7]. In that context, we showed that if the unavailability rate of the controller is smaller than a specified constant and the ATBCF is large enough, then exponential stability of the system is guaranteed. Concerning $\mathcal{H}_\infty$ disturbance attenuation, we showed that if the unavailability rate of the controller is smaller than a specified constant, then the system with an ATBCF achieves a reasonable weighted $\mathcal{H}_\infty$ disturbance attenuation level, and the weighted $\mathcal{H}_\infty$ disturbance attenuation level approaches the same normal $\mathcal{H}_\infty$ disturbance attenuation level when the ATBCF is sufficiently large. However, the results in [7] are quite conservative, and the reason is supposed to be in the use of piecewise Lyapunov functions. This observation motivates us to think about the following question: *What kind of feedback control systems have a common quadratic Lyapunov-like function* [8] *for the case where the controller works and the case where the controller fails? What kind of properties can be obtained for such systems?*

In this paper, we give a clear (though not complete) answer to the above question. More exactly, we will show that a class of symmetric LTI control systems, which are composed of a symmetric open-loop LTI system and a symmetric static output feedback, will have a common quadratic Lyapunov-like function for the case where the controller works and the case where the controller fails. Furthermore, we will show that if the unavailability rate of the controller is smaller than a specified constant, then the original system's exponential stability and $\mathcal{H}_\infty$ disturbance attenuation properties will be preserved to a reasonable level. We take symmetric systems into consideration since they appear quite often in many engineering disciplines (for example, RC or RL electrical networks, viscoelastic materials and chemical reactions) [9], [10], [11], and thus belong to an important class in control engineering.

The system we consider is described by equations of the form

$$\left\{\begin{aligned} x[k+1] &= Ax[k] + B_1 w[k] + B_2 u[k] \\ z[k] &= C_1 x[k] + Dw[k] \\ y[k] &= C_2 x[k]\,, \end{aligned}\right. \tag{1}$$

where $x[k] \in \Re^n$ is the state, $u[k] \in \Re^m$ is the control input, $w[k] \in \Re^r$ is the disturbance input, $y[k] \in \Re^p$ is the measured output, $z[k] \in \Re^q$ is the controlled output, and A, B_1, B_2, C_1, C_2, D are constant matrices of appropriate dimension. Throughout this paper, we assume

(i) the system is symmetric in the sense of satisfying

$$A = A^T\,,\ \ B_1 = C_1^T\,,\ \ B_2 = C_2^T\,,\ \ D = D^T\,; \tag{2}$$

(ii) A is not Schur stable and a symmetric static output feedback

$$u = K_s y\,, \qquad K_s = K_s^T \tag{3}$$

has been designed so that the closed-loop system composed of (1) and (3) has desired property (exponential stability with certain decay rate or certain $\mathcal{H}_\infty$ disturbance attenuation level). However, due to physical or purposeful reason, the designed controller sometimes fails with a (not constant necessarily) time interval until we recover it. In this setting, we derive the condition of controller failure time, under which the system's exponential stability and its $\mathcal{H}_\infty$ disturbance attenuation properties are preserved to a desired level. As in [4], [5], [6], we use the word "controller failure" in this paper to mean complete breakdown of the controller ($u = 0$) on certain time interval, neither as the one in [1] that part of the controller fails, nor as the one in [2] that the controller decays slowly at a rate.

To analyze stability and $\mathcal{H}_\infty$ disturbance attenuation properties of the symmetric system with controller failures, we utilize a common quadratic Lyapunov-like function approach. It is well known that Lyapunov function theory is the most general and useful approach for studying qualitative properties of various control systems. However, for the system on hand, traditional Lyapunov functions do not exist since the system is unstable when the controller fails. Instead of traditional Lyapunov functions, we construct a common quadratic Lyapunov-like function along with the situation of the controller. Although the common quadratic Lyapunov-like function proposed in this paper is similar to traditional Lyapunov functions in form, it does not meet the requirement for traditional Lyapunov functions, and thus called *common quadratic Lyapunov-like function* in this paper. It should be noted here that the idea of common quadratic Lyapunov-like functions for $\mathcal{H}_\infty$ control systems with controller failures in this paper originates from the recent paper [12], where stability and $\mathcal{L}_2$ gain properties of switched systems composed of stable symmetric LTI subsystems were analyzed. In this paper, we extend the approach in that context to symmetric $\mathcal{H}_\infty$ control systems which include the unstable situation when the controller fails.

