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QUANTIZATION IN MODEL BASED NETWORKED CONTROL SYSTEMS

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Abstract: Model-Based Networked Control Systems (MB-NCS) use a model of the plant to compensate for the lack of information between transmission times. This results in a significant reduction of the bandwidth used for stabilizing the control system. Previously published stability results for MB-NCS assume no quantization error. In this paper quantization is introduced for MB-NCS. Sufficient stability conditions for static uniform, static logarithmic, and dynamic quantizers for continuous linear time-invariant plants are derived. The results illustrate the effects of quantization over the stability of MB-NCS and suggest a design model that starts with the non-quantized MB-NCS.

Keywords: Model-Based Networked Control System, Quantization, Limited Information.

1. INTRODUCTION

The use of networks as media to interconnect the different components in control systems is rapidly increasing. These systems are commonly referred to as Networked Control System (NCS). In summary a NCS is a control system in which a data network is used as feedback media. The use of networked control systems poses, though, some challenges. One of the main problems to be addressed when considering a networked control system is the size of bandwidth required by each subsystem. In this paper, we consider the problem of reducing the bandwidth an NCS using a novel approach called Model-Based NCS (MB-NCS). MB-NCS were introduced in (Montestruque, *et al.*, 2002). The MB-NCS architecture makes explicit use of knowledge about the plant dynamics to enhance the performance of the system.

Several results have been published regarding the issues involved with quantization in NCS and sampled data problems, see (Elia, *et al.*, 2001; Ling, *et al.*, 2004; Nair, *et al.*, 2000a and 2000b). Most results characterize the stability properties of NCS when the number of bits used by each network packet is finite and small. The goal of MB-NCS is the reduction of bandwidth, but the design of the MB-NCS first attempts to reduce the bandwidth by reducing the rate at which packets are sent. A second step is to further reduce the bandwidth by reducing the number of bits used to transmit each packet. In this way the designer has a number of parameters that can be modified, namely the model uncertainty, the packet transmission times, and finally the number of bits used for each packet.

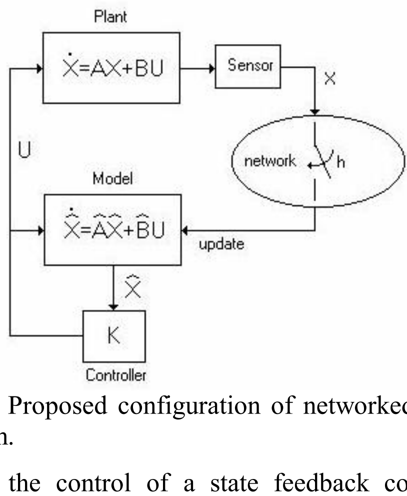

Figure 1: Proposed configuration of networked control system.

Consider the control of a state feedback continuous linear plant where the state sensor is connected to a linear controller/actuator via a network. In this case, the controller uses an explicit model of the plant that approximates the plant dynamics and makes possible the stabilization of the plant even under slow network conditions.

The plant model is used at the controller/actuator side to recreate the plant behavior so that the sensor can delay sending data since the model can provide an approximation of the plant dynamics. *The main idea is to perform the feedback by updating the model's state using the actual state of the plant that is provided by the sensor. The rest of the time the control action is based on a plant model that is incorporated in the controller/actuator and is running open loop for a period of h seconds.* The control architecture is shown in Figure 1.

If all the states are available, then the sensors can send this information through the network to update the model's vector state. For our analysis we will assume that the compensated model is stable and that the transportation delay is negligible. We will assume that the frequency at which the network updates the state in the controller is constant. The idea is to find the smallest frequency at which the network must update the state in the controller, that is, an upper bound for h, the update time.

Consider the control system of Figure 1 where plant is given by $\dot{x} = Ax + Bu$, the plant model by $\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u$, and the controller by $u = K\hat{x}$. The state error is defined as $e = x - \hat{x}$, and represents the difference between the plant state and the model state. The modeling error matrices $\tilde{A} = A - \hat{A}$ and $\tilde{B} = B - \hat{B}$ represent the difference between the plant and the model. Also define the error $e(t) = x(t) - \hat{x}(t)$. A necessary and sufficient condition for stability of the state feedback MB-NCS without quantization will now be presented.

