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RECONSTRUCTION METHODS TO OBTAIN INVERSE STABLE SAMPLED SYSTEMS

K.M. Passino and P.J. Antsaklis

Department of Electrical Engineering, University of Notre Dame
Notre Dame, IN 46556

ABSTRACT

It is shown that Fractional Order Hold and Pulse Amplitude Modulation signal reconstruction methods can be used to improve, in many cases, the zero properties of the sampled transfer function H(z) over the Zero Order Hold reconstruction. State-Space descriptions and algorithms are used throughout as they provide a convenient way to study H(z) for any sampling period T. In addition, the problem of how well the sampled H(z) models the continuous G(s) is discussed.

INTRODUCTION

Given a continuous plant G(s), we are interested in using a reconstruction circuit to obtain the sampled transfer function H(z). H(z) in this case represents the reconstruction circuit (e.g. a Zero Order Hold (ZOH)) followed in cascade by G(s) and a sampler. Note that G(s) usually includes the antialiasing filters. It is known that the reconstruction circuit can be represented for analysis purposes as an impulse reconstructor (modulator) in cascade with $G_r(s)$, the continuous representation of the reconstruction used. For example, for ZOH $G_r(s) = G_0(s)$ where [1] (T the sampling period)

$$G_0(s) = (1 - e^{-Ts})/s \tag{1}$$

It follows that H(z) is the representation of $G_rG(s)$ preceded by the impulse reconstructor and followed by the sampler, that is

$$H(z) = Z\{G_rG(s)\} \tag{2}$$

where Z{ } denotes the z-transform of the corresponding continuous time signal. Note that in the case of ZOH

$$H_0(z) = Z\{(1-e^{-Ts})G(s)/s\} = (1-z^{-1})Z\{G(s)/s\} \tag{3}$$

Zeros of H(z) inside the unit disc are very desirable in control design. Unfortunately, a G(s) with zeros inside the left-half s-plane is not necessarily transformed to an H(z) with zeros inside the unit disc. In contrast, the poles p_i of G(s) are transformed as $p_i \to \exp(p_iT)$ a transformation which maps the left-half plane onto the unit disc. This problem was studied by Åström et al. [2] for the case of ZOH reconstruction and it was shown that continuous systems G(s) with pole excess larger than two will always give sampled systems with unstable zeros provided that the sampling period T is sufficiently small. This implies that in many cases, where T must be chosen relatively small to satisfy other criteria, H(z) will be inverse unstable.

In view of (2) it is clear that the zeros of H(z) depend on G(s), the reconstruction method used $G_r(s)$, and the sampling period T. It is of interest to investigate whether the zero properties of H(z) can be improved when reconstruction methods other than ZOH are used. Realistic practical alternatives to ZOH should of course be standard perhaps "off the shelf" reconstruction circuits which can be readily implemented in digital control.

H(z) should also be an acceptable approximation of G(s). Note that good zero properties of H(z) are of course desirable, but not at the expense of the adequacy of the model H(z). For example, under certain assumptions, as $T \to \infty$ in ZOH all the zeros of H(z) go inside the unit disc [2] as desired but H(z) becomes a poor approximation of G(s) $(H(z)=G(0)z^{-1})$. Therefore, in addition to studying the zero properties of H(z) under different reconstruction methods, one should also evaluate how well H(z) models G(s). Here, the frequency responses (magnitude and phase) of G(s) and H(z) over the interval of interest

$$-\pi/T \leq \omega \leq \pi/T \tag{4}$$

are used to study this problem. Relation (2) clearly shows the challenge. H(z), which must model G(s) only, is actually the z-transform of the product $G_r(s)G(s)$. The reconstruction circuit $G_r(s)$ will, in general, distort the characteristics of G(s) and H(z) will only be an approximation.

Two reconstruction methods are studied here, the Fractional Order Hold (FROH), and a method commonly used in Communications, the Pulse Amplitude Modulation (PAM) reconstruction (Partial Duty Cycle ZOH).

State space descriptions are used throughout this paper as they provide a convenient way to study H(z) for any sampling period T. Expressions for the zero polynomials of H(z) are easily derived and the computations are carried out via computer algorithms in the state space. In addition, note that the state space approach used allows the work presented here to be easily extended to the multivariable case; it can also be used to study other reconstruction methods.