II. Stability Analysis

In this section, we set $w[k] \equiv 0$ in the system (1) to analyze stability for the system with controller failures. First, we give a definition concerning exponential stability of an autonomous system quantitatively.

Definition 1: The system $x[k+1] = f(x[k])$ with $f(0) = 0$ is said to be *exponentially stable with decay rate* $0 < \mu < 1$ if $\|x[k]\| \le c\mu^k \|x[0]\|$ holds for any $x[0]$, any $k \ge 1$ and a constant $c > 0$.

Now, we assume that a controller (3) has been designed for the system (1) so that the closed-loop system

$$x[k+1] = A_s x[k]\,,\ \ A_s = A + B_2 K_s C_2 \tag{4}$$

is exponentially stable. However, the designed controller sometimes fails and we need a (not definitely constant) time interval to recover it. Obviously, when the controller fails, the closed-loop system assumes the form of

$$x[k+1] = Ax[k]\,, \tag{5}$$

which is obtained by substituting $u = 0$ in (1). Hence, the performance of the entire system is dominated by the following piecewise difference equation:

$$x[k+1] = \begin{cases} A_s x[k] & \text{when the controller works} \\ Ax[k] & \text{when the controller fails}\,. \end{cases} \tag{6}$$

The next definition is about the unavailability rate of the controller, which plays a crucial role in this paper.

Definition 2: For any $k > 1$, we denote by $T_u(k)$ the total time interval of controller failures during $[0, k)$, and call the ratio $\frac{T_u(k)}{k}$ the *unavailability rate of the controller* in the system.

Since A_s is Schur stable and A is not Schur stable, we can always find two positive scalars $\lambda_s > 1$ and $\lambda_u > 1$ such that $\lambda_s A_s$ remains Schur stable and $\lambda_u^{-1} A$ becomes Schur stable. As can be seen later, large λ_s and small λ_u are desirable. Furthermore, since now $\lambda_s A_s$ and $\lambda_u^{-1} A$ are Schur stable, and both matrices are symmetric, we obtain

$$(\lambda_s A_s)^2 = \lambda_s^2 A_s^2 < I\,,\ \ (\lambda_u^{-1} A)^2 = (\lambda_u)^{-2} A^2 < I\,. \tag{7}$$

We define the following *common quadratic Lyapunov-like function* candidate

$$V(k) = x^T[k] x[k] \tag{8}$$

for the entire system.

Without loss of generality, we assume that the designed controller works during $[k_{2j}, k_{2j+1})$, and the controller fails during $[k_{2j+1}, k_{2j+2})$, $j = 0, 1, \cdots$, where $k_0 = 0$. Then, we obtain for any $k > 1$ that

$$V(k) \leq \begin{cases} \lambda_s^{-2(k-k_{2j})} V(k_{2j}) & \text{if } k_{2j} \leq k < k_{2j+1} \\ \lambda_u^{2(k-k_{2j+1})} V(k_{2j+1}) & \text{if } k_{2j+1} \leq k < k_{2j+2} \end{cases} \quad (9)$$

and by induction that for any $k > 1$,

$$V(k) \leq \lambda_s^{-2(k-T_u(k))} \lambda_u^{2T_u(k)} V(0)\,, \quad (10)$$

which is equivalent to

$$\|x[k]\| \leq \lambda_s^{-(k-T_u(k))} \lambda_u^{T_u(k)} \|x[0]\|\,. \quad (11)$$

If there exists a positive scalar λ satisfying $\lambda < 1$ such that

$$\frac{T_u(k)}{k} \leq \frac{\ln \lambda_s + \ln \lambda}{\ln \lambda_s + \ln \lambda_u}\,, \quad (12)$$

which is a condition on the unavailability rate of the controller, then we obtain easily from (12) that

$$(\lambda_s \lambda_u)^{T_u(k)} \leq (\lambda_s \lambda)^k \iff \lambda_s^{-(k-T_u(k))} \lambda_u^{T_u(k)} \leq \lambda^k \quad (13)$$

and thus

$$\|x[k]\| \leq \lambda^k \|x[0]\|\,. \quad (14)$$

Thus, the entire system is exponentially stable with decay rate λ.

The above discussion is summarized in the following theorem.

Theorem 1: If the unavailability rate of the controller in the system (1) is small in the sense of satisfying (12) for some positive $\lambda < 1$, then the system (1) is exponentially stable with decay rate λ.

