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EXPLICIT HYBRID OPTIMAL CONTROLLER FOR DISTURBANCE ATTENUATION IN LINEAR HYBRID SYSTEMS

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Abstract: In this paper, the disturbance attenuation properties for classes of linear hybrid systems perturbed by exterior disturbances are investigated, and a hybrid l^1 robust optimal control problem is studied. First, a procedure is developed to determine the minimal l^1 norm of linear hybrid systems. However, for general hybrid systems, the termination of the procedure is not guaranteed. Then, the decidability issues are briefly discussed. Finally, we study the robust l^1 optimal controller synthesis problem. It is shown that the optimal performance level can be achieved by a piecewise linear state feedback control law for the linear hybrid systems, and a systematic approach to design such feedback control is proposed.

Keywords: Piecewise linear controllers, State feedback, Hybrid modes, Disturbance rejection, Robust performance.

1. INTRODUCTION

The last decade has seen increasing research activities in hybrid/switched systems. However, the literature on robust control of hybrid/switched systems is still relatively sparse. In this paper, we will focus on the induced gain analysis and robust optimal control for classes of linear hybrid/switched systems which are perturbed by exterior disturbances.

There are some related works in the literature on analyzing the induced gain in switched systems. In (Zhai *et al.*, 2001), the $\mathcal{L}_2$ gain of continuous-time switched linear systems was studied using an average dwell time approach and piecewise quadratic Lyapunov functions, and the results were extended to discrete-time case in (Zhai *et al.*, 2002). In (Hespanha, 2003), the root-mean-square (RMS) gain of a continuous-time switched linear system with slow switching was computed in terms of the solutions to a collection of Riccati equations. Both of these robust performance problems are in the signal's energy sense, and assume that the disturbances are constrained to have finite energy, i.e., bounded $\mathcal{L}_2$ norm. In practice, there are disturbances that do not satisfy this condition and act more or less continuously over time. Such disturbances are called persistent, and can not be treated in the above framework. In this paper we consider l^∞ induced gains to deal with the robust performance problems in the signal's magnitude sense, i.e., time domain specifications.

The persistent disturbance attenuation properties for hybrid/switched systems have been considered in our previous work (Lin and Antsaklis, 2003*a*; Lin and Antsaklis, 2003*c*; Lin and Antsaklis, 2004). In (Lin and Antsaklis, 2003*c*), a class of uncertain switched linear systems affected by both parameter variations and exterior disturbances was considered, and the uniformly ultimate boundedness control problem was studied for both discrete-time and continuous-time case. Under the assumption that each subsystem admits a finite persistent disturbance atten-

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uation level, it was shown in (Lin and Antsaklis, 2003*c*) that, by proper switching, the closed-loop switched systems can reach a better disturbance attenuation level than any single subsystem's. An optimal disturbance attenuation property for uncertain switched systems and its decidability issue were discussed in (Lin and Antsaklis, 2003*a*). The results for optimal disturbance attenuation property analysis were extended to classes of general uncertain hybrid systems in (Lin and Antsaklis, 2004). All of these previous works are analysis results on the disturbance attenuation property. This paper is an extension of (Lin and Antsaklis, 2004) and addresses the robust optimal hybrid controller synthesis problem.

This paper is organized as follows. In Section 2, we first define the linear hybrid systems with persistent external disturbances. Then, the l^∞ induced gain analysis problem for the linear hybrid system and its robust optimal controller synthesis problem are formulated. After introducing some necessary preliminary results in Section 3, the l^∞ induced gain analysis problem is investigated in Section 4, and a bisection based procedure is proposed to determine a non-conservative bound on the optimal disturbance attenuation level. The decidability issues of the proposed procedure are briefly discussed. The robust optimal controller synthesis problem is studied in Section 5. A systematic approach to design an explicit hybrid state feedback control law is introduced. It is interesting that the optimal performance level can be achieved by a piecewise linear state feedback control law, which is deduced from a conic partition of a (non-convex) performance region, for linear hybrid systems. The techniques are based on polyhedral algebra and linear programming. Finally, concluding remarks are made.