FRACTIONAL ORDER HOLD (FROH) SIGNAL RECONSTRUCTION

Consider the Fractional Order Hold (FROH) signal reconstruction method described by

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$$\hat{u}(t) = u(kT) + \beta[\frac{u(kT)-u(kT-T)}{T}](t-kT) \quad (5)$$

for $kT \le t < kT + T$, where the approximation $\hat{u}(t)$ is formed from the samples $u(kT)$ of the signal $u(t)$ with T the sampling period; β is a real number. If $\beta=1$, (5) describes the First Order Hold (FOH) reconstruction while for $\beta=0$ it becomes

$$\hat{u}(t) = u(kT), \; kT \le t < kT + T \quad (6)$$

which is the well known "staircase" approximation called the Zero Order Hold (ZOH).

The continuous transfer function $G_\beta(s)$ of the FROH reconstruction circuit can be shown to be [3]

$$G_\beta(s) = (1-\beta e^{-Ts})\frac{1-e^{-Ts}}{s} + \frac{\beta}{Ts^2}(1-e^{-Ts})^2 \quad (7)$$

which readily reduces for $\beta=0$ to the ZOH transfer function $G_o(s)$ of (1).
Using sampled data system analysis [1], the FROH transformation of the given plant G(s) can be shown to be

$$H_\beta(z) = (1-z^{-1})(1-\beta z^{-1})Z\{G(s)/s\} + (\beta/T)(1-z^{-1})^2 Z\{G(s)/s^2\} \quad (8)$$

We are interested in studying the location of the zeros of the discrete transfer function $H_\beta(z)$. $H_\beta(z)$ is given by the rather difficult to evaluate expression (8). The study of zero locations can perhaps be more conveniently carried out if state-space representations are introduced and this is done in the following. An additional important advantage of the state-space approach is the availability and accuracy of existing numerical techniques.

State-Space Analysis. Let

$$\dot{x}(t) = Fx(t) + Gu(t) \;,\; y(t) = Hx(t) \quad (9)$$

be a minimal realization of G(s). The state x(t) at the sampling instants is given by

$$x(kT+T) = e^{FT}x(kT) + \int_{kT}^{kT+T} e^{F(kT+T-\tau)}Gu(\tau)d\tau \quad (10)$$

where $u(t) = \hat{u}(t)$ of (5). Substituting, the sampled system is described by

$$\begin{bmatrix} x(kT+T) \\ x_1(kT+T) \end{bmatrix} = \begin{bmatrix} \phi & \beta\Lambda \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(kT) \\ x_1(kT) \end{bmatrix} + \begin{bmatrix} \Gamma - \beta\Lambda \\ 1 \end{bmatrix} u(kT)$$

$$y(kT) = [H \quad 0] \begin{bmatrix} x(kT) \\ x_1(kT) \end{bmatrix} \quad (11)$$

where $x_1(kT) \overset{\Delta}{=} u(kT-T)$ was introduced so that the delayed input will not appear in the state equation. Notice that the sampled system description under ZOH($\beta=0$) is

$$x(kT+T) = \phi x(kT) + \Gamma u(kT), \; y(kT) = Hx(kT). \quad (12)$$

The matrices in (11) are defined by [1,4]:

$$\phi=e^{FT} \;,\; \Gamma=[\int_0^T e^{F\eta}d\eta]G, \; \Lambda=[-\int_0^T(1-\eta/T)e^{F\eta}d\eta]G \quad (13)$$

These matrices can be easily evaluated using the computer. In particular, write $\phi=I+FT\psi$ and $\Gamma=\psi TG$, where ψ is defined by

$$\psi = \sum_{i=o}^{\infty} F^iT^i/(i+1)! \quad (14)$$

and evaluated using the algorithms suggested in [1, Appendix A of Chapter 6]. Similarly, Λ can be written as

$$\Lambda = [-\sum_{i=o}^{\infty} F^iT^{i+1}/(i+2)!]G \quad (15)$$

which can also be expressed in terms of ψ for computer implementation [4].

The pole polynomial $P_\beta(z)$ of (11) is $P_\beta(z) = z|zI-\phi| = z\Pi(z-\exp(p_iT)) = zP_o(z)$. Note that the pole at the origin reflects the delayed sample $u(kT-T)$ used in FROH (5). The zero polynomial $Z_\beta(z)$ of (11) is the determinant of its Rosenbrock's System matrix [5], which can be written as

$$Z_\beta(z) = z\begin{vmatrix} zI-\phi & \Gamma-\beta\Lambda \\ -H & 0 \end{vmatrix} + \begin{vmatrix} zI-\phi & \beta\Lambda \\ -H & 0 \end{vmatrix}$$

$$= zZ_o(z) + \beta(z-1)Z_{\beta1}(z) \quad (16)$$

with

$$Z_o(z)=\begin{vmatrix} zI-\phi & \Gamma \\ -H & 0 \end{vmatrix}, \; Z_{\beta1}(z)= -\begin{vmatrix} zI-\phi & \Lambda \\ -H & 0 \end{vmatrix} \quad (17)$$

where properties of the determinants were used in the derivation. Therefore an alternative to (8) expression for $H_\beta(z)$ is

$$H_\beta(z)=Z_\beta(z)/P_\beta(z)=[\beta(z-1)Z_{\beta1}(z)+zZ_o(z)]/zP_o(z) \quad (18)$$

