

## Stabilization and Performance Analysis for a Class of Switched Systems

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**Abstract**—This paper investigates stability and control design problems with performance analysis for discrete-time switched linear systems. The switched Lyapunov function method is combined with Finsler's Lemma to generate various tests in the enlarged space containing both the state and its time difference, allowing extra degree of freedom for stability analysis and control design. Two performance measures being considered are the decay rate and the input-output performance. A new LMI based stability test for the existence of switched Lyapunov functions is first developed. If a switched Lyapunov function exists, asymptotic stability of the switched system also implies its exponential stability. An LMI optimization problem is then formulated to find a bound on the decay rate of the system. To attain the bound, state feedback control gains are designed. Using the same framework and the well-known  $S$ -procedure, a generalized sufficient LMI condition is obtained which guarantees a  $\gamma$ -performance of the closed-loop switched systems subject to input disturbances.

### I. INTRODUCTION

In recent years, considerable research efforts have been devoted to the study of switched systems. The motivation for studying switched systems comes from the fact that many practical systems are inherently multimodal [8], and the fact that some of intelligent control methods are based on the idea of switching between different controllers [9], [11], [18], [20]. The existence of systems that cannot be asymptotically stabilized by a single static continuous feedback controller [4] also motivates the study. A survey of basic problems in stability and design of switched systems is given in [15].

One of the basic problems is to find conditions which guarantee that switched systems are asymptotically stable under arbitrary switching sequences. Stability analysis of switched systems is usually carried out in the Lyapunov framework [3], [9]. For switched linear systems, stability under arbitrary switching is equivalent to the existence of a common Lyapunov function [15]. Although progress has been recently made [5], [16], [19], [25], finding a common Lyapunov function is still an open problem. Quite often, a linear matrix inequality (LMI) problem formulation is used to obtain sufficient stability conditions by constructing a set of quadratic Lyapunov-like functions [12], [23]. More recently, the switched Lyapunov function (SLF) method and less conservative LMI based conditions were developed in [6] for stability analysis and control design for switched linear systems.

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For switched linear systems, the existence of SLF is a weaker condition than the solvability of Lie algebra, which implies the existence of a common quadratic Lyapunov function [1]. The Lie algebra approach, however, can be generalized to study switched nonlinear systems [17] while the extension of SLF approach to the nonlinear context is not straightforward. In the SLF method, we also assume that the SLF strictly decrease along the solutions of the systems for all time instances. This restrictive assumption can be relaxed for certain classes of switched linear systems. One can deduce asymptotic stability using multiple Lyapunov functions whose Lie derivative are only negative semi-definite [10]. The SLF method has been applied to solve different problems of switched linear systems [7], [14], [28]. In particular, the input-output performance problem was studied in [7]. There are some related works in the literature on analyzing the input-output properties of switched systems. For example, the  $\mathcal{L}_2$  analysis [29] and the  $l^\infty$  disturbance attenuation problem [13] have been studied.

Motivated by the work in [6], we combine the SLF method with Finsler's Lemma [21], [26] to study stability and control design problems with performance analysis for discrete-time arbitrarily switching linear systems. Two performance measures considered in this paper are the decay rate and the input-output performance. First, a new LMI-based necessary and sufficient condition, which generalizes the one in [6], is obtained to check the existence of a SLF. Our new method is conceptually simple. Here, difference equations are considered as constraints and these dynamical constraints are incorporated into the stability analysis condition through the use of matrix Lagrange multipliers. The key idea is to increase the dimension of the LMIs and to introduce new matrix variables, allowing extra degree of freedom for stability analysis and control design. We show that if a SLF exists, the switched system is not only asymptotically stable but also exponentially stable. Of particular interest is the formulation of an LMI optimization problem to find a sharp estimate of the decay rate to the origin for switched linear systems. To attain a bound on the decay rate, switched state feedback control design is investigated. By switched control design, we mean the design of state feedback control gains for each subsystem such that the closed-loop switched system is asymptotically (or exponentially) stable. The closed-loop switched system is the one corresponding to the closed-loop subsystems under an arbitrary switching rule. Finally, we study the robustness of the feedback control law when the system is subject to input disturbances from a  $\gamma$ -performance point of view, which provides an upper bound on the

worst case energy amplitude gain for switched systems over all possible inputs and switching signals. Using the same mathematical tool and the well-known  $\mathcal{S}$ -procedure [2], a new LMI-based sufficient condition is obtained to ensure the asymptotic stability of the switched system while satisfying a  $\gamma$ -performance condition. Better performance level than the one in [7] is guaranteed due to extra matrix variables introduced in the new LMI condition. Most of the introductory material on SLF can be found in [6].