Remark 1: According to the unavailability rate condition (12), we see that comparatively long controller failure time $T_u(k)$ is tolerable for large λ_s and small λ_u. This is reasonable since the closed-loop system has large decay rate (thus good stability property) when the controller works with large λ_s, and the open-loop system does not diverge greatly when the controller fails with small λ_u. Therefore, if we concentrate on stability of the system, we should design the controller so that a large λ_s can be obtained.

Remark 2: Although we concentrated on the case of complete controller breakdown ($u = 0$) in this paper, it is an easy matter to extend the discussion to the case where due to various reason the controller (3) decays in the sense of $u \to \alpha u$ with α being a fixed constant satisfying $0 \leq \alpha < 1$. This case is very common in recent works on control systems which are controlled by limited rate of data or information. In that case, if the closed-loop system composed of (1) and $u = \alpha K_s y$ is unstable, the discussions up to now are the same by making some notation modification. If this is not the case, then the entire system can be considered as a switched system composed of two stable subsystems, and thus it is always exponential stable no matter how long the unavalability time of the controller is; see the detailed discussions in [12].

III. $\mathcal{H}_\infty$ Disturbance Attenuation Analysis

In this section, we assume that a controller (3) has been designed for the system (1) so that the closed-loop system

$$\begin{cases} x[k+1] = A_s x[k] + B_1 w[k] \\ z[k] = C_1 x[k] + D w[k]\,, \end{cases} \quad (15)$$

is Schur stable and the $\mathcal{H}_\infty$ norm of the transfer function from w to z is less than a prespecified constant γ. Since our interest here is to analyze $\mathcal{H}_\infty$ disturbance attenuation property of the system, we assume $x[0] = 0$.

Also, we suppose that the designed controller sometimes fails and we need a (not constant necessarily) time interval to recover it. When the controller fails, the closed-loop system assumes the form of

$$\begin{cases} x[k+1] = A x[k] + B_1 w[k] \\ z[k] = C_1 x[k] + D w[k]\,. \end{cases} \quad (16)$$

Then, the behavior of the entire system is dominated by the piecewise LTI system: the system (15) when the controller works and the system (16) when the controller fails.

Since A_s is Schur stable and $\|C_1(zI - A_s)^{-1} B_1 + D\|_\infty < \gamma$, according to the Bounded Real Lemma [13], there exists $P_s > 0$ such that

$$\begin{bmatrix} -P_s & P_s A_s & P_s B_1 & 0 \\ A_s^T P_s & -P_s & 0 & C_1^T \\ B_1^T P_s & 0 & -\gamma I & D^T \\ 0 & C_1 & D & -\gamma I \end{bmatrix} < 0\,. \quad (17)$$

To proceed, we need the following lemma.

Lemma 1: P_s can be replaced with I in (17), i.e.,

$$\begin{bmatrix} -I & A_s & B_1 & 0 \\ A_s & -I & 0 & C_1^T \\ B_1^T & 0 & -\gamma I & D \\ 0 & C_1 & D & -\gamma I \end{bmatrix} < 0\,. \quad (18)$$

Proof: Since $P_s > 0$, there always exists a nonsingular matrix U satisfying $U^T = U^{-1}$ such that

$$U^T P_s U = \Sigma_0 = \mathrm{diag}\{\sigma_1,\ \sigma_2,\ \cdots,\ \sigma_n\} \quad (19)$$
$$\sigma_i > 0\,, \quad i = 1, 2, \cdots, n\,.$$

Pre- and post-multiplying the inequality (17) respectively by diag$\{U^T, U^T, I, I\}$ and diag$\{U, U, I, I\}$, we obtain

$$\begin{bmatrix} -\Sigma_0 & \Sigma_0 \bar{A}_s & \Sigma_0 \bar{B}_1 & 0 \\ \bar{A}_s \Sigma_0 & -\Sigma_0 & 0 & \bar{B}_1 \\ \bar{B}_1^T \Sigma_0 & 0 & -\gamma I & D \\ 0 & \bar{B}_1^T & D & -\gamma I \end{bmatrix} < 0\,, \quad (20)$$

where $\bar{A}_s = U^T A_s U$, $\bar{B}_1 = U^T B_1$, and we replaced C_1 with B_1^T. Furthermore, pre- and post-multiplying the first and second rows and columns in (20) by Σ_0^{-1} leads to

$$\begin{bmatrix} -\Sigma_0^{-1} & \bar{A}_s\Sigma_0^{-1} & \bar{B}_1 & 0 \\ \Sigma_0^{-1}\bar{A}_s & -\Sigma_0^{-1} & 0 & \Sigma_0^{-1}\bar{B}_1 \\ \bar{B}_1^T & 0 & -\gamma I & D \\ 0 & \bar{B}_1^T\Sigma_0^{-1} & D & -\gamma I \end{bmatrix} < 0\,. \tag{21}$$

