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Convergence Rate of Quantization Error in Networked Control Systems

Hui Fang and Panos J. Antsakls *

Abstract

In this paper, we investigate the problem of convergence rate of the quantization error in networked control systems. Given bit rate R, we derive the best convergence rate of the quantization error together with a corresponding rate allocation strategy. Specifically, using a min-max eigenvalue problem (MMEP) formulation, it is shown that the error convergence rate is determined by the ratio between $|\det A|$ *and R where A is the state matrix. It is also shown that the bit rate R can be optimized through a dynamic quantization policy among the unstable modes of the system, such that the convergence rate of the quantization error approaches the optimal solution to the MMEP problem.*

1. Introduction

There is strong ongoing interest in the problem of control under communication constraints; attempting to bring together classical control theory and practical communication theoretical issues in the design of control system. There have been several results on stability [1], [2]. A number of results have focused on the minimum bit rate required for stability, using different formulations and techniques [3], [4], [5], [6], [7], [8], [9], [10]. Specifically, the necessary and sufficient condition on the rate for asymptotic stabilization in a linear, discrete time system is $R > \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\}$ where R is the bit rate, and $\lambda(A)$ denotes the eigenvalues of the state matrix A. This result is independent of the particular quantization strategy, and the minimum R for stability is totally determined by the magnitude of the unstable eigenvalues of the plant. Stabilization of nonlinear systems has been studied by [12], [13] and a Slepian-Wolf coding scheme for stabilizing decentralized linear systems under rate constraints has also been considered in [14].

Besides stability, there is some research on the performance of control systems under communication constraints. The initial steps toward a methodology for the design of controllers, in the presence of communication constraints, has been addressed in [15]. In a recent paper by Tatikonda [16], the classical linear quadratic Gaussina problem is reconsidered in this framework; under some mild assumption of no "dual-effect", it is shown that the optimal LQG cost decomposes into two terms: a full knowledge cost and a sequential rate distortion cost introduced by the communication constraints.

Another recent area of investigation is the analysis in the presence of disturbance and uncertainty. In [17], stability in the presence of disturbances and operator theoretic uncertainty is investigated. For a particular class of channels, the work by [18] has shown that the extra rate $C - \sum_{\lambda(A)} \max\{0, \log(|\lambda(A)|)\}$ is critical for performance, as measured by the expected power of the state of the plant. The work by [20] has shown that the coarsest is logarithmic, and can be computed by solving a special linear quadratic regulator (LQR) problem. Research in this area currently focuses on understanding the fundamental limitations of control performance under communication constraints.

In this paper, instead of viewing the channel capacity as a communication constraint, we regard R as an available resource to minimize the quantization error. We investigate the optimization problem of maximizing the convergence rate of quantization error $\|e[k]\|$, which is equivalent to minimizing the following asymptotic convergence factor,

$$r_{asym} = \sup_{\|e\| \neq 0} \lim_{k \to \infty} \left(\frac{\|e[k]\|}{\|e[0]\|} \right)^{1/k} \tag{1}$$

We formulate this problem as a min-max eigenvalue problem, and from solution of this optimization problem, we find

- The optimal communication rate allocation strategy to maximize the quantization error convergence rate of the quantized systems.
- A quantization policy which stabilizes the entire system and approaches the solution of the optimal strategy.

It is shown that the error asymptotic convergence factor for the quantized system is determined by the ratio between $|\det A|$ and R. For the optimal strategy, the communication resources must be appropriately distributed among the different unstable modes of the system [6], [7]. It is further proved that a quantization policy presented in [7] ap-

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proaches this optimal solution. Related problems on the design of a ”communication sequence” to stabilize multiple systems whose feedback loops are closed over one common shared medium, can be found in [22].