In view of (12) and the definition of $P_o(z)$ and $Z_o(z)$, it is clear that $H_o(z)$, the ZOH discrete equivalent of G(s), is given by

$$H_o(z) = Z_o(z)/P_o(z) \quad (19)$$

Observe that when $\beta=0$ in (16), the zero at the origin (in $zZ_o(z)$) is not a transmission zero but it is an output decoupling zero [5]; alternatively, when $\beta=0$ the eigenvalue at the origin corresponds to an unobservable mode. This is to be expected as the description for the ZOH case are given by (12) and (19).

Åström et al [2] (also [6,7]) studied the zeros of H(z) under ZOH reconstruction. They showed [2, Theorem 1] that given G(s) with m zeros and n poles (m<n), as $T\to0$, m zeros of H(z) go to 1 while the remaining n-m-1 zeros of H(z) go to the zeros of $B_{n-m}(z)$. These polynomials $B_n(z)$ are the numerator polynomials of the ZOH discrete equivalent of $G(s) = 1/s^n$. In particular

$$H_o(z) = (T^n/n!)B_n(z)/(z-1)^n \quad (20)$$

where $B_1(z) = 1$, $B_2(z) = z+1$, $B_3(z) = z^2+4z+1$, etc (see [2]); notice that the polynomials $B_n(z)$ have zeros outside or on the unit circle for $n \ge 2$. It can be easily seen that the poles and zeros of the

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ZOH discrete equivalent $H_0(z)$ of a strictly proper transfer function $G(s)$ go to the poles and zeros of the ZOH equivalent of $1/s^{n-m}$ as $T \to 0$; alternatively, as $T \to 0$, $G(s)$ can be approximated by its high frequency model $1/s^{n-m}$, leading to the same ZOH equivalent [7].

In [4], the zero polynomial of the FROH discrete equivalent of $G(s) = 1/s^n$ has been determined explicitly via a minimal realization $\{F,G,H\}$ of $G(s)$ in controllable companion form [8] using the fact that the matrices in (17), the determinant of which must be calculated, are in the Hessenberg form [9]. In particular it has been shown that

$$Z_\beta(z) = \frac{T^n}{n!}(z-\beta)B_n(z) + \frac{\beta T^n}{(n+1)!}B_{n+1}(z) \ ,$$

$$Z_0(z) = \frac{T^n}{n!}B_n(z) \qquad (21)$$

In [4] $B_n(z)$ were derived from the state space descriptions; in addition, a novel recursive defining relation was introduced, namely:

$$B_n(z) = \sum_{k=0}^{n-1}\binom{n}{k}(z-1)^{n-k-1}B_k(z), B_0(z) \triangleq 1 \qquad (22)$$

The following Proposition has been shown in the above analysis.

Proposition 1: Let $\{F,G,H\}$ a minimal realization of $G(s)$. Under FROH reconstruction, the zero polynomial $Z_\beta(z)$ of the discrete equivalent $H_\beta(z)$ is given by (16). In the special case of $G(s) = 1/s^n$ an explicit expression for $Z_\beta(z)$ is given by (21).

Root Locus Analysis: The zero polynomial $Z_\beta(z)$ is given by (16) where β is a real number. Notice that the zeros of $Z_0(z)$ and $Z_{\beta 1}(z)$ are the zeros of the systems $\{\phi,\Gamma,H\}$ and $\{\phi,\Lambda,H\}$; this can be seen from (17) where the matrices involved are the Rosenbrock's System matrices [5] of the above state space descriptions. These zeros can be determined using any appropriate state-space algorithm. In [4], the multivariable root-locus algorithm of [10] was successfully implemented to determine the zeros of $Z_0(z)$ and $Z_{\beta 1}(z)$.