Notation: The letters $\mathcal{E}, \mathcal{P}, \mathcal{S} \cdots$ denote sets, $\partial\mathcal{P}$ the boundary of set $\mathcal{P}$, and $int\{\mathcal{P}\}$ its interior. A polytope (bounded polyhedral set) $\mathcal{P}$ will be presented either by a set of linear inequalities $\mathcal{P} = \{x : F_i x \leq g_i,\ i = 1, \cdots, s\}$, and compactly by $\mathcal{P} = \{x : Fx \leq g\}$, or by the dual representation in terms of the convex hull of its vertex set $\{x_j\}$, denoted by $Conv\{x_j\}$. For $x \in \mathbb{R}^n$, the l^1 and l^∞ norms are defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$ and $\|x\|_\infty = \max_i |x_i|$ respectively. l^∞ denotes the space of bounded vector sequences $h = \{h(k) \in \mathbb{R}^n\}$ equipped with the norm $\|h\|_{l^\infty} = \sup_i \|h_i(k)\|_\infty < \infty$, where $\|h_i(k)\|_\infty = \sup_{k\geq 0} |h_i(k)|$.

2. PROBLEM FORMULATION

We consider discrete-time piecewise linear hybrid systems of the form

$$x(t+1) = A_q x(t) + B_q u(t) + Ed(t), \text{ if } x \in \mathcal{P}_q \quad (1)$$

where $x(t) \in \mathbb{R}^n$ ($t \in \mathbb{Z}^+$) is the state variable, $u(t) \in \mathcal{U}_q \subset \mathbb{R}^m$ is the control input, and the disturbance input $d(t)$ is contained in $\mathcal{D} \subset \mathbb{R}^r$, the l^∞ unit ball, i.e., $\mathcal{D} = \{d : \|d\|_{l^\infty} \leq 1\}$. It is assumed that $\mathcal{U}_q$ is polytopes assigned to each mode q. Let the finite set Q stand for the collection of discrete modes q. The partition of the state space $\mathcal{X}$ is given as a finite set of polyhedra $\{\mathcal{P}_q : q \in Q\}$, where $\mathcal{P}_q \subseteq \mathcal{X}$ and $\bigcup_{q\in Q} \mathcal{P}_q = \mathcal{X}$.

A possible evolution of the piecewise linear hybrid systems from a given initial condition $x_0 \in \mathcal{X}$ can be described as follows. First, there exists at least one discrete mode $q_0 \in Q$ such that $x_0 \in \mathcal{P}_{q_0}$; let the mode q_0 be called the feasible mode for state x_0. Then the next continuous variable state is given by the transition $x_1 = A_{q_0} x_0 + B_{q_0} u + Ed$ for some possible disturbance $d \in \mathcal{D}$ and specific $u \in \mathcal{U}_{q_0}$. Then the above procedure is repeated for state x_1 to determine the next possible state x_2, and so on.

Associated with the piecewise linear hybrid system (1), a controlled output $z(t)$ is considered.

$$z(t) = Cx(t) \quad (2)$$

where $C \in \mathbb{R}^{p\times n}$ and $z(t) \in \mathbb{R}^p$.

For this linear hybrid system (1)-(2), we are interested in determining a non-conservative bound for the l^∞ induced norm from $d(t)$ to $z(t)$, which is defined as

$$\mu_{inf} = \inf\{\mu | \exists q \in Q, u \in \mathcal{U}_q : \|z(t)\|_{l^\infty} \leq \mu, \forall \|d(t)\|_{l^\infty} \leq 1\}$$

The first problem considered in this paper can be formulated as follows.