2. Problem Formulation

This paper studies discrete-time LTI systems of the following form

$$x[k+1] = Ax[k]+Bu[k] \tag{2}$$
$$u[k] = Kx^q[k] \tag{3}$$

where the state $x[k] \in \Re^n$, the control signal $u[k] \in \Re^m$, $A \in \Re^{n\times n}, B \in \Re^{n\times m}$ and $K \in \Re^{m\times n}$. The state $x[k]$ is quantized and encoded into a symbol $s[k]$ from a discrete set $\{1,2,\ldots,Q\}$. $s[k]$ is transmitted to the decoder over a perfect communication channel with fixed rate R bps. The decoder uses the received symbols to compute an estimate, $x^q[k]$ of the plant’s true state $x[k]$. The controller uses this estimate $x^q[k]$ to compute the control signal $u[k] = Kx^q[k]$. Besides, we assume that the feedback gain matrix K is properly designed to satisfy performance specifications and that the close loop state matrix $(A+BK)$ is stable when there are no communication contraints.

We are interested in the stability of the system which gurantees

$$\lim_{k\to\infty} \|x[k]\| = 0$$

for any $x[0] \in \Re^n$, where $\|\cdot\|$ denotes Euclidean 2-norm. We study stability under the following assumptions,

1. (A,B) is controllable. $A = \mathrm{diag}(J_1,J_2,\ldots,J_p)$, where J_i is an $n_i \times n_i$ Jordan block. We assume throughout this paper that A has only unstable real eigenvalues, i.e, $|\lambda_i| \geq 1$.
2. The initial condition $x[0]$ lies in a ”known” parallelogram $P[0]$.
3. Both the encoder and decoder know the system matrices A,B, the coding-decoding policy and control law. They also agree upon the initial uncertainty set, i.e. $P[0]$.

Assumption (2) requires the initial state to lie within a known parallelogram $P[0]$, which can be written as

$$P[0] = x^q[0]+U[0]$$

where $x^q[0] \in \Re^n$ is the centroid of $P[0]$, and $U[0]$ is a parallelogram centered at the origin. Similarly, the state $x[k]$ at time k is quantized with respect to a parallelogram $P[k]$ representing the quantization ”uncertainty”, i.e,

$$P[k] = x^q[k]+U[k] \tag{4}$$

where $x^q[k] \in \Re^n$ is the centroid of the $P[k]$ and $U[k]$ is a parallelogram with center at the origin. At time instance k, we know that

$$x[k] \in x^q[k]+U[k]$$

where $x^q[k]$ denote the current estimate of the state $x[k]$, and $U[k]$ is used to represent the uncertainty associated with such an estimate.

The quantization error $e[k]$ is defined as the difference between the actual state $x[k]$ and the estimated state $x^q[k]$

$$e[k] = x[k]-x^q[k] \tag{5}$$

and from our assumption, $e[k] \in U[k]$.

The parallelogram $U[k]$ is used to model the quantization error which can be formally characterized by a set of vectors $v_{i,j}[k] \in \Re^{n_i}$ where $i = 1,\ldots,p$ and $j = 1,\ldots,n_i$. The parallelogram associated with the i-th Jordan block in A is denoted as the convex hull

$$S_i[k] = \mathrm{Co}\left\{ v : v = \sum_{j=1}^{n_i} \pm\frac{1}{2} v_{i,j}[k] \right\} \tag{6}$$

The entire parallelogram $U[k]$ may therefore be expressed as the Cartesian product of the sides, $S_i[k]$, i.e:

$$U[k] = \prod_{i=1}^{p} S_i[k] \tag{7}$$

The volume of U is defined as $\mathrm{vol}(U) = \int_{x\in U} 1\cdot dx$. The ”size” of $U[k]$ is measured by its diameter, which is defined as

$$d_{max}(U) = \sup_{x,y\in U} \|x-y\|$$

As proved in [5] by an argument of the triangular inequality, the convergence of the diameter of $U[k]$ is equivalent to the asymptotic stability of the quantized system if the closed-loop state matrix $(A+BK)$ is stable, which implies that if the diameter of $U[k]$ converges to zero, then the quantized system is asymptotically stable.