The paper is organized as follows. Section II gives the problem formulation. Stability analysis of switched systems under arbitrary switching is addressed in Section III using the SLF method combined with Finsler's Lemma. Exponential stability of switched systems is studied in Section IV. A bound on the decay rate to the origin is found by solving an LMI optimization problem. Switched state feedback control gains are then designed to attain the bound. In Section V, the input-output performance and synthesis problems are investigated. Section VI concludes the paper.

**Notation:** The notation is standard.  $\mathbb{Z}$  is the set of integer numbers.  $\mathbb{R}^{m \times n}$  denotes the set of  $m \times n$  real matrices.  $\mathbb{S}^n$  denotes the set of  $n \times n$  real symmetric matrices and  $\mathbb{S}_+^n$ , the set of  $n \times n$  real symmetric positive-definite matrices.  $\lambda(M)$  stands for the eigenvalue of a matrix  $M$ .  $M^T$  is the transpose of the matrix  $M$ .  $M > 0$  ( $M < 0$ ) means that  $M$  is positive definite (negative definite).

## II. PROBLEM FORMULATION

Consider the class of switched linear systems

$$x(t+1) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), \quad t \in \mathbb{Z}_+ \quad (1)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $u(t) \in \mathbb{R}^m$  is the control input, and  $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$ . The switching signal  $\sigma(t) : \mathbb{Z}_+ \rightarrow \mathcal{I} = \{1, \dots, N\}$  is a piecewise constant function of time  $t$ , which is unknown *a priori*. (For notational simplicity, we may not explicitly mention the time-dependence of the switching signal below.) Here,  $(A_i, B_i)$  ( $i \in \mathcal{I}$ ) are constant matrices of appropriate dimensions denoting the subsystems, and  $N \geq 2$  is the number of subsystems.

We are interested in the following important questions.

- Q1) *Sufficient conditions for asymptotic stability:* Is it possible to specify conditions such that the autonomous switched system is asymptotically stable?
- Q2) *Decay rate:* If the answer to Q1) is yes, can a bound on the decay rate to the origin be found?
- Q3) *Control design:* If the answer to Q2) is yes, does a control  $u(t)$  exist such that the bound is actually attained?
- Q4) *Robustness of the control law:* Is the control robust when subject to input disturbances?

In the sequel we show that answers to all the above questions are indeed yes.

## III. STABILITY ANALYSIS

In this section, we investigate the stability of the origin of an autonomous switched system. The asymptotic stability of

the system is verified by means of a set of LMIs formulated in terms of subsystem A-matrices. If feasible, these LMIs provide a set of Lyapunov matrices that can be combined to form a switched quadratic Lyapunov function.

The autonomous switched system is given by

$$x(t+1) = A_{\sigma(t)}x(t). \quad (2)$$

Define the indication function

$$\xi(t) = [\xi_1(t), \dots, \xi_N(t)]^T \quad (3)$$

with

$$\xi_i(t) = \begin{cases} 1, & \sigma(t) = i \\ 0, & \text{otherwise} \end{cases}$$

Then, the switched system (2) can also be written as

$$x(t+1) = \sum_{i=1}^N \xi_i(t) A_i x(t). \quad (4)$$

To check asymptotic stability of system (2), a SLF with a structure similar to that of the system description was used [6]:

$$V(t, x(t)) = x^T(t) \left( \sum_{i=1}^N \xi_i(t) P_i \right) x(t), \quad (5)$$

where  $P_i \in \mathbb{S}_+^n$ , ( $i = 1, \dots, N$ ). If such a positive-definite Lyapunov function exists and

$$\Delta V(t, x(t)) = V(t+1, x(t+1)) - V(t, x(t)) \quad (6)$$

is negative definite along the solutions of (2), then the origin of the switched system (2) is asymptotically stable.