In (21), we exchange the first and second rows and columns, and then exchange the thrid and fourth rows and columns, to obtain

$$\begin{bmatrix} -\Sigma_0^{-1} & \Sigma_0^{-1}\bar{A}_s & \Sigma_0^{-1}\bar{B}_1 & 0 \\ \bar{A}_s\Sigma_0^{-1} & -\Sigma_0^{-1} & 0 & \bar{B}_1 \\ \bar{B}_1^T\Sigma_0^{-1} & 0 & -\gamma I & D \\ 0 & \bar{B}_1^T & D & -\gamma I \end{bmatrix} < 0\,. \tag{22}$$

Since $\sigma_1 > 0$, there always exists a scalar λ_1 such that

$$0 < \lambda_1 < 1\,, \quad \lambda_1\sigma_1 + (1-\lambda_1)\sigma_1^{-1} = 1\,. \tag{23}$$

Then, by computing $\lambda_1\times$(20) + $(1-\lambda_1)\times$(22), we obtain

$$\begin{bmatrix} -\Sigma_1 & \Sigma_1\bar{A}_s & \Sigma_1\bar{B}_1 & 0 \\ \bar{A}_s\Sigma_1 & -\Sigma_1 & 0 & \bar{B}_1 \\ \bar{B}_1^T\Sigma_1 & 0 & -\gamma I & D \\ 0 & \bar{B}_1^T & D & -\gamma I \end{bmatrix} < 0\,, \tag{24}$$

where

$$\begin{aligned} \Sigma_1 \;=&\; \mathrm{diag}\left\{\lambda_1\sigma_1 + (1-\lambda_1)\sigma_1^{-1}\,, \lambda_1\sigma_2 + (1-\lambda_1)\sigma_2^{-1}\,,\right. \\ &\left. \cdots, \lambda_1\sigma_n + (1-\lambda_1)\sigma_n^{-1}\right\} \\ \triangleq&\; \mathrm{diag}\left\{1\,, \bar{\sigma}_2\,, \cdots, \bar{\sigma}_n\right\} > 0\,. \end{aligned} \tag{25}$$

In the similar way to obtain (22), we can obtain

$$\begin{bmatrix} -\Sigma_1^{-1} & \Sigma_1^{-1}\bar{A}_s & \Sigma_1^{-1}\bar{B}_1 & 0 \\ \bar{A}_s\Sigma_1^{-1} & -\Sigma_1^{-1} & 0 & \bar{B}_1 \\ \bar{B}_1^T\Sigma_1^{-1} & 0 & -\gamma I & D \\ 0 & \bar{B}_1^T & D & -\gamma I \end{bmatrix} < 0 \tag{26}$$

from (24). Since $\bar{\sigma}_2 > 0$, there exists a scalar λ_2 such that

$$0 < \lambda_2 < 1\,, \quad \lambda_2\bar{\sigma}_2 + (1-\lambda_2)\bar{\sigma}_2^{-1} = 1\,. \tag{27}$$

Then, the combination $\lambda_2\times$(24) + $(1-\lambda_2)\times$(26) results in

$$\begin{bmatrix} -\Sigma_2 & \Sigma_2\bar{A}_s & \Sigma_2\bar{B}_1 & 0 \\ \bar{A}_s\Sigma_2 & -\Sigma_2 & 0 & \bar{B}_1 \\ \bar{B}_1^T\Sigma_2 & 0 & -\gamma I & D \\ 0 & \bar{B}_1^T & D & -\gamma I \end{bmatrix} < 0\,, \tag{28}$$

where

$$\begin{aligned} \Sigma_2 \;=&\; \mathrm{diag}\left\{1\,, \lambda_2\bar{\sigma}_2 + (1-\lambda_2)\bar{\sigma}_2^{-1}\,,\right. \\ &\left. \cdots, \lambda_2\bar{\sigma}_n + (1-\lambda_2)\bar{\sigma}_n^{-1}\right\} \\ \triangleq&\; \mathrm{diag}\left\{1\,, 1\,, \cdots, \tilde{\sigma}_n\right\} > 0\,. \end{aligned} \tag{29}$$