Problem 1. (Robust Performance Analysis). Given the piecewise linear hybrid system (1)-(2), determine the minimal l^∞ induced gain from $d(t)$ to $z(t)$ that can be achieved by some admissible control law.

The second problem is to construct such a *control law*, including a rule for active mode selecting and an admissible continuous variable control signal, that guarantees the l^∞ induced gain from $d(t)$ to $z(t)$, μ_{inf}, being obtained (assume proper initial conditions). This problem can be formulated as follows.

Problem 2. (Robust Optimal Control). Given linear hybrid systems (1)-(2), construct an admissible control law, such that the minimal l^∞ induced gain from $d(t)$ to $z(t)$, μ_{inf}, is achieved.

The robust optimal controller synthesis problem is studied in Section 5. It is interesting to notice that the optimal performance level can be achieved by a piecewise linear state feedback control law.

3. PRELIMINARY RESULTS

The basic idea employed in this paper is to translate the required level of performance into constraints on the controlled system, which can be dealt with by the invariant set theory. Therefore, we introduce the controlled robust invariant set for the linear hybrid systems as follows.

Definition 3. The set $\Omega \subset \mathcal{X}$ is *controlled robust invariant* for the linear hybrid system (1)-(2) if for all the initial condition $x_0 \in \Omega$, there exist feasible modes and admissible control inputs, such that $x(t) \in \Omega$, $\forall t \geq 0$, despite disturbances.

Invariant set theory has been studied in the literature for decades, see for example the survey paper (Blanchini, 1999). In the literature of hybrid systems, a similar concept, maximal safety set, was studied for example in (Lygeros *et al.*, 1999; Vidal *et al.*, 2001; Berardi *et al.*, 2003). In this paper, the invariance checking and calculation for Ω is based on the backward reachability analysis and robust predecessor operator, which is defined below.

Definition 4. The *robust one-step predecessor set*, $pre(\Omega)$, is the set of states in $\mathcal{X}$, for which there exist feasible modes and admissible control inputs to drive these states into Ω in one step, despite disturbances, i.e.,

$$pre(\Omega) = \{x(t) \in \mathcal{X} | \exists q \in Q, u(t) \in \mathcal{U}_q : x(t) \in \mathcal{P}_q, \\ A_q x(t) + B_q u(t) + Ed(t) \in \Omega, \ \forall d(t) \in \mathcal{D}\}$$

We can also define the one-step predecessor set under the q-th mode, $pre_q(\Omega)$, as the set of all states $x \in \mathcal{P}_q$, for which an admissible control input $u \in \mathcal{U}_q$ exists and guarantees that the system will be driven to Ω by the transformation $A_q x + B_q u + Ed$ for all allowable disturbances. It is easy to verify that

$$pre(\Omega) = \bigcup_{q \in Q} pre_q(\Omega) \tag{3}$$

Therefore, we only need to calculate the one-step predecessor set for each q-th subsystem. The predecessor set of a piecewise linear set Ω under a single mode, i.e., $pre_q(\Omega)$, has been studied extensively in the literature and can be computed by Fourier-Motzkin elimination (Motzkin, 1952) and linear programming techniques (Blanchini, 1999; Kerrigan, 2000). Notice that the difficulty in calculating $pre_q(\Omega)$ comes mainly from the fact that the region Ω is typically non-convex. Even if one starts with convex sets, the procedure deduces non-convex sets for hybrid systems after an one-step predecessor operation. Although the convexity is not preserved, the one-step predecessor set for a (non-convex) piecewise linear set Ω, $pre(\Omega)$, is still a piecewise linear set and can be written as finite union of polyhedra (Lin and Antsaklis, 2003*b*). Therefore, one can apply the predecessor operation recursively, which will be explored in the next section.