In the following, a new LMI-based necessary and sufficient condition is obtained by combining SLF method with Finsler's Lemma [21], [26]. This condition is more general than those conditions of Theorem 2 in [6] and no matrix inversion is involved in the construction of SLF.

We first introduce Finsler's Lemma, which has been previously used in the control literature mainly with the purpose of eliminating design variables in matrix inequalities. In this context, Finsler's Lemma is usually referred to as *Elimination Lemma*.

*Lemma 1 (Finsler's Lemma):* Let  $x \in \mathbb{R}^n$ ,  $P \in \mathbb{S}^n$ , and  $H \in \mathbb{R}^{m \times n}$  such that  $\text{rank}(H) = r < n$ . The following statements are equivalent:

- 1)  $x^T P x < 0, \forall Hx = 0, x \neq 0$ .
- 2)  $\exists X \in \mathbb{R}^{n \times m} : P + XH + H^T X^T < 0$ .

In Lemma 1, item 1) has a *constrained* quadratic form in  $\mathbb{R}^n$  while item 2) provides an *unconstrained* quadratic form, where the constraint is taken into account by introducing multiplier  $X$ .

Recall that the requirement  $\Delta V(t, x(t)) < 0, \forall x(t) \neq 0$  can be stated as  $\exists P_i, P_j \in \mathbb{S}_+^n$  such that

$$\begin{bmatrix} x(t)^T & x(t+1)^T \end{bmatrix} \begin{bmatrix} -P_i & 0 \\ 0 & P_j \end{bmatrix} \begin{bmatrix} x(t) \\ x(t+1) \end{bmatrix} < 0, \\ \forall \begin{bmatrix} A_i & -I \end{bmatrix} \begin{bmatrix} x(t) \\ x(t+1) \end{bmatrix} = 0, \begin{bmatrix} x(t) \\ x(t+1) \end{bmatrix} \neq 0,$$

assuming that  $\sigma(t) = i$  and  $\sigma(t+1) = j$ .

It now becomes clear that Lemma 1 can be applied to the switched system under study. In the new procedure, the dynamic difference equations that characterize the system are seen as constraints, which are naturally incorporated into the stability condition using Finsler's Lemma. In contrast with standard space methods, where stability is carried in the space of the state vector, the stability test is generated in the enlarged space containing both the state and its time difference.

*Theorem 1:* There exists a Lyapunov function of the form (5) whose difference is negative definite, proving asymptotic stability of (2) if and only if there exist  $P_i \in \mathbb{S}_+^n$ , and matrices  $F_i, G_i \in \mathbb{R}^{n \times n}$  ( $i = 1, \dots, N$ ), satisfying

$$\begin{bmatrix} A_i F_i^T + F_i A_i^T - P_i & A_i G_i - F_i \\ G_i^T A_i^T - F_i^T & P_j - G_i - G_i^T \end{bmatrix} < 0, \quad (7)$$

$\forall (i, j) \in \mathcal{I} \times \mathcal{I}$ .

The Lyapunov function is then given by (5).

*Proof:* Apply Lemma 1 with

$$\begin{aligned} x &\leftarrow \begin{bmatrix} x(t) \\ x(t+1) \end{bmatrix}, P \leftarrow \begin{bmatrix} -P_i & 0 \\ 0 & P_j \end{bmatrix}, \\ H^T &\leftarrow \begin{bmatrix} A_i^T \\ -I \end{bmatrix}, X \leftarrow \begin{bmatrix} F_i \\ G_i^T \end{bmatrix}, \\ P_i &\in \mathbb{S}_+^n, F_i, G_i \in \mathbb{R}^{n \times n} (i \in \mathcal{I}). \end{aligned}$$

The equivalence between asymptotic stability of (2) and the following feasibility test is then established:

$$\exists P_i \in \mathbb{S}_+^n, F_i, G_i \in \mathbb{R}^{n \times n} (i \in \mathcal{I}) : \begin{bmatrix} A_i^T F_i^T + F_i A_i - P_i & A_i^T G_i - F_i \\ G_i^T A_i - F_i^T & P_j - G_i - G_i^T \end{bmatrix} < 0. \quad (8)$$

The result follows by transposing  $A_i$  in (8).  $\blacksquare$

The key idea behind Theorem 1 is to increase the dimension of the LMIs and to introduce new matrix variables  $F_i$  and  $G_i$ , here identified as Lagrange multipliers, allowing some degree of freedom to verify (5) and (6). With special choices of  $F_i$  and  $G_i$ , we have the following corollary.