In view of (16), it is clear that the standard Root-Locus approach can be used to determine the range of β for which the zeros of $Z_\beta(z)$ lie inside the unit disc. If such β exists, the corresponding FROH reconstruction will give an $H_\beta(z)$ which is inverse stable. This method was applied to several transfer functions. As an example, let $G(s) = 1/(s+1)^3$. It was shown in [2] that the ZOH equivalent $H_0(z)$ has a zero outside the unit disc if $0 \leq T < 1.8399$. Using FROH reconstruction, the minimum T for stable zeros, T_{min}, can be significantly reduced. One can choose any $T < 1.8399$ and then determine via our Root-Locus analysis the appropriate range for β. For example if $T = .5$, for $-.77 < \beta < -.7572$ the zeros are inside the unit disc (although quite close to the unit circle: for $\beta = -.76$ the zeros are at $-.976 \pm j.0999$, $-.36$).

In summary, our study has shown that in many cases the zero locations of $H_\beta(z)$ can be improved by appropriately choosing β; the T_{min} for stable zeros can be reduced compared to the ZOH reconstruction. The FROH circuit is of course more complex than the ZOH; actually it can be implemented via a configuration involving two ZOH circuits [4]. The additional complexity can perhaps be justified by the improved zero properties of $H_\beta(z)$.

There is another important issue which needs to be addressed when a discrete equivalent $H(z)$ of $G(s)$ is used, namely, how well $H(z)$ models $G(s)$. It is understood that the degree of the approximation required depends very much on the requirements of the design to be carried out and many times successful designs do not involve highly accurate signal approximations. On the other hand, if in studying the zero properties of $H(z)$ the accuracy of the approximation is ignored, one could be led to erroneous conclusions such as choosing an $H(z)$ with excellent zero properties which has little to do with the given $G(s)$ (see example in Introduction). We have chosen to use the frequency response (magnitude and phase plots) as a measure to study how well $H(z)$ models $G(s)$. This study will also help us understand why zeros appear outside the unit disc.

Frequency Response Analysis: Consider first the ZOH reconstruction. $G_0(s)$ (in (1)) has magnitude and phase given by

$$|G_0(j\omega)| = T|\mathrm{sinc}(\omega T/2)|, \ \mathrm{Phase}[G_0(j\omega)] = -\omega T/2. \qquad (23)$$

$H_0(z)$ which should match $G(s)$ over the range $-\pi/T \leq \omega \leq \pi/T$, is determined from G_0G. Notice the amplitude attenuation and the phase shift caused to G as $\omega \to \pi/T$ (for $\omega = \pi/T$, there is a phase shift of $-90°$) which implies that $H_0(\exp(j\omega T))$ will only approximate $G(j\omega)$ with the approximation becoming worse as ω approaches π/T. In addition, note that the range of interest becomes larger (π/T) as T becomes smaller. Consider $G(s) = 1/s^3$ with $T = .5$ in which case $H_0(z) = (1/48)(z^2+4z+1)/(z-1)^3$. The magnitude of H_0 matches the magnitude of G quite well up to π/T. There is not much aliasing due to the low pass character of $G(s)$. There is considerable distortion in the phase due to the ZOH circuit. Noticeable phase lag appears at .1 rad/sec and at π/T the phase lag is $-\pi/2$. Notice that $H_0(z)$ has three poles (at +1) and two zeros. These poles and zeros (plus the gain) contribute to magnitude and phase and they give the plots of H_0 which match those of G_0G. Let's examine what happens at a particular frequency, say $\pi/2T$. The poles are fixed at +1 and they contribute $-405°$ phase. The phase of G_0G at this frequency is $-315°$, $-270°$ due to $G(s)$ and $-45°$ due to G_0. The two zeros of H_0(at a_1 and a_2) must make this correction and contribute to $90°$ phase, that is $\tan^{-1}[1/a_1] + \tan^{-1}[1/a_2] = 90°$. If they are both inside the unit disc they contribute too much phase; closer examination shows that we must have $a_2 = 1/a_1$ which implies that one zero must be outside the unit disc. This type of analysis can be applied to any strictly proper $G(s)$ (of order n). $H_0(z)$ always has n poles (at $\exp(p_iT)$) and n-1 zeros. For each T, the location of the n-1 zeros will be so that together with the poles and the gain, the frequency (magnitude and phase) characteristics of $H_0(z)$ match those of $G_0G(s)$.