Theorem #1

The State Feedback MB-NCS without quantization is globally exponentially stable around the solution $z = \begin{bmatrix} x & e \end{bmatrix}^T = \mathbf{0}$ *if and only if the eigenvalues of* $M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda h} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ *are strictly inside the unit circle.*

A detailed proof for Theorem **1** can be found in (Montestruque, *et al.*, 2002 and 2003).

In this paper stability conditions for MB-NCS under popular quantization schemes are derived. The paper is organized as follows, in Section 2 the stability of MB-NCS with Static Quantizers is addressed. Then is Section 3 MB-NCS with Dynamic Quantizers are discussed. Conclusions are presented at the end.

2. STATIC QUANTIZATION

In this subsection we address the stability analysis of a state feedback MB-NCS using a static quantizer. Static quantizers have defined quantization regions that do not change with time. They are an important class of quantizers since they are simple to implement in both hardware and software and are not computationally expensive as their dynamic counterparts. Two types of quantizers are analyzed here, namely uniform quantizers and logarithmic quantizers. Each quantizer is associated with two popular data representations. The uniform quantizer is associated with the fixed-point data representation. Indeed, fixed-point numbers have a constant maximum error regardless of how close is the actual number to the origin. Logarithmic quantizers on the other hand are associated with floating-point numbers, this allows the maximum error to decrease as the actual number is close to origin.

2.1. Uniform Quantizers

We will define a uniform quantizer as function $q : \mathbb{R}^n \to \mathbb{R}^n$ with the following property:

$$\left\| z - q(z) \right\| \le \delta,\ z \in \mathbb{R}^n, \delta > 0 \qquad (1)$$

Theorem 2

Assume that the state feedback MB-NCS networked system without quantization is stable and satisfies:

$$\left(e^{(\hat{A}+\hat{B}K)^T h} + \Delta(h)^T \right) P \left(e^{(\hat{A}+\hat{B}K)h} + \Delta(h) \right) - P = -Q_D \qquad (2)$$

with Q_D and P symmetric and positive definite. Then when using the uniform quantizer defined by (1), the state feedback MB-NCS plant state will enter and remain in the region $\|x\| \le R$ defined by:

$$R = \left(e^{\bar{\sigma}(\hat{A}+\hat{B}K)h} + \Delta_{\max}(h) \right) r + \left(e^{\bar{\sigma}(A)h} + \Delta_{\max}(h) \right) \delta$$

$$\text{where } r = \sqrt{\frac{\lambda_{\max}\left(\left(e^{Ah} - \Delta(h) \right)^T P \left(e^{Ah} - \Delta(h) \right)^T \right) \delta^2}{\lambda_{\min}(Q_D)}}$$

$$\text{and } \Delta_{\max}(h) = \int_0^h e^{\bar{\sigma}(A)(h-\tau)} \bar{\sigma}\left(\tilde{A} + \tilde{B}K \right) e^{\bar{\sigma}(\hat{A}+\hat{B}K)\tau} d\tau$$

Proof:

The response for the error is given now by:

$$\begin{aligned} e(t) &= e^{A(t-t_k)} e(t_k) + \Delta(t-t_k)\hat{x}(t_k^+) \\ &= \left(e^{A(t-t_k)} - \Delta(t-t_k) \right) e(t_k) + \Delta(t-t_k) x_k \end{aligned} \qquad (3)$$

where $\Delta(t-t_k) = \int_0^{t-t_k} e^{A(t-t_k-\tau)} \left(\tilde{A} + \tilde{B}K \right) e^{(\hat{A}+\hat{B}K)\tau} d\tau$

The contribution due to $e(t_k)$ initial value will grow exponentially with time and with a rate that corresponds to the uncompensated plant dynamics. So at time $t \in [t_k, t_{k+1}]$ the plant state is:

$$\begin{aligned} x(t) &= \hat{x}(t) + e(t) \\ &= e^{(\hat{A}+\hat{B}K)(t-t_k)} x_k + \left(e^{A(t-t_k)} - \Delta(t-t_k) \right) e(t_k) \\ &\quad + \Delta(t-t_k) x_k \end{aligned} \qquad (4)$$