*Corollary 1:* The following statements are equivalent.

- i) There exists a Lyapunov function of the form (5) whose difference is negative definite, proving asymptotic stability of (2).
- ii) There exist  $P_i \in \mathbb{S}_+^n$ , ( $i = 1, \dots, N$ ), satisfying

$$\begin{bmatrix} P_i & A_i^T P_j \\ P_j A_i & P_j \end{bmatrix} > 0, \quad \forall (i, j) \in \mathcal{I} \times \mathcal{I}. \quad (9)$$

The Lyapunov function is then given by (5).

- iii) There exist  $P_i \in \mathbb{S}_+^n$  and  $G_i \in \mathbb{R}^{n \times n}$  ( $i = 1, \dots, N$ ), satisfying

$$\begin{bmatrix} -P_i & A_i G_i \\ G_i^T A_i^T & P_j - G_i - G_i^T \end{bmatrix} < 0, \quad \forall (i, j) \in \mathcal{I} \times \mathcal{I}. \quad (10)$$

The Lyapunov function is given by (5).

TABLE I

NUMERICAL COMPLEXITY ASSOCIATED WITH THREE STABILITY TESTS

Stability Tests	$K$ (scalar variables)	$L$ (LMI rows)
Cor. 1-ii)	$\frac{Nn(n+1)}{2}$	$Nn + 2N^2n$
Cor. 1-iii)	$\frac{Nn(n+1)}{2} + Nn^2$	$Nn + 2N^2n$
Theorem 1	$\frac{Nn(n+1)}{2} + 2Nn^2$	$Nn + 2N^2n$

*Proof:* The equivalence of i)-iii) follows from Theorem 1 by making  $F_i = G_i = 0$  for item ii) and  $F_i = 0$  for item iii), respectively in condition (8).  $\blacksquare$

It is not difficult to see that Corollary 1 is essentially equivalent to Theorem 2 in [6] while the proof here is more straightforward. Moreover, with no matrix inversion involved in the Lyapunov function, Theorem 1 allows us to formulate an LMI optimization problem to find a sharp estimate of exponential convergence rate of (2) as illustrated in Section IV, provided that (7) is feasible.

*Remark 1:* The numerical complexity associated with the LMI conditions can be computed in terms of the number  $K$  of scalar variables and number  $L$  of LMI rows. As discussed in [2], the number of floating point operation or the time required to test the feasibility of the set of LMIs is proportional to  $K^3L$ . Table I shows  $K$  and  $L$  as a function of  $n$  (states) and  $N$  (subsystems) for three tests presented here. (For a practical purpose, only those instances with  $N \leq 10$ ,  $n \leq 10$  is tractable.) In the case of restrictive switching signals, we can modify these conditions or invoke the  $\mathcal{S}$ -procedure to improve the conservatism of these conditions. Take for instance, a system which does not allow arbitrary transitions between subsystems will have the set of all ordered pairs  $(i, j)$  of subsystem indices much smaller than  $\mathcal{I} \times \mathcal{I}$ .

#### IV. ATTAINABLE BOUNDS ON DECAY RATE

##### A. A Bound on Decay Rate

From Theorem 1, the asymptotic stability of a switched linear system under arbitrary switching can be checked with the feasibility test (7). Indeed, we can say more: if the switched system (2) is asymptotically stable, it is also exponentially stable about the origin, i.e.,  $\exists \kappa > 0$  and  $0 \leq \xi < 1$ , such that

$$\|x(t)\| \leq \kappa \cdot \xi^t \|x(0)\|. \quad (11)$$

for all initial conditions  $x(0)$  and for all  $t \geq 0$ . In fact, the Lyapunov function (5) is positive definite, decrescent, and radially unbounded since  $V(t, 0) = 0, \forall t \geq 0$ , and

$$\eta \|x(t)\|^2 \leq V(t, x(t)) \leq \rho \|x(t)\|^2 \quad (12)$$

for all  $x(t) \in \mathbb{R}^n$  with  $\eta = \min_{i \in \mathcal{I}} \lambda_{\min}(P_i)$  and  $\rho = \max_{i \in \mathcal{I}} \lambda_{\max}(P_i)$  positive scalars. Furthermore,

$$\Delta V \leq -\nu \|x(t)\|^2 \quad (13)$$

with  $\nu = \min_{(i,j) \in \mathcal{I} \times \mathcal{I}} \lambda_{\min}(P_i - A_i^T P_j A_i) < \rho$ . With the above observations, the exponential stability of (2) about the origin immediately follows from the well-known fact [24]:

*Theorem 2:* The sequence  $x(t)$  is exponentially stable about the origin if there exists a Lyapunov function  $V(t, x(t))$  such that  $\forall t \geq 0$

$$\begin{aligned} \eta \|x(t)\|^2 &\leq V(t, x(t)) \leq \rho \|x(t)\|^2, \\ V(t+1, x(t+1)) - V(t, x(t)) &\leq -\nu \|x(t)\|^2, \end{aligned} \quad (14)$$

$\eta, \rho, \nu > 0$ . Then  $\|x(t)\| \leq \kappa \cdot \xi^t \|x(0)\|$ , where  $\kappa^2 = \rho/\eta$  and  $\xi^2 = 1 - \nu/\rho$ .

We obtain the following corollary.

*Corollary 2:* If the LMIs given in Theorem 1 or Corollary 1 are satisfied, the autonomous switched system (2) is exponentially stable about the origin.

In the case of verifying exponential stability of (2), it may be desirable not only find feasible solutions to (10) or (7) but to search for solutions that give an estimate of the decay rate  $\xi$  in Theorem 2. To this end, we want to maximize the ratio  $\nu/\rho$ . Since  $\Delta V = x^T(t) (A_i^T P_j A_i - P_i) x(t)$ , constraints (12) and (13) can be rewritten as:

$$\begin{aligned} \eta I &< P_i < \rho I, \\ \begin{bmatrix} A_i F_i^T + F_i A_i^T - P_i + \nu I & A_i G_i - F_i \\ G_i^T A_i^T - F_i^T & P_j - G_i - G_i^T \end{bmatrix} &< 0, \end{aligned} \quad (15) \\ \forall (i, j) \in \mathcal{I} \times \mathcal{I},$$

Note that inequalities (15) represent strict LMIs but the constraints (12) and (13) are non-strict. Recall that minimization under non-strict LMI constraints gives the same result as minimization under strict LMI constraints when both strict and non-strict LMI constraints are feasible [2]. This is the case for (15).

To obtain a well-posed optimization problem, we should normalize  $\rho$  to 1 since the ratio  $\nu/\rho$  can be made arbitrarily small by choosing sufficiently large  $\rho$  without violating constraints (15). With

$$\begin{aligned} 1 &\leftarrow \rho, \\ \nu &\leftarrow \nu/\rho, \eta \leftarrow \eta/\rho, G_i \leftarrow G_i/\rho, P_i \leftarrow P_i/\rho, \end{aligned}$$

the following optimization problem is proposed:

maximize  $\nu$   
subject to

$$\begin{aligned} \eta I &< P_i < I, \\ \begin{bmatrix} A_i F_i^T + F_i A_i^T - P_i + \nu I & A_i G_i - F_i \\ G_i^T A_i^T - F_i^T & P_j - G_i - G_i^T \end{bmatrix} &< 0, \end{aligned} \quad (16) \\ \nu, \eta > 0, \forall (i, j) \in \mathcal{I} \times \mathcal{I},$$

*Remark 2:* Constraints  $\eta I < P_i < I$  ( $i \in \mathcal{I}$ ) limit the condition number of  $P_i$  to  $1/\eta$ . The advantage of the condition number limit is that it will prevent the LMI solution algorithm from converging to  $P_i$  that could lead to roundoff problems.

The following theorem answers the Q2):

*Theorem 3:* Switched system (2) is exponentially stable with a decay rate  $\xi = (1 - \nu)^{\frac{1}{2}}$  if the problem (16) is feasible. The assured bound on the decay rate is given by  $(1 - \nu_{opt})^{\frac{1}{2}}$  with  $\nu_{opt}$  the optimal value of optimization problem (16).