The change of $\mathrm{vol}(U[k])$ involves two parts: one is associated with quantization when $P[k]$ is partitioned into 2^R small uncertainty sets, described by:

$$\mathrm{vol}(U^q[k]) = \frac{\mathrm{vol}(U[k])}{2^R} \tag{8}$$

while the other is due to the dynamics of the underlying plant, namely:

$$\mathrm{vol}(U[k]) = |\mathrm{det}A|\cdot \mathrm{vol}(U^q[k-1]); \tag{9}$$

here we use the notation $U^q[k]$ to denote the uncertainty set $U[k]$ right after quantization. Equations (8), (9) provide the insight on the necessary low bound needed for stabilization

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since we have the following iteration relationship in terms of the ”measure” of the uncertainty set $U[k]$,

$$\mathrm{vol}(U[k]) = \frac{|\mathrm{det}A|}{2^R} \cdot \mathrm{vol}(U[k-1]) \tag{10}$$

Without going into details (see [9]), we can conclude from (10) a necessary condition for stabilization is

$$R > \sum_{\lambda(A)} \max\{0, \log|\lambda(A)|\} \tag{11}$$

However, when the volume of the uncertainty set $U[k]$ converges to zero, it does not necessarily mean that the diameter of $U[k]$ converges to zero, because the edges of $U[k]$ can be quite ”irregular” (such that some length of the supporting edge of $\|v_{i,j}[k]\|$ never converges). Various techniques have been used to ”segment” the edge of $U[k]$ and prove that the bound is sharp [6], [5], [7].

In this paper, given bit rate R, we want to find the best policy to maximize the convergence rate of quantization error. This can be posed as minimizing the asymptotic convergence factor. Throughout this paper, the terms ”maximum convergence rate” and ”minimum asymptotic convergence factor” are used interchangeably.

3. Minimum asymptotic convergence factor of quantization error

As we have mentioned, $U[k]$ can be represented by the Cartesian product of convex hull of $v_{i,j}[k]$. If $q_{i,j}$ level is used to ”quantize” $v_{i,j}[k]$ (8), then after quantization, $v_{i,j}[k]$ is divided by $q_{i,j}$, and $v^q_{ij}[k]$ is used to denote the vector $v_{ij}[k]$ right after quantization,

$$v^q_{i,j}[k] = \frac{1}{q_{i,j}} v_{i,j}[k] \tag{12}$$

According to the dynamics of the system (9), $v_{i,j}[k]$ will evolve as follows,

$$v_{i,j}[k] = J_i v^q_{i,j}[k-1]$$

The new uncertainty convex hull $S_i[k+1]$ associated with J_i can be represented as

$$S_i[k] = \mathrm{Co}\left\{ v : v = \sum_{j=1}^{n_i} \pm \frac{1}{2} \frac{J_i}{q_{i,j}} v^q_{i,j}[k-1] \right\}$$

where Co denote the convex hull generate by the associated vectors. Let’s introduce the augmented state vector for $v^a = [v^a_1, \ldots, v^a_p] \in \Re^{\Sigma_i^p (n_i)^2}$, where

$$v^a_i = [[v^q_{i,1}]^T, \cdots, [v^q_{i,n_i}]^T]^T \tag{13}$$

Assume that $q_{i,j}$ is the quantization level associated with $v_{i,j}[k]$, then

$$\begin{aligned} v^a_i[k+1] &= \begin{bmatrix} \frac{J_i}{q_{i,1}} & & \\ & \ddots & \\ & & \frac{J_i}{q_{i,n_i}} \end{bmatrix} v^a_i[k] \\ &= A^q_i v^a_i[k] \end{aligned}$$