We can therefore evaluate the Lyapunov function at any instant in time $t \in [t_k, t_{k+1}]$. It is know that for uniformly exponential stability we require (Ye, *et al.*, 1998) that:

$$\frac{1}{h}\left(V\left(x(t_{k+1}) \right) - V\left(x(t_k) \right) \right) \le -c\left(\left\| x(t_k) \right\|^2 \right),\ c \in \mathbb{R}^+ \qquad (5)$$

We are interested in its value at t_{k+1}:

$$
\begin{aligned}
&V\left(x\left(t_{k+1}\right)\right)=x\left(t_{k+1}\right)^T Px\left(t_{k+1}\right)\\
&=x_k^T\left(\mathrm{e}^{\left(\hat{A}+\hat{B}K\right)h}+\Delta(h)\right)^T P\left(\mathrm{e}^{\left(\hat{A}+\hat{B}K\right)h}+\Delta(h)\right)x_k \qquad (6)\\
&\quad+e_k^T\left(\mathrm{e}^{Ah}-\Delta(h)\right)^T P\left(\mathrm{e}^{Ah}-\Delta(h)\right)e_k
\end{aligned}
$$

where $h=h_k=t_{k+1}-t_k>0,\ e_k=e\left(t_k\right)$

So from (6) we obtain:

$$
\begin{aligned}
&V\left(x\left(t_{k+1}\right)\right)-V\left(x\left(t_k\right)\right)\\
&=e_k^T\left(\mathrm{e}^{Ah}-\Delta(h)\right)^T P\left(\mathrm{e}^{Ah}-\Delta(h)\right)e_k-x_k^T Q_D x_k
\end{aligned}
\tag{7}
$$

We can bound (7) by:

$$
\begin{aligned}
&e_k^T\left(\mathrm{e}^{Ah}-\Delta(h)\right)^T P\left(\mathrm{e}^{Ah}-\Delta(h)\right)e_k-x_k^T Q_D x_k\\
&\le\lambda_{\max}\left(\left(\mathrm{e}^{Ah}-\Delta(h)\right)^T P\left(\mathrm{e}^{Ah}-\Delta(h)\right)\right)\delta^2 \qquad (8)\\
&\quad-\lambda_{\min}\left(Q_D\right)\left\|x_k\right\|^2
\end{aligned}
$$

The sampled value of the state of the plant at the update times will enter the region $\|x\|\le r$ where:

$$
r=\sqrt{\frac{\lambda_{\max}\left(\left(\mathrm{e}^{Ah}-\Delta(h)\right)^T P\left(\mathrm{e}^{Ah}-\Delta(h)\right)\right)\delta^2}{\lambda_{\min}\left(Q_D\right)}}
\tag{9}
$$

The plant state vector might exit this region between samples. The maximum magnitude the state plant can reach between samples after reaching the sphere $\|x\|\le R$ is given by:

$$
\begin{aligned}
&\|x(t)\|\\
&=\left\|\left(\mathrm{e}^{\left(\hat{A}+\hat{B}K\right)\left(t-t_k\right)}+\Delta\left(t-t_k\right)\right)x_k+\left(\mathrm{e}^{A\left(t-t_k\right)}-\Delta\left(t-t_k\right)\right)e_k\right\|\\
&\le\left(\mathrm{e}^{\bar{\sigma}\left(\hat{A}+\hat{B}K\right)h}+\Delta_{\max}(h)\right)r+\left(\mathrm{e}^{\bar{\sigma}(A)h}+\Delta_{\max}(h)\right)\delta
\end{aligned}
$$

where $\Delta_{\max}(h)=\int_0^h \mathrm{e}^{\bar{\sigma}(A)(h-\tau)}\bar{\sigma}\left(\tilde{A}+\tilde{B}K\right)\mathrm{e}^{\bar{\sigma}\left(\hat{A}+\hat{B}K\right)\tau}d\tau$

♦

2.2. Logarithmic Quantizers

We will define a logarithmic quantizer as function $q:\mathbb{R}^n\to\mathbb{R}^n$ with the following property:

$$
\|z-q(z)\|\le\delta\|z\|,\ z\in\mathbb{R}^n,\delta>0
\tag{10}
$$

Theorem 3

Assume that the state feedback MB-NCS without quantization is stable and satisfies:

$$
\left(\mathrm{e}^{\left(\hat{A}+\hat{B}K\right)^T h}+\Delta(h)^T\right)P\left(\mathrm{e}^{\left(\hat{A}+\hat{B}K\right)h}+\Delta(h)\right)-P=-Q_D
$$

with Q_D and P symmetric and positive definite. Then when using the logarithmic quantizer defined by (10), the state feedback MB-NCS is exponentially stable if:

$$
\delta<\sqrt{\frac{\lambda_{\min}\left(Q_D\right)}{\lambda_{\max}\left(\left(\mathrm{e}^{Ah}-\Delta(h)\right)^T P\left(\mathrm{e}^{Ah}-\Delta(h)\right)\right)}}
$$