When the FROH reconstruction is used, the poles of $H_\beta(z)$ are at the same locations as before ($\exp(p_iT)$) with the addition of a pole at the origin. This is for any $\beta(\beta \neq 0)$. Varying β we alter the frequency characteristics of G_β and of $G_\beta G$ which, in view of the fact that the poles of $H_\beta(z)$ are fixed, causes a shift in the zero locations of

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$H_\beta(z)$. Indeed, for $G(s) = 1/(s+1)^3$, by using FROH with $-.77 < \beta < -.7572$ and $T = .5$ (as it was already discussed), $H_\beta(z)$ is inverse stable, while using ZOH, $H_0(z)$ was inverse unstable for $0 \leq T < 1.8399$. Comparing the magnitude and phase plots of $H_\beta(z)$ and $G(s)$ in this case ($\beta=-.76$, $T=.5$) of Figure 2 as well as with the plots of the ZOH reconstruction $H_0(z)$ ($T=.5$) of Figure 1 it becomes clear that the discrete equivalent $H_0(z)$ is a better approximation of $G(s)$ than $H_\beta(z)$; that is, stable zeros were obtained at the expense of worsening the discrete approximation.

In summary, the discrete equivalent $H(z)$ is determined from $G_rG(s)$ (see (2)). The reconstruction circuit $G_r(s)$ generally distorts $G(s)$ and the frequency response of $H(z)$ only approximates $G(s)$ over $-\pi/T \leq \omega \leq \pi/T$. The amount of distortion depends on the type of reconstruction circuit used and the sampling period T and if the minimum distortion possible is not acceptable another reconstruction circuit should be used. The zero locations of $H(z)$ depend on (the magnitude and phase of) $G_rG(s)$. Note that the number of poles and zeros of $H(z)$ is fixed and for given T the pole locations are also fixed. As it was shown, using FROH with fixed T, by varying β one alters the zero locations of $H(z)$; G_β contributes magnitude and phase to $G_\beta G$ which causes the zeros of $H(z)$ to shift. The contribution (distortion) of G_β can be such that the zero locations of $H(z)$ improve. The amount of improvement depends on $G(s)$ and its relation to the magnitude and phase contributed by the FROH. It should be noted that in some cases $H(z)$ becomes inverse stable only when considerable distortion is introduced by the reconstruction circuit, and this was shown above by example. It is therefore clear that for FROH, $H(z)$ becoming a better model of a minimum phase $G(s)$, it does not necessarily imply that $H_\beta(z)$ will become inverse stable. This fact can be used sometimes in our advantage to obtain inverse stable $H_\beta(z)$ from a nonminimum phase $G(s)$.

PULSE AMPLITUDE MODULATION (PAM) RECONSTRUCTION

Consider the Pulse Amplitude Modulation (PAM) [11] signal reconstruction method described by

$$\hat{u}(t) = \begin{cases} u(kT) & kT \leq t < kT+\tau \\ 0 & kT+\tau \leq t < kT+T \end{cases} \tag{24}$$

This signal reconstruction is also known as Partial Duty Cycle ZOH as the value of the signal $u(kT)$ remains constant for only part (τ) of the cycle of length T. It can be easily shown that the continuous transfer function $G_p(s)$ of the PAM reconstruction is:

$$G_p(s) = (1-e^{-\tau s})/s \quad , \quad 0 < \tau \leq T \tag{25}$$

As expected, if $\tau=T$, $G_p(s) = G_0(s)$ of the ZOH reconstruction. The PAM discrete equivalent of $G(s)$ is

$$H_p(z) = Z\{(1-e^{-\tau s})G(s)/s\} \tag{26}$$

and the state-space descriptions will again be used to facilitate the analysis.

<u>State-Space Analysis</u>: Let $\{F,G,H\}$ in (9) be a minimal realization of $G(s)$. The state $x(t)$ at the sampling instants is given by

$$x(kT+T) = e^{FT}x(kT) + \int_{kT}^{kT+\tau} e^{F(kT+T-\sigma)}Gu(\sigma)d\sigma \tag{27}$$

where, in view of (26), $u(t) = u(kT)$. The sampled system is then described by

$$x(kT+T) = \phi x(kT) + \Gamma_1 u(kT), \; y(kT) = Hx(kT) \tag{28}$$

$$\text{where} \quad \phi = e^{FT} \quad \text{and} \quad \Gamma_1 = [\int_{T-\tau}^{T} e^{F\eta}d\eta]G \tag{29}$$

Similarly to (13), these matrices can be expressed in a series form. Let

$$\psi(a) = \sum_{i=0}^{\infty} F^i a^i/(i+1)! \tag{30}$$

Then $\psi(T) = \psi$ as defined in (14), $\phi = \phi(T)$ where $\phi(a) = e^{Fa} = I + a\psi(a)F$ and $\Gamma_1 = \tau\phi(T-\tau)\psi(\tau)G$. ϕ and Γ_1 can be evaluated [4] using the algorithms of [1, Appendix A of Chapter 6]. Notice that these expressions are similar to the ones describing sampled delay systems in the state space.