*Remark 3:* It is possible to produce a better estimate of the decay rate of the switched system. However, a finer partitioning is needed [27]. A SLF having the same switching signals as the switched system may not be sufficient.

### B. Switched State Feedback

An important aspect of the new conditions (7) given in Theorem 1 is that they are LMIs in  $P_i$ ,  $F_i$  and  $G_i$  where there is no cross product between the matrix  $A_i$  and the Lyapunov matrix  $P_i$  ( $i \in \mathcal{I}$ ). This fact has an impact on the synthesis problem considered below.

Let us consider the switched systems

$$x(t+1) = A_\sigma x(t) + B_\sigma u(t) \quad (17)$$

where  $u(t)$  is the control and the switching signal is available in real-time. The stabilizing state feedback control problem is to find

$$u(t) = K_\sigma x(t) \quad (18)$$

such that the corresponding closed-loop switched system

$$x(t+1) = (A_\sigma + B_\sigma K_\sigma) x(t) \quad (19)$$

is stable.

The following theorem gives a sufficient condition to build a switched state feedback controller, which ensures the exponential stability of the closed-loop switched system. Moreover, this controller is optimal in the sense that the bound on the decay rate of the system is attained. Q3) is thus solved.

*Theorem 4:* If there is a solution to  
maximize  $\nu$   
subject to

$$\begin{aligned} \eta I &< P_i < I, \\ \begin{bmatrix} -P_i + \nu I & A_i G_i + B_i R_i \\ G_i^T A_i^T + R_i^T B_i^T & P_j - G_i - G_i^T \end{bmatrix} &< 0, \\ \nu, \eta > 0, \forall (i, j) \in \mathcal{I} \times \mathcal{I}, \end{aligned}$$

then the state feedback control given by (18) with

$$K_i = R_i G_i^{-1}, \forall i \in \mathcal{I} \quad (20)$$

exponentially stabilizes the system (17). The decay rate of the system is given by  $\xi = (1 - \nu)^{\frac{1}{2}}$ .

*Proof:* Conditions of Theorem 4 lead to

$$\begin{aligned} \eta I &< P_i < I, \\ \begin{bmatrix} -P_i + \nu I & (A_i + B_i K_i) G_i \\ G_i^T (A_i + B_i K_i)^T & P_j - G_i - G_i^T \end{bmatrix} &< 0, \\ \nu, \eta > 0, \forall (i, j) \in \mathcal{I} \times \mathcal{I}, \end{aligned}$$

which are equivalent to condition (16) written for the closed-loop system (19) with  $F_i = 0$ . The result then

follows from Theorem 3. Note that satisfying (20) implies  $P_j - G_i - G_i^T < 0$  and matrices  $G_i$  are non-singular. Hence the following feedback gain  $K_i = R_i G_i^{-1}$  is always available whenever (20) are feasible. ■

*Remark 4:* The condition given in Theorem 4 can also be adapted to the switched static output feedback control as shown in [6].

## V. INPUT-OUTPUT PERFORMANCE

The next step is naturally to consider the robustness of the switched state feedback control. This is the last question, Q4) proposed in Section II. We study the robustness from a  $\gamma$ -performance point of view, that is the designed switched feedback control ensures that the worst case energy amplitude gain of the closed loop system is less than or equal to some specified positive level  $\gamma$ .

### A. $\gamma$ -performance

Consider an autonomous discrete-time switched system given by

$$\begin{cases} x(t+1) &= A_\sigma x(t) + B_\sigma^w w(t) \\ z(t) &= C_\sigma^z x(t) + D_\sigma^{zw} w(t) \end{cases} \quad (21)$$

where  $x(t) \in \mathbb{R}^n$  is the state,  $x(0) = 0$ ,  $w(t) \in \mathbb{R}^q$  is the disturbance and  $z(t) \in \mathbb{R}^p$  is an output vector. The switching rule  $\sigma$  is defined as previously. Similarly, the matrices  $(A_\sigma, B_\sigma^w, C_\sigma^z, D_\sigma^{zw})$  are allowed to take values, at an arbitrary time, in the finite set

$$\{(A_1, B_1^w, C_1^z, D_1^{zw}), \dots, (A_N, B_N^w, C_N^z, D_N^{zw})\}.$$

Given  $\gamma > 0$ , the  $\gamma$ -performance for the switched system (21) is defined as below.