Proof:

The difference between the values of the plant's state Lyapunov function at two consecutive update times is given by:

$$
\begin{aligned}
&V\left(x\left(t_{k+1}\right)\right)-V\left(x\left(t_k\right)\right)\\
&=e_k^T\left(\mathrm{e}^{Ah}-\Delta(h)\right)^T P\left(\mathrm{e}^{Ah}-\Delta(h)\right)e_k-x_k^T Q_D x_k
\end{aligned}
\tag{11}
$$

We can now bound (11) using the quantizer property given in (10) by:

$$
\begin{aligned}
&e_k^T\left(\mathrm{e}^{Ah}-\Delta(h)\right)^T P\left(\mathrm{e}^{Ah}-\Delta(h)\right)e_k-x_k^T Q_D x_k\\
&\le\lambda_{\max}\left(\left(\mathrm{e}^{Ah}-\Delta(h)\right)^T P\left(\mathrm{e}^{Ah}-\Delta(h)\right)\right)\delta^2\left\|x_k\right\|^2 \qquad (12)\\
&\quad-\lambda_{\min}\left(Q_D\right)\left\|x_k\right\|^2
\end{aligned}
$$

This allows us to ensure exponential stability as in (5) if:

$$
\lambda_{\max}\left(\left(\mathrm{e}^{Ah}-\Delta(h)\right)^T P\left(\mathrm{e}^{Ah}-\Delta(h)\right)\right)\delta^2-\lambda_{\min}\left(Q_D\right)<0
$$

or equivalently (assuming $\left(\mathrm{e}^{Ah}-\Delta(h)\right)^T P\left(\mathrm{e}^{Ah}-\Delta(h)\right)\neq 0$):

$$
\delta<\sqrt{\frac{\lambda_{\min}\left(Q_D\right)}{\lambda_{\max}\left(\left(\mathrm{e}^{Ah}-\Delta(h)\right)^T P\left(\mathrm{e}^{Ah}-\Delta(h)\right)\right)}}
\tag{13}
$$

♦

The previously shown sufficient conditions for static quantizers relate the stability of the MB-NCS with the update time, the plant uncertainties, and the robustness of the non quantized MB-NCS characterized in this case by $\lambda_{\min}\left(Q_D\right)$.

3. DYNAMIC QUANTIZATION

In this subsection we will consider the case of dynamic quantization, where the quantized region and quantization error vary at each transmission time. It has been shown that these type of quantizers can achieve the smallest bit count per packet while maintaining stability (Ling, *et al*., 2004; Nair, *et al*., 2000a and 2000b). This comes with the price of quantizer complexity, while the static quantizers did required a relatively small amount of computations, the dynamic quantizers need to compute new quantization regions and detect the plant state presence with in this regions. Yet dynamic quantizers are an attractive alternative when the number of bits available per transmission is minimum.

We will assume that the plant model matrix $\hat{A}$ has distinct real unstable eigenvalues. This assumption can be relaxed at the expense of more complex notation and problem geometry. We will also assume that the compensated model is stable.

Previous results (Ling, *et al*., 2004; Hespanha, *et al*., 2002) consider a similar case but our result is novel in that it incorporates the plant-model mismatch within our Model-Based Networked Control Systems approach.

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Namely, at transmission time t_k the encoder partitions the hyper parallelogram R_k^- containing the plant state $x(t_k)$ into 2^N smaller hyper parallelograms and sends the decoder the symbol (encoded as N bit word) identifying the partition R_k within R_k^- that contains the plant state. The controller then uses the center c_k of R_k to update the plant model generates the control signal using the plant model until time t_{k+1}^-. At this point, using the plant model and plant-model uncertainties both encoder and decoder calculate a new hyper parallelogram R_{k+1}^- that should contain the plant state by evolving or propagating forward the initial region R_k. The process is then repeated. Stability will be ensured if the radius and center of the hyper parallelograms converge to zero with time. We will show now how the hyper parallelogram R_{k+1}^- is obtained from R_k.