4. ROBUST PERFORMANCE ANALYSIS

In this section, we will focus on the first problem and determine the minimal l^∞ induced gain from $d(t)$ to $z(t)$ that can be achieved by some admissible control laws for the closed-loop linear hybrid systems. For such purpose, we first introduce the performance level μ set as

$$\Omega_\mu = \{x : \|Cx\|_\infty \leq \mu\} = \{x : \begin{bmatrix} C \\ -C \end{bmatrix} x \leq \begin{bmatrix} \bar{\mu} \\ \bar{\mu} \end{bmatrix}\}$$

where $\bar{\mu}$ stands for a column vector with μ as its elements. Note that Ω_μ is a polytope containing the origin in its interior.

A value $\mu < +\infty$ is said to be admissible if $\mu > \mu_{inf}$. Clearly, a sufficient condition for μ to be admissible is that the hybrid performance level set Ω_μ is controlled robust invariant. Therefore, the l^∞ induced gain analysis problem is transformed into checking the controlled robust invariance of the disturbance attenuation performance level set.

In order to get necessary and sufficient condition for the admissibility of μ, we introduce the following definition.

Definition 5. The set $\mathcal{C}_\infty(\Omega_\mu)$ is the *maximal controlled robust invariant set* contained in Ω_μ for the linear hybrid system (1)-(2) if $\mathcal{C}_\infty(\Omega_\mu)$ is controlled robust invariant and contains all the controlled robust invariant sets contained in Ω_μ.

The uniqueness of the maximal controlled robust invariant set $\mathcal{C}_\infty(\Omega_\mu)$, if non-empty, follows immediately from the fact that the union of two controlled robust invariant sets is still controlled robust invariant, and that $\mathcal{C}_\infty(\Omega_\mu)$ is a subset of Ω_μ. In order to calculate the maximal controlled robust invariant set in Ω_μ, we introduce the one-step controllable set of Ω_μ as

$$\mathcal{C}_1(\Omega_\mu) = pre(\Omega_\mu) \cap \Omega_\mu. \tag{4}$$

and recessively define the i-step controllable set $\mathcal{C}_i(\Omega_\mu)$ as

$$\mathcal{C}_i(\Omega_\mu) = \mathcal{C}_1(\mathcal{C}_{i-1}(\Omega_\mu)) = pre(\mathcal{C}_{i-1}(\Omega_\mu)) \cap \mathcal{C}_{i-1}(\Omega_\mu)$$

for $i \geq 2$. The sequence of finite-step controllable sets $\mathcal{C}_i(\Omega_\mu)$ has the following property.

Proposition 6. The sequence of finite step controllable sets $\mathcal{C}_i(\Omega_\mu)$ is decreasing in the sense of

$$\mathcal{C}_i(\Omega_\mu) \subseteq \mathcal{C}_{i-1}(\Omega_\mu),$$

for $i \geq 1$ and $\mathcal{C}_0(\Omega_\mu) = \Omega_\mu$. The maximal controlled invariant set in Ω_μ for the piecewise linear hybrid system (1) is given by

$$\mathcal{C}_\infty(\Omega_\mu) = \bigcap_{i=0}^{\infty} \mathcal{C}_i(\Omega_\mu).$$

Based on the maximal controlled robust invariant set $\mathcal{C}_\infty(\Omega_\mu)$, we state now the basic result of this section which will be used to give a solution

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to the disturbance attenuation property analysis problem.

Proposition 7. A value μ ($< +\infty$) is admissible, i.e., $\mu > \mu_{inf}$, *if and only if* the maximal controlled robust invariant subset of Ω_μ, $\mathcal{C}_\infty(\Omega_\mu)$, is non-empty.

This result suggests the following constructive procedure for finding a robust performance bound.