By repeating this process, we see that $\Sigma_n = I$ also satisfies (20), i.e.,

$$\begin{bmatrix} -I & \bar{A}_s & \bar{B}_1 & 0 \\ \bar{A}_s & -I & 0 & \bar{B}_1 \\ \bar{B}_1^T & 0 & -\gamma I & D \\ 0 & \bar{B}_1^T & D & -\gamma I \end{bmatrix} < 0\,. \tag{30}$$

Pre- and post-multiplying this inequality respectively by diag$\{U, U, I, I\}$ and diag$\{U^T, U^T, I, I\}$, we obtain (18). This completes the proof.

When the controller works, we rewrite (18) as

$$\begin{bmatrix} -I & A_s & B_1 & 0 \\ A_s & -I & 0 & C_1^T \\ B_1^T & 0 & -\gamma I & D \\ 0 & C_1 & D & -\gamma I \end{bmatrix} < 0\,. \tag{31}$$

It is easy to confirm that this inequality is equivalent to

$$\begin{bmatrix} A_s^2 + \frac{1}{\gamma}C_1^TC_1 - I & A_sB_1 + \frac{1}{\gamma}C_1^TD \\ B_1^TA_s + \frac{1}{\gamma}DC_1 & B_1^TB_1 + \frac{1}{\gamma}D^2 - \gamma I \end{bmatrix} < 0\,, \tag{32}$$

and thus there exists a positive scalar $\lambda_s < 1$ such that

$$\begin{bmatrix} A_s^2 + \frac{1}{\gamma}C_1^TC_1 - \lambda_s I & A_sB_1 + \frac{1}{\gamma}C_1^TD \\ B_1^TA_s + \frac{1}{\gamma}DC_1 & B_1^TB_1 + \frac{1}{\gamma}D^2 - \gamma I \end{bmatrix} < 0\,. \tag{33}$$

Next, we consider the case when the controller fails. In this case, we can always find a scalar η satisfying $0 < \eta < 1$ such that ηA is Schur stable and the $\mathcal{H}_\infty$ norm of the system $(\eta A, \eta B_1, \eta C_1, \eta D)$ is smaller than γ. Since symmetricity of this adjusted system remains the same, we use the proof of Lemma 1 to obtain

$$\begin{bmatrix} -I & \eta A & \eta B_1 & 0 \\ \eta A & -I & 0 & \eta C_1^T \\ \eta B_1^T & 0 & -\gamma I & \eta D \\ 0 & \eta C_1 & \eta D & -\gamma I \end{bmatrix} < 0\,, \tag{34}$$

and by some simple calculation,

$$\begin{bmatrix} A^2 + \frac{1}{\gamma}C_1^TC_1 - \eta^{-2}I & AB_1 + \frac{1}{\gamma}C_1^TD \\ B_1^TA + \frac{1}{\gamma}DC_1 & B_1^TB_1 + \frac{1}{\gamma}D^2 - \gamma\eta^{-2}I \end{bmatrix} < 0\,. \tag{35}$$

In this inequality, we find a positive scalar $\lambda_u \geq \eta^{-2}$ such that

$$\begin{bmatrix} A^2 + \frac{1}{\gamma}C_1^TC_1 - \lambda_u I & AB_1 + \frac{1}{\gamma}C_1^TD \\ B_1^TA + \frac{1}{\gamma}DC_1 & B_1^TB_1 + \frac{1}{\gamma}D^2 - \gamma I \end{bmatrix} < 0\,. \tag{36}$$

This is always possible since $B_1^TB_1 + \frac{1}{\gamma}D^2 - \gamma I < 0$ is guaranteed by (32) and the (1,1)-block of the left side $A^2 + \frac{1}{\gamma}C_1^TC_1 - \lambda_u I$ in (36) will be negative definite "enough" with a large scalar λ_u.