Assume the plant model matrix $\hat{A} \in \mathbb{R}^{nxn}$ has n distinct unstable eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ with n corresponding linearly independent normalized eigenvectors $v_1, v_2, ..., v_n \in \mathbb{R}^n$. We will also assume that at t=0 both encoder and decoder agree in a hyper parallelogram R_0 containing the initial state of the plant. Denote a hyper parallelogram as the (n+1)-tuple where c is the center of the hyper parallelogram and η_i are its axis. In particular:

$$R(c, \eta_1, \eta_2, ..., \eta_n) = \left\{ \begin{aligned} & x \in \mathbb{R}^n, \sum_{i=1}^{n} \alpha_i \eta_i = x - c, \\ & \eta_i \in \mathbb{R}^n, \alpha_i \in [-1,1] \text{and } c \in \mathbb{R}^n \end{aligned} \right\}$$

Let each hyper parallelogram R_k with center c_k be defined as follows:

$R_K = R(c_k, \eta_{k,1}, \eta_{k,2}, ..., \eta_{k,n})$; $\eta_{k,i} = b_{k,i} v_i$ and $b_{k,i} \in \mathbb{R}$

Therefore it can be easily verified that according to the plant dynamics the region R_k evolves into a hyper parallelogram R_{k+1}^p defined by:

$$\begin{aligned} & R_{k+1}^p = R\left(c_{k+1}^p, \eta_{k+1,1}^p, \eta_{k+1,2}^p, ..., \eta_{k+1,n}^p\right) \\ & \text{with } \eta_{k+1,i}^p = e^{Ah} \eta_{k,i} \\ & \text{and } c_{k+1}^p = \left(e^{Ah} + \int_0^h e^{A(h-s)} BK e^{(\hat{A}+\hat{B}K)s} ds \right) c_k \end{aligned} \tag{14}$$

Correspondingly, according to the plant model dynamics the hyper parallelogram R_k should evolve into a different hyper parallelogram R_{k+1}^m:

$$\begin{aligned} & R_{k+1}^m = R\left(c_{k+1}^m, \eta_{k+1,1}^m, \eta_{k+1,2}^m, ..., \eta_{k+1,n}^m\right) \\ & \text{with } \eta_{k+1,i}^m = e^{\lambda_i h} \eta_{k,i}, \text{ and } c_{k+1}^m = e^{(\hat{A}+\hat{B}K)h} c_k \end{aligned} \tag{15}$$

According to equation (15) the hyper parallelogram R_{k+1}^m has edges that are parallel to those of the original hyper parallelogram R_k but are longer by a factor of $e^{\lambda_i h}$ for each corresponding edge. Also the center of the parallelogram has shifted. Note that the hyper parallelogram R_{k+1}^m doesn’t necessarily contain the plant state. We will now express R_{k+1}^p in terms of the parameters of R_{k+1}^m. By manipulating the expressions in (14) we can obtain:

$$e^{Ah} = e^{\hat{A}h} + \int_0^h e^{A(h-s)} \tilde{A} e^{\hat{A}s} ds \quad \text{and}$$

$$\begin{aligned} & e^{Ah} + \int_0^h e^{A(h-s)} BK e^{(\hat{A}+\hat{B}K)s} ds \\ & = e^{(\hat{A}+\hat{B}K)h} + \int_0^h e^{A(h-s)} \left(\tilde{A} + \tilde{B}K\right) e^{(\hat{A}+\hat{B}K)s} ds \end{aligned}$$

Therefore the parameters of R_{k+1}^p can be expressed in terms of the parameters of R_{k+1}^m:

$$\begin{aligned} \eta_{k+1,i}^p &= e^{Ah} \eta_{k,i} = \left(e^{\hat{A}h} + \int_0^h e^{A(h-s)} \tilde{A} e^{\hat{A}s} ds \right) \eta_{k,i} \\ &= e^{\lambda_i h} \eta_{k,i} + \Delta_\eta(h) \eta_{k,i} = \eta_{k+1,i}^m + \Delta_\eta(h) \eta_{k,i} \\ c_{k+1}^p &= \left(e^{Ah} + \int_0^h e^{A(h-s)} BK e^{(\hat{A}+\hat{B}K)s} ds \right) c_k \\ &= e^{(\hat{A}+\hat{B}K)h} c_k + \left(\int_0^h e^{A(h-s)} \left(\tilde{A} + \tilde{B}K\right) e^{(\hat{A}+\hat{B}K)s} ds \right) c_k \\ &= c_{k+1}^m + \Delta_c(h) c_k \end{aligned} \tag{16}$$