Procedure 1. Checking whether $\mu > \mu_{inf}$

(1) Initialization: Set $i = 0$ and set $\mathcal{C}_0 = \Omega_\mu$.
(2) Compute the set $\mathcal{C}_{i+1}(\Omega_\mu) = pre(\mathcal{C}_i(\Omega_\mu)) \cap \mathcal{C}_i(\Omega_\mu)$.
(3) If $0 \notin \mathcal{C}_{i+1}$ then stop, the procedure has failed. Thus, the output does not robustly meet the performance level μ.
(4) If the $\mathcal{C}_i(\Omega_\mu) = \mathcal{C}_{i-1}(\Omega_\mu)$, then stop, and set $\mathcal{C}_\infty(\Omega_\mu) = \mathcal{C}_i(\Omega_\mu)$.
(5) Set $i = i + 1$ and go to step 1.

This procedure can then be used together with a bisection method on μ to approximate the optimal value μ_{inf} arbitrarily close, which solves the disturbance attenuation property analysis problem. If the procedure stops at step 3, we conclude that $\mu < \mu_{inf}$ and we can increase the value of the output bound μ. This comes from the fact that if $\mathcal{C}_\infty(\Omega_\mu) \neq \emptyset$ then $0 \in \mathcal{C}_{i+1}$. Else, if the procedure stops at step 4, we have determined an admissible bound for the output, say $\mu > \mu_{inf}$, that can be decreased.

Unfortunately, the reachability problem for general hybrid systems is undecidable (Alur *et al.*, 2000). Therefore, the bisection method on μ that approximates the optimal value μ_{inf} can not be guaranteed to terminate in finite number of steps. Hence, a natural question is under what condition the procedure can terminate in finite number of steps, i.e., decidable. To specify the decidable subclass of linear hybrid systems for the robust performance problems, two kinds of simplification may be employed. One way is to simplify the continuous variable dynamics of the hybrid systems, see for example (Alur *et al.*, 2000; Vidal *et al.*, 2001). However, this approach may not be attractive to control applications, where simple continuous variable dynamics may not be adequate to capture the system's dynamics. Alternatively, one may restrict the discrete event dynamics of the uncertain linear hybrid systems. In (Lin and Antsaklis, 2003*a*), we followed the second route and obtained a decidable subclass of the linear hybrid systems, called switched linear systems, by simplifying the discrete event dynamics. In particular, for the switched linear systems, we do not consider partition of the state space $\mathcal{P}_q$, i.e., set $\mathcal{P}_q$ to be $\mathbb{R}^n$. In other words, the transitions between any two modes may happen at any point in the state space.

5. HYBRID CONTROLLER DESIGN

Our objective in this section is to design a hybrid control law, $\{q(t), u(t)\}$, such that the closed-loop hybrid systems achieve the possible minimal l^∞ induced gain from $d(t)$ to $z(t)$, μ_{inf}. It has been shown in the previous section that the disturbance attenuation problem is solved if and only if the set Ω_μ has nonempty controlled invariant subset, $\mathcal{C}_\infty(\Omega_\mu)$. In addition, we know that the robust optimal control problem can be solved if and only if the closed loop trajectories remain in the maximal invariant subset of the performance level set $\mathcal{C}_\infty(\Omega_{\mu_{inf}})$. In this section, we will present a systematic procedure for the hybrid controller design, which robustly drives the system, with proper initial conditions, to guarantee that the states remain within $\mathcal{C}_\infty(\Omega_{\mu_{inf}})$ despite the disturbances. For notational simplicity, we denote the maximal controlled invariant subset $\mathcal{C}_\infty(\Omega_{\mu_{inf}})$ as $\mathcal{C}$ in the sequel. Note that $\mathcal{C}$ is a (maybe non-convex) piecewise linear set.

A similar invariant control problem has been considered in (Lin and Antsaklis, 2003*b*), in which a receding horizon control based approach was proposed to obtain the appropriate discrete modes and control signals by solving a collection of linear programming problems at each step. However, the computation burden is usually heavy for practical applications. Therefore, an explicit state feedback controller is desirable, especially for large dimensional systems or applications with fast dynamics.