Now, we consider the difference of the common quadratic Lyapunov-like function (8) along the trajectories of the system. When the controller works,

$$\begin{aligned}
&V(k+1)-V(k)\\
&=(A_sx[k]+B_1w[k])^T(A_sx[k]+B_1w[k])-x^T[k]x[k]\\
&=\tilde{x}^T[k]\begin{bmatrix} A_s^2-I & A_s^TB_1\\ B_1^TA_s & B_1^TB_1\end{bmatrix}\tilde{x}[k]\\
&\le -\tilde{x}^T[k]\begin{bmatrix} \frac{1}{\gamma}C_1^TC_1+(1-\lambda_s)I & \frac{1}{\gamma}C_1^TD\\ \frac{1}{\gamma}DC_1 & \frac{1}{\gamma}D^2-\gamma I\end{bmatrix}\tilde{x}[k]\\
&=-\frac{1}{\gamma}\Gamma(k)-(1-\lambda_s)V(k)\,, \qquad (37)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{x}[k] &\triangleq \begin{bmatrix} x^T[k] & w^T[k]\end{bmatrix}^T\\
\Gamma(k) &\triangleq z^T[k]z[k]-\gamma^2w^T[k]w[k]\,,
\end{aligned} \qquad (38)$$

and (33) was used to obtain the inequality. Therefore, in the case where the designed controller works,

$$V(k+1)\le\lambda_sV(k)-\frac{1}{\gamma}\Gamma(k)\,. \qquad (39)$$

When the controller fails,

$$\begin{aligned}
&V(k+1)-V(k)\\
&=(Ax[k]+B_1w[k])^T(Ax[k]+B_1w[k])-x^T[k]x[k]\\
&=\tilde{x}^T[k]\begin{bmatrix} A^2-I & A^TB_1\\ B_1^TA & B_1^TB_1\end{bmatrix}\tilde{x}[k]\\
&\le -\tilde{x}^T[k]\begin{bmatrix} \frac{1}{\gamma}C_1^TC_1+(1-\lambda_u)I & \frac{1}{\gamma}C_1^TD\\ \frac{1}{\gamma}DC_1 & \frac{1}{\gamma}D^2-\gamma I\end{bmatrix}\tilde{x}[k]\\
&=-\frac{1}{\gamma}\Gamma(k)-(1-\lambda_u)V(k)\,, \qquad (40)
\end{aligned}$$

where (36) was used to obtain the inequality. Therefore, in the case where the designed controller fails,

$$V(k+1)\le\lambda_uV(k)-\frac{1}{\gamma}\Gamma(k)\,. \qquad (41)$$

As in the previous section, we assume that the designed controller works during $[k_{2j},k_{2j+1})$, and the controller fails during $[k_{2j+1},t_{2j+2})$, $j=0,1,\cdots$, where $k_0=0$. Then, for any $k\ge1$ in the interval $[k_{2j},k_{2j+1})$, we use the well known difference theory to obtain from (39) that

$$V(k)\le\lambda_s^{k-k_{2j}}V(k_{2j})-\frac{1}{\gamma}\sum_{m=k_{2j}}^{k-1}\lambda_s^{k-1-m}\Gamma(m)\,, \qquad (42)$$

and similarly for any $k\in[k_{2j+1},t_{2j+2})$,

$$V(k)\le\lambda_u^{k-k_{2j+1}}V(k_{2j+1})-\frac{1}{\gamma}\sum_{m=k_{2j+1}}^{k-1}\lambda_u^{k-1-m}\Gamma(m)\,. \qquad (43)$$

By induction, we obtain that for any $k\ge1$,

$$\begin{aligned}
V(k)&\le\lambda_s^{k-T_u(k)}\lambda_u^{T_u(k)}V(0)\\
&\quad-\frac{1}{\gamma}\sum_{m=0}^{k-1}\lambda_s^{k-1-m-(T_u(k)-T_u(m))}\lambda_u^{T_u(k)-T_u(m)}\Gamma(m)\,, \qquad (44)
\end{aligned}$$

and thus from $x(0)=0$ and $V(k)\ge0$ that

$$\begin{aligned}
&\sum_{m=0}^{k-1}\lambda_s^{k-1-m-(T_u(k)-T_u(m))}\lambda_u^{T_u(k)-T_u(m)}\Gamma(m) \qquad (45)\\
&=\sum_{m=0}^{k-1}(\lambda_s^{-1})^{-(k-1-m-(T_u(k)-T_u(m)))}\lambda_u^{T_u(k)-T_u(m)}\Gamma(m)\le0\,.
\end{aligned}$$

According to the discussion in the previous section, if the unavailability rate of the controller satisfies the inequality

$$\frac{T_u(k)}{k}\le\frac{\ln(\lambda_s^{-1})+\ln\lambda}{\ln(\lambda_s^{-1})+\ln\lambda_u}=\frac{\ln\lambda-\ln\lambda_s}{\ln\lambda_u-\ln\lambda_s}\,, \qquad (46)$$

for some positive scalar $\lambda_s\le\lambda<1$, then

$$\lambda_s^{k-1-m-(T_u(k)-T_u(m))}\lambda_u^{T_u(k)-T_u(m)}\le\lambda^{k-1-m}\,. \qquad (47)$$