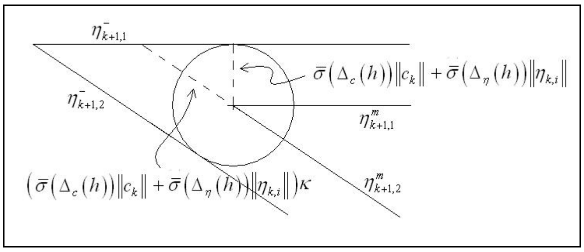

Figure 2. Construction of hyper parallelogram R_{k+1}^- from R_{k+1}^m.

Since matrices $\Delta_c(h)$ and $\Delta_\eta(h)$ are unknown, the hyper parallelogram R_{k+1}^p cannot be constructed. Instead we will use the expressions in equation (16) and the bounds over the norms of $\Delta_c(h)$ and $\Delta_\eta(h)$ to construct a hyper parallelogram that will contain the plant state i.e. it will contain R_{k+1}^p. This is depicted in Figure 2.

$$\begin{aligned} & R_{k+1}^- = R\left(c_{k+1}^-, \eta_{k+1,1}^-, \eta_{k+1,2}^-, ..., \eta_{k+1,n}^-\right) \\ & \text{with } \eta_{k+1,i}^- = \left(\begin{aligned} & 1 + \bar{\sigma}(\Delta_c(h)) \|c_k\| \frac{\kappa}{\|\eta_{k+1,i}^m\|} \\ & + \bar{\sigma}(\Delta_\eta(h)) \|\eta_{k,i}\| \frac{\kappa}{\|\eta_{k+1,i}^m\|} \end{aligned} \right) \eta_{k+1,i}^m \end{aligned} \tag{17}$$

$c_{k+1}^- = c_{k+1}^m$,where $\kappa = 1/\det([v_1\ v_2 \dots v_n])$, $\|v_i\| = 1$

Note that bounds over $\bar{\sigma}(\Delta_c(h))$ and $\bar{\sigma}(\Delta_\eta(h))$ can be obtained based on the norms over the error matrices $\tilde{A}$ and $\tilde{B}$. Note also that R_{k+1}^- is a hyper parallelogram with edges larger but parallel to those of R_{k+1}^m. At this

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time the encoder will divide R_{k+1}^- into smaller parallelograms and transmits to the decoder the symbol that identifies the one that contains the plant state R_{k+1}. And the process repeats itself again. This process is depicted below, also see Figure 3:

$$R_k^- \xrightarrow{\text{encoder}} R_k \xrightarrow[\text{h seconds}]{\text{plant}} R_{k+1}^- \xrightarrow{\text{encoder}} R_{k+1} \qquad (18)$$

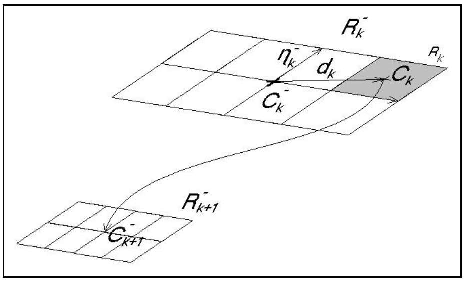

Figure 3. Evolution of quantized regions.

In Figure 3 the term d_k represents the displacement of the center of R_{k+1} with respect to the center of R_{k+1}^-. We will now establish the relationship between the evolution of the hyper parallelograms parameters and stability. It is clear that in order to ensure the stability of the system we require that the center and radius of the hyper parallelograms must converge to zero with time.