In this section, we will design the hybrid control law in an explicit state feedback form, i.e., $\{q(x(t)), u(x(t))\}$. For such purpose, we partition the region $\mathcal{C}$ into a finite number of convex subregions. First, we coarsely divide $\mathcal{C}$ into a finite union of convex piecewise linear sets $\mathcal{C}^i$, i.e., $\mathcal{C} = \bigcup_{i=1}^m \mathcal{C}^i$, which satisfy the property

$$pre_q\left(\bigcup_{i=1}^{m} \mathcal{C}^i\right) = \bigcup_{i=1}^{m} pre_q(\mathcal{C}^i), \tag{5}$$

for all feasible mode q. It is easy to show the existence of such partition. It is assumed that the polytopic region $\mathcal{C}^i$ can be represented as

$$\mathcal{C}^i = \{x : F^i x \le g^i\},$$

with proper dimensional matrix F^i and vector g^i, for all $i = 1, 2, \cdots, m$.

Secondly, we refine the above polyhedral partition by subdividing each polyhedra $\mathcal{C}^j$ into

$$\mathcal{C}_q^{i,j} = pre_q(\mathcal{C}^i) \bigcap \mathcal{C}^j, \; i,j = 1,\cdots,m, \; q \in Q. \tag{6}$$

Note that $\mathcal{C}_q^{i,j}$ is a convex polyhedral set and it has the following property.

Proposition 8. A controlled invariant set $\mathcal{C}$ can be written as the following polytopic partition

$$\mathcal{C} = \bigcup_{q\in Q} \bigcup_{i,j} \mathcal{C}_q^{i,j}, \tag{7}$$

where $\mathcal{C}_q^{i,j}$ is defined in (6).

Proof : First, $\bigcup_{i,j} \mathcal{C}_q^{i,j} = \bigcup_{i,j}(pre_q(\mathcal{C}^i)\bigcap \mathcal{C}^j) = \bigcup_i(pre_q(\mathcal{C}^i))\bigcap(\bigcup_j \mathcal{C}^j) = pre_q(\bigcup_i \mathcal{C}^i)\bigcap \mathcal{C}$ $= pre_q(\mathcal{C})\bigcap \mathcal{C}$. Secondly, because $pre_q(\mathcal{C})\bigcap \mathcal{C} \subseteq \mathcal{C}$, for all $q \in Q$, so $\bigcup_{q\in Q}(pre_q(\mathcal{C})\bigcap \mathcal{C}) \subseteq \mathcal{C}$. On the other hand, $\mathcal{C}$ is controlled invariant, so $\mathcal{C} \subseteq pre(\mathcal{C}) = \bigcup_{q\in Q} pre_q(\mathcal{C})$. □

These polytopic subregion $\mathcal{C}_q^{i,j}$ has the following property. For all the states x contained in the polytopic subregion $\mathcal{C}_q^{i,j}$, there exist admissible control signals $u \in \mathcal{U}_q$ such that drive x into $\mathcal{C}^i$ (not $\mathcal{C}_q^{i,j}$ itself) along mode q for all admissible disturbances. This property comes from the definition that $\mathcal{C}_q^{i,j} = pre_q(\mathcal{C}^i)\bigcap \mathcal{C}^j$, so $x \in \mathcal{C}_q^{i,j}$ implies $x \in pre_q(\mathcal{C}^i)$. The possible next step state x', which is guaranteed to be contained in $\mathcal{C}^i$, also falls into another polytopic subregion $\mathcal{C}_{q'}^{i',i}$, for some $q' \in Q$ and $i' \in \{1, 2, \cdots, m\}$. For the state x', there also exist control signals $u \in \mathcal{U}_{q'}$ to drive x' into $\mathcal{C}^{i'}$ along mode q'. The procedure repeated for the next step state, and so on. Therefore, the state trajectories under such control signals are contained in the region $\mathcal{C} = \bigcup_i \mathcal{C}^i$ despite disturbances. This observation suggests that one may pick the mode q as the active mode for a polytopic subregion $\mathcal{C}_q^{i,j}$, and the existence of the admissible continuous-variable control signal $u \in \mathcal{U}_q$ is guaranteed, which makes the region $\mathcal{C}$ $(= \bigcup_{q\in Q} \bigcup_{i,j} \mathcal{C}_q^{i,j})$ robust controlled invariant. In the sequel, we will propose a systematic method to construct such continuous-variable control signals. In particular, a linear state feedback control law is designed for each polytopic subregion $\mathcal{C}_q^{i,j}$.