It is of interest to determine what happens to $H_p(z)$ as $\tau\to0$. For this, we shall assume that the PAM circuit also includes a (normalization) gain $1/\tau$. The transfer function (25) now becomes

$$G_p(s)=(1/\tau s)(1-e^{-\tau s})=[1-(\tau s)/2!+(\tau s)^3/3!\ldots] \tag{31}$$

where the series expansion of $e^{-\tau s}$ was used. We are interested in frequencies satisfying $-\pi/T \leq \omega \leq \pi/T$. So for $\tau\to0$, $\tau s\to0$ and $G_p(s) \to 1$. Therefore, the PAM discrete equivalent in this case becomes

$$H_p(z) \to Z\{G(s)\}. \tag{32}$$

This relation implies that evaluating the zeros of the PAM equivalent $H_p(z)$ when $\tau\to0$ is quite straightforward. $Z\{G(s)\}$ can be either found in transform tables directly or after using partial fractions. In addition note that the z-transform of $G(s)$ can also be found from the state space description $\{F,G,H\}$. In particular, in view of the definition of the z-transform,

$$Z\{G(s)\} = Z[r(t)] = \sum_{k=0}^{\infty} r(kT)z^{-k} \tag{33}$$

and the fact that $r(t) = He^{Ft}G$ is the impulse response,

$$Z\{G(s)\} = z[H(zI-\phi)^{-1}G] \tag{34}$$

where $\phi = e^{FT}$. Therefore:

<u>Lemma</u>: Let $\{F,G,H\}$ a minimal realization of $G(s)$. The z-transform of $G(s)$ is given by (34).

This Lemma shows that the zeros of the z-transform $Z\{G(s)\}$ can be determined using our state-space algorithms [4] and the Rosenbrock's System matrix [5] of $\{\phi,G,H\}$ with $\phi = e^{FT}$.

Returning to the PAM reconstruction, the following results have been shown above:

<u>Proposition 2</u>: Let $\{F,G,H\}$ a minimal realization of $G(s)$. Under PAM reconstruction, the zero polynomial of $H_p(z)$ is given by the determinant of the Rosenbrock's System matrix of (28). Furthermore, as $\tau\to0$, $H_p(z) \to Z\{G(s)\}$.

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As it was shown, if $\tau \to 0$, $H_p(z) \to Z\{G(s)\}$ and for $\tau = T$ $H_p(z) = H_o(z)$ the ZOH discrete equivalent. In view of the fact that the ZOH (3) involves $Z\{G(s)/s\}$, the zero properties of $H_p(z)$ as $\tau \to 0$ will, in many cases, improve because they involve $Z\{G(s)\}$, a lower relative degree transfer function. Depending on G(s) this improvement may lead to zeros of $H_p(z)$ inside the unit disc. As an example consider $G(s) = 1/s^n$. In view of (20) and (32),

$$H_p(z) = H_o(z) = \frac{T^n}{n!}\frac{B_n(z)}{(z-1)^n} \qquad (\tau = T)$$
$$H_p(z) \to Z\{G(s)\} = \frac{T^{n-1}}{(n-1)!}\frac{zB_{n-1}(z)}{(z-1)^n} \qquad (\tau \to 0) \tag{35}$$

which shows that by allowing $\tau \to 0$ we improve the zero properties by moving the zeros from the zeros of B_n to B_{n-1}. This implies that for n=2, the inverse unstable ZOH equivalent $H_o(z)$ becomes inverse stable if PAM with $\tau \to 0$ is used.

The above suggests the following procedure: Given G(s), determine the zeros of the ZOH equivalent $H_o(z)$ ($\tau = T$). If improvement is desirable, determine the zeros of $Z\{G(s)\}$($\tau \to 0$). Then choosing τ between 0 and T one could achieve good zero properties with acceptable τ. Note that all these calculations can be easily carried out using the computer algorithms described above. In the analysis, $T(\tau)$ can be kept fixed and $\tau(T)$ can be varied. It should be noted that in applications of PAM to Communications, $\tau/T = 1/16$ is quite common.