Define

$$A_q = \mathrm{diag}\{A^q_1, \ldots, A^q_p\}$$

then the system equation for the augmented vectors v^a to represent the uncertain set $U[k]$ is

$$v^a[k+1] = A_q v^a[k] \tag{14}$$

Here we allow $q_{i,j} \in \Re^+$, with the constraint,

$$\prod_{i=1}^{p} \prod_{j=1}^{n_i} q_{i,j} \leq Q = 2^R \tag{15}$$

Obviously, $\{\|v^a[k]\|\}$ converges to zero if and only if $\{\|v_{i,j}[k]\|\}$ converges to zero. The convergence factor of $\{d_{max}(U[k])\}$ is determined by the maximum eigenvalue of A_q [24],i.e,

$$r_{asmp} = \lambda_{max}(A_q) \tag{16}$$

In [23], a similar optimization problem for analyzing the minimum rate to achieve boundedness of the quantization error with a time-invariant quantizer is formulated with the quantization levels $q_{i,j}$ being integers. However, we allow $q_{i,j}$ to be positive reals in the following optimization; In this way we can overcome the difficulties associated with the integer programming problem. In the next section, we will prove that a dynamic quantization policy will use an average $q_{i,j}(q_{i,j} \in \Re^+)$ quantization level per step if the time varying integer levels of $q_{i,j}$ are averaged over time. Before we begin any practical quantization design, let us first find the solution to the following minimization problem, which is equivalent to minimizing the maximum eigenvalue of A_q; the equivalence follows from the structure of A_q.

Min-Max Eigenvalue Problem(MMEP):

$$\min\ t \tag{17}$$

subject to

$$\begin{aligned} \frac{\lambda_i}{q_{i,j}} &\leq t \\ \prod_{i=1}^{p} \prod_{j=1}^{n_i} q_{i,j} &\leq Q \end{aligned}$$

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Theorem 1 *The optimal solution to the above MMEP problem is*

$$t_{min} = t^* = \left[\frac{|\det A|}{Q}\right]^{\frac{1}{n}} \tag{18}$$

with

$$q^*_{i,j} = |\lambda_i|\left[\frac{Q}{|\det A|}\right]^{\frac{1}{n}} \tag{19}$$

Proof 1 *This is a standard generic programming [25] problem. To form the Lagrangian, we introduce the $n+1$ multiplier $\zeta_{i,j}$, ν for the equality constraints, then we obtain:*

$$L(t,q,\zeta,\nu) = t + \sum_i^p \sum_j^{n_i} \zeta_{i,j}\left(\frac{\lambda_i}{q_{i,j}} - t\right) + \nu\left(\prod_{i=1}^p \prod_{j=1}^{n_i} q_{i,j} - Q\right) \tag{20}$$

where $q = (q_{i,j})^T, \zeta = (\zeta_{i,j}) \in \Re^n, \nu \in \Re$.
Apply the Kuhn-Tucker condition to $L(t,q,\zeta,\nu)$:

$$\begin{aligned}
0 = \frac{\partial L}{\partial t} &= 1 - \sum_i^p \sum_j^{n_i} \zeta_{i,j} \\
0 = \frac{\partial L}{\partial q_{i,j}} &= -\zeta_{i,j}\lambda_i q_{i,j}^{-2} + \nu \prod_{i=1}^p \prod_{j=1, j\neq i}^{n_i} q_{i,j} \\
0 = \frac{\partial L}{\partial \zeta_{i,j}} &= \frac{\lambda_i}{q_{i,j}} - t \\
0 = \frac{\partial L}{\partial \nu} &= \prod_{i=1}^p \prod_{j=1}^{n_i} q_{i,j} - Q
\end{aligned}$$

Combining these equations, we get

$$\begin{aligned}
t^* &= \left[\frac{|\det A|}{Q}\right]^{\frac{1}{n}} \\
q^*_{i,j} &= |\lambda_i|\left[\frac{Q}{|\det A|}\right]^{\frac{1}{n}}
\end{aligned}$$

while the multiplier $\zeta_{i,j} = \frac{1}{n}$, $\nu = \frac{1}{nQ}\left[\frac{|\det A|}{Q}\right]^{\frac{1}{n}}$.

It is straightforward to check that t^* is the optimal solution of the above MMEP problem.