*Definition 1 ([7]):* The autonomous system (21) is said to have a  $\gamma$ -performance if it is asymptotically stable and

$$\sum_{t=0}^{\infty} z^T(t)z(t) < \gamma^2 \sum_{t=0}^{\infty} w^T(t)w(t), \quad (22)$$

$$\forall w(t) \in \mathcal{L}_2, \text{ i.e., } \sum_{t=0}^{\infty} w^T(t)w(t) < \infty.$$

Now, define the same Lyapunov function  $V(t, x(t)) > 0$ ,  $x \neq 0$  considered in Section III, and the modified Lyapunov stability conditions

$$\begin{aligned} \Delta V(t, x(t)) &< 0, \quad \gamma^2 w^T(t)w(t) \leq z^T(t)z(t), \\ \forall(x(t), x(t+1), w(t), z(t)) \text{ satisfying (21),} \\ (x(t), x(t+1), w(t), z(t)) &\neq 0, \end{aligned} \quad (23)$$

for a given  $\gamma$ . The  $\mathcal{S}$ -procedure [2] is invoked to generate the equivalent condition

$$\begin{aligned} \Delta V(t, x(t)) &< \gamma^2 w^T(t)w(t) - z^T(t)z(t), \\ \forall(x(t), x(t+1), w(t), z(t)) \text{ satisfying (21),} \\ (x(t), x(t+1), w(t), z(t)) &\neq 0. \end{aligned} \quad (24)$$

If (24) is feasible for some  $0 < \gamma < \infty$  then it is possible to conclude that the system (21) is internally asymptotically

stable and has a  $\gamma$ -performance since

$$\begin{aligned} 0 &< V(t+1, x(t+1)) \\ &< \gamma^2 \sum_{k=0}^t w^T(k)w(k) - \sum_{k=0}^t z^T(k)z(k), \end{aligned}$$

which is valid for all  $t > 0$ . In particular, take  $t \rightarrow \infty$ ,

$$0 < V(\infty) < \gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k) - \sum_{k=0}^{\infty} z^T(k)z(k), \quad (25)$$

which implies  $\gamma^2 \sum_{k=0}^{\infty} w^T(k)w(k) > \sum_{k=0}^{\infty} z^T(k)z(k)$ .

We are now ready to state the following theorem which gives a sufficient condition to check if the autonomous system (21) has a  $\gamma$ -performance.

*Theorem 5:* The system (21) has a  $\gamma$ -performance if there exist  $P_i \in \mathbb{S}_+^n$ ,  $F_{1i}, G_{1i} \in \mathbb{R}^{n \times n}$ ,  $F_{2i}, G_{2i} \in \mathbb{R}^{n \times p}$ ,  $H_{1i} \in \mathbb{R}^{p \times n}$ ,  $J_{1i} \in \mathbb{R}^{q \times n}$ ,  $H_{2i} \in \mathbb{R}^{p \times p}$ ,  $J_{2i} \in \mathbb{R}^{q \times p}$ ,  $i = \{1, \dots, N\}$ , such that

$$P + U + U^T < 0, \quad (26)$$

where

$$P = \begin{bmatrix} P_i & 0 & 0 & 0 \\ 0 & -P_j & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}, \quad \forall(i, j) \in (\mathcal{I} \times \mathcal{I}), \quad (27)$$

$$U = \begin{bmatrix} F_{1i}A_i + F_{2i}C_i^z & -F_{1i} & -F_{2i} & F_{1i}B_i^w + F_{2i}D_i^{zw} \\ G_{1i}A_i + G_{2i}C_i^z & -G_{1i} & -G_{2i} & G_{1i}B_i^w + G_{2i}D_i^{zw} \\ H_{1i}A_i + H_{2i}C_i^z & -H_{1i} & -H_{2i} & H_{1i}B_i^w + H_{2i}D_i^{zw} \\ J_{1i}A_i + J_{2i}C_i^z & -J_{1i} & -J_{2i} & J_{1i}B_i^w + J_{2i}D_i^{zw} \end{bmatrix}.$$