Combining (45) and (47), we obtain

$$\sum_{m=0}^{k-1}\lambda_s^{k-1-m}z^T[m]z[m]\le\gamma^2\sum_{m=0}^{k-1}\lambda^{k-1-m}w^T[m]w[m]\,. \qquad (48)$$

Summing both sides of the above inequality from $k=1$ to $k=\infty$ (by rearranging the double-summation area) leads to

$$\frac{1}{1-\lambda_s}\sum_{m=0}^{\infty}z^T[k]z[k]\le\frac{\gamma^2}{1-\lambda}\sum_{m=0}^{\infty}w^T[k]w[k]\,, \qquad (49)$$

which means an $\mathcal{H}_\infty$ disturbance attenuation level $\sqrt{\frac{1-\lambda_s}{1-\lambda}}\gamma$ is achieved under the unavailability rate condition (46).

We summarize the above discussions in the following theorem.

Theorem 2: If the unavailability rate of the controller in the system (1) is small in the sense of satisfying (46) for some $0<\lambda<1$, then the entire system achieves an $\mathcal{H}_\infty$ disturbance attenuation level $\sqrt{\frac{1-\lambda_s}{1-\lambda}}\gamma$.

Remark 3: If $\lambda\to\lambda_s$, which means from (46) that the controller's failure time is close to zero, then we obtain from Theorem 2 that the achieved $\mathcal{H}_\infty$ disturbance attenuation level $\sqrt{\frac{1-\lambda_s}{1-\lambda}}\gamma$ also approaches the original γ. In this sense, $\sqrt{\frac{1-\lambda_s}{1-\lambda}}\gamma$ is a reasonable estimation in the situation where controller failures exist.

Remark 4: It is an easy task to extend the discussions here to the case where the controller (3) decays in the sense of $u\to\alpha u$ with α being a fixed constant satisfying $0\le\alpha<1$. In that case, if the closed-loop system composed of (1) and $u=\alpha K_sy$ is unstable, the discussions up to now are the same by making some notation change. If this is not the case, then the entire system can be viewed as a switched system composed of two stable subsystems, and thus a reasonable $\mathcal{H}_\infty$ disturbance attenuation level is achieved without considering the unavailability rate of the controller; refer to the detailed discussions in [12], [14].

Remark 5: Different from our other works [4]-[6] on controller failure time analysis for feedback control systems, we do not require any condition in Theorems 1 and 2 about average time interval between controller failures (ATBCF). In [4]-[6], we used a piecewise Lyapunov function

$$V(x) = \begin{cases} x^T P_s x & \text{when the controller works} \\ x^T P_u x & \text{when the controller fails}\,, \end{cases} \tag{50}$$

where P_s and P_u are different positive definite matrices. Then, we have to use a scalar $\mu > 1$, which satisfies both $x^T P_s x \le \mu x^T P_u x$ and $x^T P_u x \le \mu x^T P_s x$ for $\forall x$ (one choice is $\mu = \frac{\max\{\lambda_M(P_s),\lambda_M(P_u)\}}{\min\{\lambda_m(P_s),\lambda_m(P_u)\}}$, where $\lambda_M(\cdot)$ ($\lambda_m(\cdot)$) denotes the largest (smallest) eigenvalue of a symmetric matrix), in order to estimate the value change of the piecewise Lyapunov function when controller failures occur. Usually μ is much larger than 1, and thus it leads to quite conservative results and the requirement of ATBCF in [4]-[6]. In this paper, we have shown by now that we can use $P_s = P_u = I$ in (50) for symmetric control systems with controller failures. Therefore, we obtain $\mu = 1$ in this case, and thus the condition of ATBCF is not necessary again and less conservative results are obtained.

IV. Concluding Remarks

We have studied a controller failure time analysis problem for a class of symmetric $\mathcal{H}_\infty$ control systems, which are composed of a symmetric LTI system and a symmetric static output feedback. The attention has been focused on analyzing stability and $\mathcal{H}_\infty$ disturbance attenuation properties when the pre-designed controller fails from time to time due to physical or purposeful reason. We have shown that if the unavailability rate of the controller is smaller than a specified constant, then the entire system has a common quadratic Lyapunov-like function $V(k) = x^T[k]x[k]$ for the case where the controller works and the case where the controller fails, and the system's exponential stability and $\mathcal{H}_\infty$ disturbance attenuation properties are preserved to a reasonable level.