The above MMEP problem provides a clear justification for the different quantization policy [7], [9]. Intuitively, the relative magnitude of $|\det A|$ and Q determines the convergence rate of quantization error, while the relative magnitude among λ_i determines the fairness among the communication rate $\tau^*_{i,j}R$ (21).

Remark 1 *It is interesting to notice that, in order to minimize the maximum eigenvalue of A_q, all the eigenvalues are set to be equal to $\left[\frac{|\det A|}{Q}\right]^{\frac{1}{n}}$, which reflects the "balance" between the convergence rate of $\{\|v_{i,j}[k]\|\}$. This property can be used to guide our design of quantization policy. Besides, if we regard quantization Q as another optimization parameter, the feasibility problem $t_{min} < 1$ leads to the necessary condition needed for asymptotic stabilization.*

Remark 2 *The portion of R bit rate associated with $q_{i,j}$ can be computed as follows,*

$$\tau^*_{i,j} = \log_Q q_{i,j} = \frac{1}{n} + \frac{1}{n}\log_Q \frac{|\lambda_i|^n}{|\det A|} \tag{21}$$

Note that $\sum_{i=1}^p \sum_{j=1}^{n_i} \tau^*_{i,j} = 1$. $\tau^*_{ij}R$ denotes the bit rate allocated to that particular mode. From a queuing-theoretic viewpoint, $\tau_{i,j}$ is used to capture the amount of communication resource needed to decrease the error associated with $v_{i,j}$ direction as well as the "attention" needed over the available channel [21].

Remark 3 *If we can arbitrarily assign q_{ij} quantization levels to the vector $v_{i,j}[k]$, then according to Lemma 2,*

$$\begin{aligned}
\|e[k]\| &\leq d_{max}(U[k]) & (22)\\
&\leq n\kappa\left[\frac{|\det A|}{Q}\right]^{\frac{k}{n}} k^{\max n_i - 1} d_{max}(U[0]) & (23)
\end{aligned}$$

and according to (1), (16), the minimum asymptotic convergence factor r^*_{asym} is,

$$r^*_{asym} = \frac{1}{n}\frac{|\det A|}{Q} \tag{24}$$

Generally speaking, the quantization level $q_{i,j}$ should be an positive integer instead of a positive real number. We will show that a novel dynamic quantization policy introduced in [7] will assign an "average" of q^*_{ij} quantization level to the vector $v_{i,j}[k]$, and the convergence rate will approach r^*_{asym}.

4. A Policy to Achieve the Maximum Error Convergence Rate

This section introduces a theorem showing that the optimal convergence rate can be achieved by the dynamic bit assignment policy presented in [7]. On the average, the dynamic quantization policy from [7] will assign exact $\tau^*_{i,j}R$ to each of the unstable modes, which can be seen as a practical realization to achieve the theoretical bounds we have derived in the previous section. It is further shown that the $d_{max}(U[k])$ converges at a faster rate than the rate derived in [7], and this low bound is achieved by allocating an average of $\tau^*_{i,j}$ of the total resources R to $v_{i,j}$ direction.

The dynamical quantizer is a three-tuple $(x^q[k],(I,J),U[k])$ where $x^q[k] \in \Re^n$ represents the centroid, $U[k] = (S_1[k],\ldots,S_p[k]) \in \Re^n$ represents the

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dynamical range, with $S_i[k]$ defined in (6), (I,J) represents the dynamical quantizer edge index. The dynamical quantizer uses a one-step prediction: at each step, choose(I,J) according to (28) for quantization, and assign all R-bit to the vector $v_{I,J}$, thus partition the $U[k]$ along $v_{I,J}$ into boxes with side length $v_{I,J}/Q$. The mechanism of the dynamic bit assignment policy (DBAP) from [7] is described in the Appendix. The DBAP quantization policy, which seeks to ”quantize the largest edge” can be viewed as the ”contain the largest state” policy in [22] for switching between a group of unstable systems.