For such purpose, an optimization problem for each vertex of the polytopic subregion $\mathcal{C}_q^{i,j}$ is formulated to calculate an admissible control signal for the vertex $x_q^k \in vert\{\mathcal{C}_q^{i,j}\}$. Notice that the vertices can be easily determined by solving some linear programming problems once the polytopic region $\mathcal{C}_q^{i,j}$ is specified. The control signal for the vertex x_q^k can be selected as the solution to the following minmax optimization problem:

$$\min_{u\in\mathcal{U}_q} \max_{d\in\mathcal{D}} \|F^j[A_q x_q^k + B_q u + Ed]\|_\infty$$
$$s.t. \begin{cases} F^i B_q u \le g^i - F^i A_q x_q^k - \delta_q^i \\ u \in \mathcal{U}_q \end{cases} \tag{8}$$

where $\delta_q^i = \max_{d\in\mathcal{D}}(F^i Ed)$ componentwise, which incorporates the worst effects of the disturbance d. The optimal action of the controller is one that tries to minimize the maximum cost, and tries to counteract the worst disturbance and to keep the next step state inside the region $\mathcal{C}^j$ (not $\mathcal{C}_q^{i,j}$ itself). Of course, one may choose another cost functional, but the key point here is that the above constraints are linear inequalities in u and nonempty, i.e., feasible. This is simply because that $x_q^k \in \mathcal{C}_q^{i,j} \subset pre_q(\mathcal{C}^i)$.

Because of the guaranteed feasibility of the above optimization problem for each vertex of the polytope $\mathcal{C}_q^{i,j}$, the admissible control signals for each vertex x_q^k exist, which may be denoted as u_q^k. In the next step, we will construct the continuous variable control signals for the state contained in region $\mathcal{C}_q^{i,j}$ from the control signals at the vertices. The arguments are obtained through convexity.

Note that any $x \in \mathcal{C}_q^{i,j}$ can be (not uniquely) written as the convex combination of the vertices of $\mathcal{C}_q^{i,j}$, that is $x = \sum_k \alpha_q^k(x) x_q^k$, where the convex combination coefficients $\alpha_q^k(x) \ge 0$ and $\sum_j \alpha_q^k(x) = 1$. We set the control signal $u(x)$ for state x simply as the convex combination of the control signals at the vertex u_q^k. In particular,

$$u(x) = \sum_k \alpha_q^k(x) u_q^k \tag{9}$$

and $u(x) \in \mathcal{U}_q$ comes from the convexity of $\mathcal{U}_q$. And

$$\begin{aligned} &F^i[A_q x + B_q u(x)] \\ &= F^i[A_q \sum_k \alpha_q^k(x) x_q^k + B_q \sum_k \alpha_q^k(x) u_q^k] \\ &= \sum_k \alpha_q^k(x) F^i[A_q x_q^k + B_q u_q^k] \\ &\le \sum_k \alpha_q^k(x)[g^i - \delta_q^i] = g^i - \delta_q^i \end{aligned}$$

holds. In other words, for any $x \in \mathcal{C}_q^{i,j}$, the control signal $u(x)$ given in (9) will drive the next state in $\mathcal{C}^i$ (not $\mathcal{C}_q^{i,j}$ itself) despite disturbances. Therefore, the control law of the form (9) solves the robust controlled invariance problem.