Finally we note that the results of this paper can be easily extended to symmetric dynamical output feedback case with some notation change. We also note that the present results can be extended to more general symmetric control systems. For example, if the equations $TA = A^T T$, $TB_1 = C_1^T$, $TB_2 = C_2^T$, $D = D^T$ are satisfied for a constant matrix $T > 0$, then we consider the similarity transformation $A_\star = T^{\frac{1}{2}} A T^{-\frac{1}{2}}$, $B_{\star 1} = T^{\frac{1}{2}} B_1$, $B_{\star 2} = T^{\frac{1}{2}} B_2$, $C_{\star 1} = C_1 T^{-\frac{1}{2}}$, $C_{\star 2} = C_2 T^{-\frac{1}{2}}$, $D_\star = D$. Since the stability and $\mathcal{H}_\infty$ disturbance attenuation properties of the entire system in this transformation do not change and we can easily confirm that $A_\star = A_\star^T$, $B_{\star 1} = C_{\star 1}^T$ and $B_{\star 2} = C_{\star 2}^T$, we can apply the results we have obtained up to now for the system represented by $(A_\star, B_{\star 1}, B_{\star 2}, C_{\star 1}, C_{\star 2}, D_\star)$ and derive corresponding results for the original symmetric $\mathcal{H}_\infty$ control system with controller failures.

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VI. References

[1] E. Shimemura and M. Fujita, "Control systems possessing integrity," *Journal of the Society of Instrument and Control Engineers*, vol. 26, pp. 400–405, 1987. (in Japanese)

[2] A. Hassibi and S. P. Boyd, "Control of asynchronous dynamical systems with rate constraints on events," in *Proceedings of the 38th IEEE Conference on Decision and Control*, 1999, pp. 1345–1351.

[3] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems," *IEEE Control Systems Magazine*, vol. 20, pp. 84–99, 2001.

[4] G. Zhai, S. Takai, and K. Yasuda, "Controller failure time analysis for linear time-invariant systems," *Transactions of the Society of Instrument and Control Engineers*, vol. 36, pp. 1050–1052, 2000.

[5] G. Zhai, S. Takai, and K. Yasuda, "Stability analysis of linear time-invariant systems with controller failures," in *Proceedings of the European Control Conference*, 2001, pp. 138–142.

[6] G. Zhai, X. Chen, S. Takai, and K. Yasuda, "Controller failure time analysis for $\mathcal{H}_\infty$ control systems," in *Proceedings of the 40th IEEE Conference on Decision and Control*, 2001, pp. 1029–1030.

[7] G. Zhai, "Stability and $\mathcal{H}_\infty$ disturbance attenuation analysis for LTI control systems with controller failures," in *Proceedings of the 41st IEEE Conference on Decision and Control*, 2002, pp. 3869-3874.

[8] B. Hu, G. Zhai, and A. N. Michel, "Common quadratic Lyapunov-like functions with associated switching regions for two unstable second-order LTI systems," *International Journal of Control*, vol. 75, pp. 1127–1135, 2002.

[9] J. C. Willems, "Realization of systems with internal passivity and symmetry constraints," *Journal of The Franklin Institute*, vol. 301, pp. 605–621, 1976.

[10] M. Ikeda, "Symmetric controllers for symmetric plants," in *Proceedings of the 3rd European Control Conference*, 1995, pp. 988–993.

[11] M. Ikeda, K. Miki, and G. Zhai, "$\mathcal{H}_\infty$ controllers for symmetric systems: A theory for attitude control of large space structures," in *Proceedings of the International Conference on Control, Automation and Systems*, 2001, pp. 651–654.

[12] G. Zhai, X. Chen, M. Ikeda, and K. Yasuda, "Stability and $\mathcal{L}_2$ gain analysis for a class of switched symmetric systems," in *Proceedings of the 41st IEEE Conference on Decision and Control*, 2002, pp. 4395–4400.

[13] T. Iwasaki, R. E. Skelton, and K. M. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis, London, 1998.

[14] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, "Disturbance attenuation properties of time-controlled switched systems," *Journal of The Franklin Institute*, vol. 338, pp. 765–779, 2001.