We will now assume that in order to generate the hyper parallelograms R_{k+1} each edge of the hyper parallelogram R_{k+1}^- is divided in equal Q_i parts. Note that all the Q_i must be powers of 2, that is $Q_i = 2^{b_i}$ where b_i represent the number of bits assigned to each axis. The resulting bit rate is $BitRate = \left(\sum_{i=1}^{n} b_i\right)\Big/ h$. We can now present a sufficient condition for stability of MB-NCS under the described dynamic quantization.

Theorem 4.

The state feedback MB-NCS using the dynamic quantization described in (18) is globally asymptotically stable if the following conditions are satisfied:

1. The non-quantized MB-NCS is stable.

2. The test matrix T has all its eigenvalues inside the unit circle.

Where

$$T = \begin{bmatrix} T_{11a} + T_{11b} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$

with

$$T_{11a} = diag\left(\left(\frac{e^{\lambda_1 h} + \bar{\sigma}\left(\Delta_\eta(h)\right)\kappa}{Q_1} \right), \ldots, \left(\frac{e^{\lambda_n h} + \bar{\sigma}\left(\Delta_\eta(h)\right)\kappa}{Q_n} \right) \right),$$

$$T_{11b} = \begin{bmatrix} \left(\frac{Q_1-1}{Q_1}\right) & \cdots & \left(\frac{Q_n-1}{Q_n}\right) \\ \vdots & & \vdots \\ \left(\frac{Q_1-1}{Q_1}\right) & \cdots & \left(\frac{Q_n-1}{Q_n}\right) \end{bmatrix} \bar{\sigma}\left(\Delta_c(h)\right)\kappa,$$

$$T_{12} = \begin{bmatrix} \bar{\sigma}\left(\Delta_c(h)\right)\kappa \\ \vdots \\ \bar{\sigma}\left(\Delta_c(h)\right)\kappa \end{bmatrix}, \quad T_{22} = \bar{\sigma}\left(e^{\left(\hat{A}+\hat{B}K\right)h}\right) \qquad (19)$$

$$T_{21} = \begin{bmatrix} \left(\frac{Q_1-1}{Q_1}\right) & \cdots & \left(\frac{Q_n-1}{Q_n}\right) \end{bmatrix} \bar{\sigma}\left(e^{\left(\hat{A}+\hat{B}K\right)h}\right),$$

Proof.

In order to characterize the evolution of the hyper parallelograms it is convenient to establish the relationship between the sizes of edges of R_{k+1}^- and the edges of R_k^-.

$$\begin{aligned} \left\|\eta_{k+1,i}^-\right\| &= \left(\frac{e^{\lambda_i h} + \bar{\sigma}\left(\Delta_\eta(h)\right)\kappa}{Q_i} \right) \left\|\eta_{k,i}^-\right\| + \bar{\sigma}\left(\Delta_c(h)\right)\kappa \left\|c_k\right\| \\ &\le \left(\frac{e^{\lambda_i h} + \bar{\sigma}\left(\Delta_\eta(h)\right)\kappa}{Q_i} \right) \left\|\eta_{k,i}^-\right\| + \bar{\sigma}\left(\Delta_c(h)\right)\kappa \left\|c_k^-\right\| \\ &\quad + \bar{\sigma}\left(\Delta_c(h)\right)\kappa \left\|d_k\right\| \end{aligned} \qquad (20)$$

Equation (20) is a scalar discrete linear system. It is dependent on $\left\|c_k^-\right\|$. The evolution of c_k is given by:

$$c_{k+1}^- = e^{\left(\hat{A}+\hat{B}K\right)h} c_k = e^{\left(\hat{A}+\hat{B}K\right)h} c_k^- + e^{\left(\hat{A}+\hat{B}K\right)h} d_k \qquad (21)$$

The term $\left\|d_k\right\|$ is bounded by:

$$\left\|d_k\right\| \le \sum_{i=1}^{N} \left(\left\|\eta_{k+1,i}^-\right\| \left(\frac{Q_i - 1}{Q_i} \right) \right) \qquad (22)$$

We will now bound $\left\|c_k^-\right\|$:

$$\begin{aligned} \left\|c_{k+1}^-\right\| &\le \bar{\sigma}\left(e^{\left(\hat{A}+\hat{B}K\right)h}\right) \left\|c_k^-\right\| \\ &\quad + \bar{\sigma}\left(e^{\left(\hat{A}+\hat{B}K\right)h}\right) \sum_{i=1}^{N} \left(\left\|\eta_{k+1,i}^-\right\| \left(\frac{Q_i - 1}{Q_i} \right) \right) \end{aligned} \qquad (23)$$