In summary, to make the performance level set $\mathcal{C}$ controlled invariant, the control law is given as follows. For $x(t) \in \mathcal{C}_q^{i,j}$, the discrete mode is selected as $q(t) = q$. This is always possible since

$$x(t) \in \mathcal{C}_q^{i,j} = pre_q(\mathcal{C}^i)\bigcap \mathcal{C}^j \subset \mathcal{P}_q.$$

Secondly, the continuous variable control signal, $u(t)$, is of the form (9). In this expression, $\alpha_q^k(x)$ is the convex combination coefficients of $x(t)$ by the vertices of $\mathcal{C}_q^{i,j}$, and u_q^k is the control signal for the corresponding vertices of $\mathcal{C}_q^{i,j}$. It has been shown that the vertex control signal u_q^k can be derived by solving a linear programming problem, which can be solved off-line.

A control law of the above form (9) can be implemented as a *piecewise linear state feedback controller*. For example, let $X_{i,j}^q$ be a matrix whose columns are formed by the vertex vector of $\mathcal{C}_q^{i,j}$. The columns of matrix $U_{i,j}^q$ are the calculated continuous variable control vector, u_q^k, corresponding to each vertex of $\mathcal{C}_q^{i,j}$. A piecewise linear state feedback controller is then obtained by applying the control

$$u(x) = \sum_k \alpha_q^k(x) u_q^k = U_{i,j}^q (X_{i,j}^{q^T} X_{i,j}^q)^{-1} X_{i,j}^{q^T} x \quad (10)$$

where $(\cdot)^T$ stands for transpose, and $(\cdot)^{-1}$ inverse of matrix. The convex combination coefficients $\alpha_q^k(x)$ can be calculated as $(X_{i,j}^{q^T} X_{i,j}^q)^{-1} X_{i,j}^{q^T} x$ if $(X_{i,j}^{q^T} X_{i,j}^q)$ is invertible. Otherwise another procedure is needed to generate the convex combination vector coefficients $\alpha_q^k(x)$. Note that all the calculations to derive the matrix $X_{i,j}^q$ and $U_{i,j}^q$ can be done off-line by linear programming techniques. The implementation of the control law only needs to calculate the convex combination coefficients vector $\alpha_q^k(x)$, which can be done by solving some linear equations. Therefore, this computational advantage makes the above method a good candidate to deal with high dimensional hybrid systems.

Some remarks are in order. First, for some states $x(t) \in C$, there may exist more than one feasible modes and admissible control signals. Then some criteria could be designed for the selection of $(q(x(t)), u(x(t)))$, e.g. the magnitude or energy of $u(x(t))$ etc. This flexibility may also lead to optimal control with respect to other kinds of cost functions. Secondly, the procedure developed here answers the robust optimal control problem in a decidable way even for the general hybrid systems (1)-(2) under the assumption that C is controlled invariant. In addition, although we only synthesize a hybrid control law to guarantee optimal l^∞ induced gain μ_{inf} here, the procedure can be directly used to achieve any admissible disturbance attenuation level, $\mu > \mu_{inf}$.

6. CONCLUDING REMARKS

In this paper, we put the robust performance analysis problems of linear hybrid/switched systems into the framework of invariant set theory. The robust performance problem was transformed into robust controlled invariance problems for a specific region decided by the disturbance attenuation level. A bisection based procedure was proposed to determine the optimal disturbance attenuation level μ_{inf}. The decidability issue of the robust performance analysis problem was briefly discussed. Finally, a systematic procedure for explicit hybrid l^1 optimal controller design was given, which was based on polyhedral algebra and linear programming techniques. The robust optimal controller is in the form of a piecewise linear state feedback control law. It is interesting to note that in (Bemporad *et al.*, 2002) piecewise linear state feedback control law, which is obtained through multi-parametric programming, solves a variety of optimal control problems of piecewise linear systems.

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