The magnitude and phase of the PAM reconstruction $G_p(s)$ are

$$|G_p(j\omega)| = \tau\,|\mathrm{sinc}(\omega\tau/2)|, \quad \mathrm{Phase}[G_p(j\omega)] = -\omega\tau/2 \tag{36}$$

As $\tau \to 0$ (assuming $1/\tau$ gain for normalization) it is clear that the influence of $G_p(s)$ on G(s) will become negligible as expected. This shows that PAM is a reconstruction circuit which, for small values of the parameter, introduces no distortion in $G_pG(s)$, thus leading to a better approximation H(z) of G(s). In addition, as we saw, we expect the zero properties of $H_p(z)$ to improve as τ becomes smaller.

As an example, consider $G(s) = 1/(s+1)^3$ which was also studied under FROH reconstruction. When T=.5 the zeros of the ZOH equivalent $H_o(z)$ are outside the unit disc, namely at -2.58 and -.183. Using PAM reconstruction with τ=.1, the zeros of $H_p(z)$ are inside the unit disc at -.873 and -.007106 with poles at the same locations as for the ZOH. When $\tau = T/16 = .03125$ the zeros are at -.68444 and $-.7516 \times 10^{-3}$. In addition, the smaller τ is the better model of G(s) $H_p(z)$ is over $-\pi/T \leq \omega \leq \pi/T$. See Figure 3 and compare with Figures 1 and 2.

PAM reconstruction can be implemented by a standard "off the shelf" Digital to Analog converter used for ZOH. To force the output to go to zero after τ secs a special device could be designed. In many applications this will not be necessary; the signal processor can just set its output equal to zero τ secs after the sample has appeared in the output. Zeroing the input to the plant G(s) might not be practical in certain cases; however, the antialiasing filter will tend to smooth out the actual input to the plant thus reducing this undesirable effect.

CONCLUDING REMARKS

Motivated by the work of Åström et al. in [2], two alternatives to the Zero Order Hold (ZOH) reconstruction method, the Fractional Order Hold (FROH) and the Pulse Amplitude Modulation (PAM) were studied. It was shown that they can be used to improve the zero properties of the discrete equivalent H(z) in many cases. H(z) should also be a good model of the continuous plant G(s), the desired accuracy of course depending on the particular application. This problem was studied in this paper using magnitude and phase plots. In FROH, an example was given to show that while the zero properties were improved over ZOH, the discrete equivalent was a poorer approximation of G(s). It was shown that, one can obtain better models H(z) when PAM is used; furthermore the zero locations can also be improved. Note that both reconstruction methods studied reduce to the ZOH for special values of the parameters.

State-space descriptions and algorithms were used throughout this paper and efficient ways to numerically study the zero properties of H(z) for any value of the sampling period T (not only when $T \to 0$ or $T \to \infty$) were introduced. Note that similar methods were used in [4] to study delay systems under ZOH and PAM reconstruction. It should be pointed out that these state space methods can also be used to study other reconstruction methods applied to more general (e.g. multivariable) systems.

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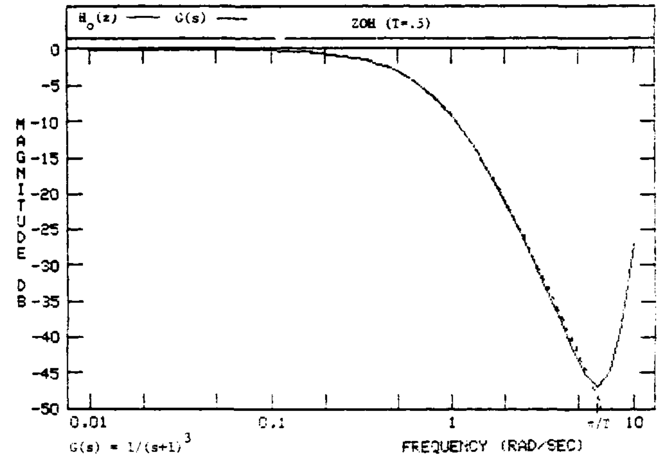