*Proof:* Assign

$$\begin{aligned} x \leftarrow \begin{bmatrix} x(t) \\ x(t+1) \\ z(t) \\ w(t) \end{bmatrix}, \quad P \leftarrow \begin{bmatrix} -P_i & 0 & 0 & 0 \\ 0 & P_j & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}, \\ H^T \leftarrow \begin{bmatrix} A_i^T & C_i^T \\ -I & 0 \\ 0 & -I \\ B_i^T & D_i^T \end{bmatrix}, \quad X \leftarrow \begin{bmatrix} F_{1i} & F_{2i} \\ G_{1i} & G_{2i} \\ H_{1i} & H_{2i} \\ J_{1i} & J_{2i} \end{bmatrix}, \end{aligned}$$

and apply Lemma 1 on (24). ■

### B. Switched Feedback Control with $\gamma$ -performance

In this section, we consider switched linear systems given by

$$\begin{cases} x(t+1) &= A_\sigma x(t) + B_\sigma u(t) + B_\sigma^w w(t) \\ z(t) &= C_\sigma^z x(t) + D_\sigma^z u(t) + D_\sigma^{zw} w(t) \end{cases} \quad (28)$$

A  $\gamma$ -performance state feedback problem is to find a control (18) such that the corresponding closed-loop switched system

$$\begin{cases} x(t+1) &= (A_\sigma + B_\sigma K_\sigma)x(t) + B_\sigma^w w(t) \\ z(t) &= (C_\sigma^z + D_\sigma^z K_\sigma)x(t) + D_\sigma^{zw} w(t) \end{cases} \quad (29)$$

has a  $\gamma$ -performance. The following theorem gives a sufficient condition to build such a controller.

**Theorem 6:** There exists a switched state feedback control (18) such that the closed-loop switched system (29) has a  $\gamma$ -performance if there exist  $P_i \in \mathbb{S}_+^n$ ,  $G_{1i} \in \mathbb{R}^{n \times n}$ ,  $F_{2i}, G_{2i} \in \mathbb{R}^{m \times n}$ ,  $H_{2i} \in \mathbb{R}^{m \times m}$ ,  $J_{2i} \in \mathbb{R}^{m \times q}$ ,  $R_i \in \mathbb{R}^{m \times n}$ ,  $i = \{1, \dots, N\}$ , such that

$$P + U + U^T < 0, \quad (30)$$

where  $P$  is given by (27) and

$$U = \begin{bmatrix} B_i^w F_{2i} & A_i G_{1i} + B_i R_i + B_i^w G_{2i} & B_i^w H_{2i} & B_i^w J_{2i} \\ 0 & -G_{1i} & 0 & 0 \\ -F_{2i} & -G_{2i} & -H_{2i} & -J_{2i} \\ D_i^{z^w} F_{2i} & C_i G_{1i} + D_i R_i + D_i^{z^w} G_{2i} & D_i^{z^w} H_{2i} & D_i^{z^w} J_{2i} \end{bmatrix}$$

The  $\gamma$ -performance state feedback control law is given by (18) with  $K_i = R_i G_{1i}^{-1}$ .

*Proof:* Use a transposed version of Theorem 5 with

$$F_{1i} = 0, \quad H_{1i} = 0, \quad J_{1i} = 0, \quad (31)$$

and  $R_i = K_i G_{1i}$ . ■

In order to obtain a convex condition (30), we made the choice in (31) but it is not unique. There are other choices that can also lead to a convex condition.

**Remark 5:** By setting  $F_{2i} = 0$ ,  $G_{2i} = 0$ ,  $H_{2i} = I$ , and  $J_{2i} = 0$ , we can recover the corresponding condition in [7]. Without restrictions on  $F_{2i}$ ,  $G_{2i}$ ,  $H_{2i}$ , and  $J_{2i}$ , a better  $\gamma$ -performance level with respect to the one in [7] can be obtained. However, this came at expense of a more intensive computation; see also [22].

**Remark 6:** The best upper bound on the  $\mathcal{L}_2$ -induced gain can be achieved by minimizing  $\gamma$  subject to the constraints defined by (30), which is a classical eigenvalue problem [2].

## VI. CONCLUSION

In this paper the SLF method has been combined with Finsler's Lemma to study switched linear systems. New and less conservative LMI conditions are developed for stability and control design problems with performance analysis. Output feedback control problem can also be treated in a similar way by using the technique developed in this paper.

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