From (20), (22), and (23) it is clear that stability is guaranteed if *T* has its eigenvalues inside the unit circle. ♦

Note that if the plant model is exact, then $\tilde{A} = 0$ and $\tilde{B} = 0$ then, $\Delta_c(h) = 0$ and $\Delta_\eta(h) = 0$. This implies that if $\bar{\sigma}\left(e^{\left(\hat{A}+\hat{B}K\right)h}\right) < 1$ then stability is guaranteed if $\max_i \left(e^{\lambda_i h} / Q_i\right) < 1$ which is a well-established result (Nair, *et al.*, 2000a and 2000b). In order to enforce the condition that $\bar{\sigma}\left(e^{(\hat{A}+\hat{B}K)h}\right)<1$ it is convenient to apply a similarity transformation that diagonalizes $\hat{A}+\hat{B}K$.

Next an example is presented. This example depicts the way a MB-NCS can be designed, namely first a non-quantized MB-NCS is designed and then a suitable quantization scheme is added and tested for stability.

Example

Consider the plant represented by the following state space matrices:

$$A=\begin{bmatrix}0 & 1\\ a_{21} & 0.5\end{bmatrix}\quad B=\begin{bmatrix}0.1\\0.2\end{bmatrix} \tag{24}$$

Where $a_{11}\in[-0.01,\,0.01]$ represents the uncertainty in the A matrix. The plant model is defined as the nominal plant, that is:

$$\hat{A}=\begin{bmatrix}0 & 1\\ 0 & 0.5\end{bmatrix}\quad \hat{B}=\begin{bmatrix}0.1\\0.2\end{bmatrix} \tag{25}$$

A feedback gain $K=[-3.3333 \quad -8.3333]$ is selected so to place the eigenvalues of the plant model at $(-0.5,-1)$. An update time of $h=1$ sec is used. The following similarity transformation that diagonalizes $\hat{A}+\hat{B}K$ is applied to the system:

$$x_{new}=Px,\ \text{where}\ P=\begin{bmatrix}1.8856 & 0.4714\\ 1.3744 & 1.3744\end{bmatrix} \tag{26}$$

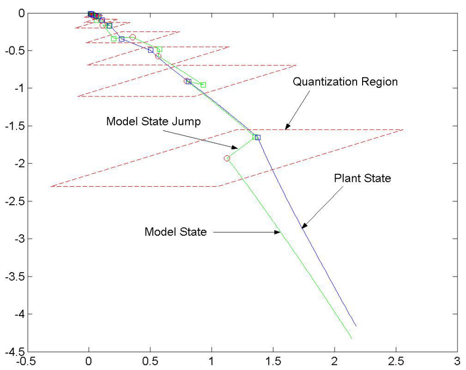

Figure 4. Trajectories for Plant State and Plant Model State showing the evolution of quantized regions.

Finally, the quantized levels are defined as $n_1=1$ bit and $n_2=2$ bits for the eigenvectors corresponding to the eigenvalues at –0.5 and –1 respectively. The bounds for the norms of uncertainty matrices are calculated in the transformed space by searching along the parameter a_{21}: $\bar{\sigma}(\Delta_c(h))\le 0.1354,\ \bar{\sigma}(\Delta_\eta(h))\le 0.0961$.

The maximum eigenvalue for the test matrix T is at 0.9531 indicating that the quantized system is stable. Next a simulation of the system is presented. In this simulation the parameter a_{21} is chosen randomly to be 0.0034, the starting region with center $[2 \quad -3]^T$, edges with length 1, and the plant state randomly placed within this region. The plots are in the non-transformed original space.

4. CONCLUSIONS

This paper characterizes the stability properties of a state feedback MB-NCS under different quantization schemes. First the computationally inexpensive static quantizers are considered, associated with them are two popular data representations, namely fixed point and floating point representations. Finally, the computationally intensive dynamic quantizer is considered. The assumptions on this paper include availability of the state and negligible transport times but the results can be extended to output feedback and networks with transport delays thanks to the MB-NCS unified framework (Montestruque, *et al*., 2003). The results quantitatively shows how the system stability degrades as the update time and plant uncertainties increase for a given quantization scheme.

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