Figure 1a

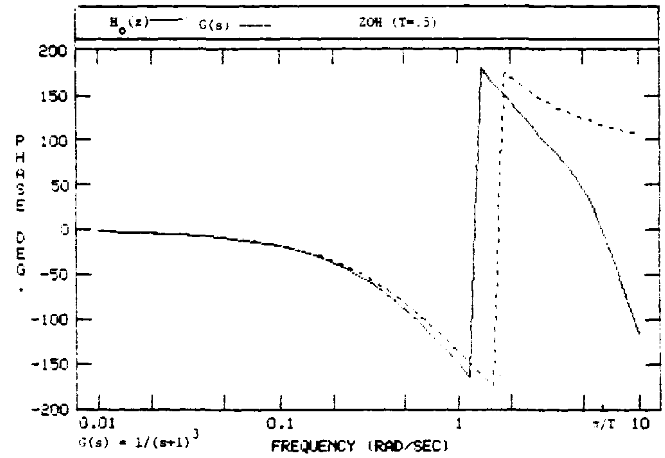

Figure 1b

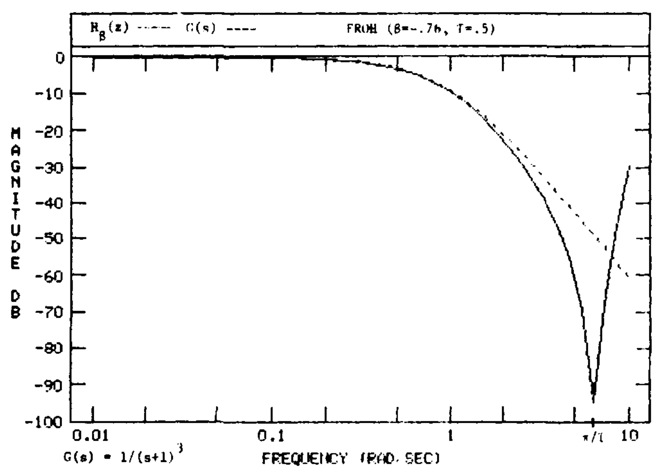

Figure 2a

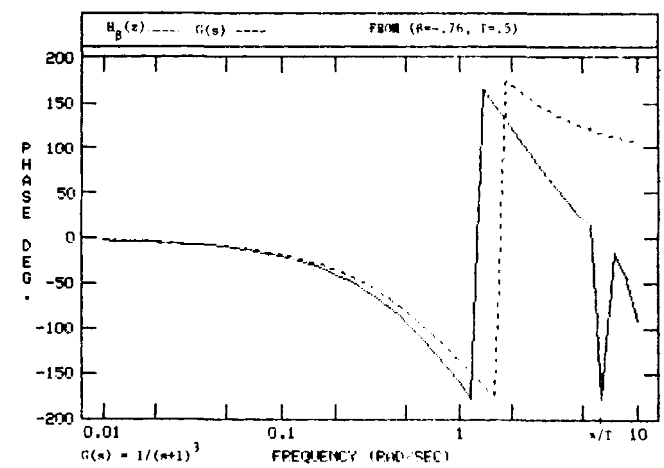

Figure 2b

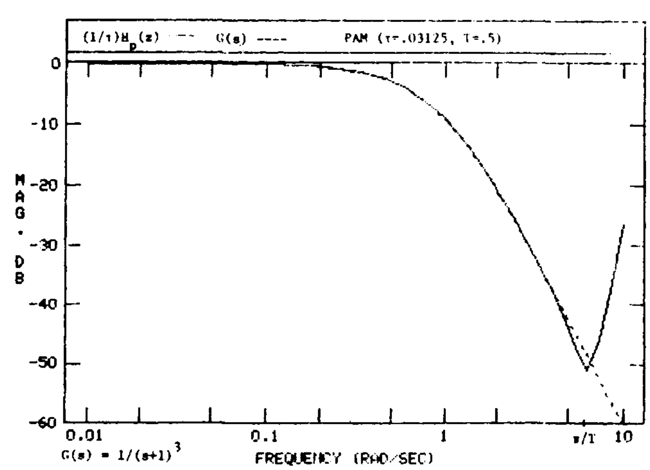

Figure 3a

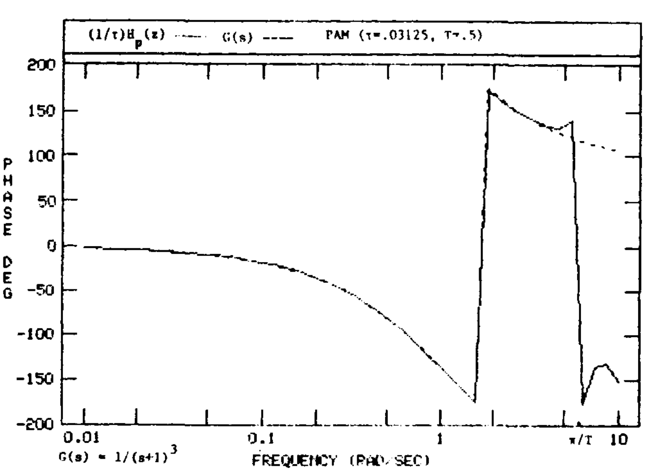

Figure 3b