Let us define the average bit-rate associated with $v_{i,j}[k]$ using the DBAP policy. Suppose from time instance 0 to k, $v_{i,j}$ has been quantized $n_{i,j}$ times. The average bit-rate associated with $v_{i,j}$ is defined as

$$R_{i,j} = \lim_{k\to\infty} \frac{n_{i,j}}{k} R \tag{25}$$

assuming that the above limit exists.

In the following theorem, we show how the DBAP quantization policy relates to our MMEP problem as we investigate the average bit-rate assigned to the edge $v_{i,j}[k]$.

Theorem 2 *Let $R^*_{i,j}$ be the average bit rate for the DBAP policy, then*

$$R^*_{i,j} = \tau^*_{i,j} R \tag{26}$$

where $\tau^*_{i,j}$ is given in (21) through the optimal solution of the MMEP problem. Furthermore, The diameter of uncertainty set $U[k]$ converges at the following rate:

$$\|d_{max}(U[k])\| \sim n\kappa \left[\frac{|\det A|}{Q}\right]^{\frac{k}{n}} k^{\max n_i - 1} d_{max}(U[0]) \tag{27}$$

where κ is a time-independent constant. Furthermore, the asymptotic convergence factor for $\{\|e[k]\|\}$ is r^*_{asmp} in (24).

Proof 2 *The proof of this theorem can be found in the Appendix.*

Remark 4 *In [9], a similar bound in terms of the convergent rate of the quantization error is derived, namely*

$$\|e[k]\| \le \kappa k^{(\max_i n_i - 1)} 2^{-k(\min_i (R_i - \log|\lambda_i(A)|))}$$

Note that the DBAP policy will force the error to converge at a faster rate because of the dynamic and balanced property of the DBAP quantization policy. Besides, the convergence rate of $d_{max}(U[k])$ is actually tighter compared to the one found in [7] and approaches r^*_{asym} from the MMEP problem.

5. Conclusion

In this paper, given bit rate R, we derived the best convergence rate for quantization error and the associated resource allocation strategy via the solution of a min-max eigenvalue problem associated with the quantized system. It is further proved that the result from our MMEP formulation is achievable through a dynamic bit assignment policy (DBAP) [7]. Our result provides the theoretical foundation for the dynamic quantization policy to increase its error convergence rate.

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6. Appendix

Dynamic Bit Assignment Policy (DBAP):
Encoder/Decoder Initialization:
Initialize $X^q[0]$ and $\{v_{i,j}[0]\}$ so that $x[0] \in x^q[0] + U[0]$ and set $k = 0$.
Encoder Task:

1. Select the indexes (I,J) by
$$(I,J) = arg \max_{i,j} \|J_i v_{i,j}[k]\|_2. \tag{28}$$
2. Quantize the state $x[k]$ by setting $s[k] = s$ if and only if
$$x[k] \in x^q[k] + x_s^{(I,J)} + U^{(I,J)}[k]$$
where
$$x_s^{(I,J)} = \begin{bmatrix} 0 & \cdots & 0 & v^T & 0 & \cdots & 0 \end{bmatrix} \tag{29}$$
and $v = \frac{-Q+(2s-1)}{2Q} v_{I,J}[k]$ for $s \in 1,\dots,Q$.
3. Transmit the quantized symbol $s[k]$ and wait for acknowledgment.
4. Update the variable
$$\begin{aligned} v_{i,j}[k+1] &= J_i v_{i,j}[k] \\ x^q[k+1] &= (A+BK)x^q[k] \end{aligned}$$
5. If decode ack received:
$$\begin{aligned} v_{I,J}[k+1] &= \frac{1}{Q} v_{I,J}[k+1] \\ x^q[k+1] &= x^q[k+1] + A x_s^{(I,J)}[k] \end{aligned}$$
where $x_{s[k]}^{(I,J)}$ is defined as in (29).

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6. Update time, $k := k+1$ and return to step 1.

Decode Task:

1. Update the variables

$$\begin{aligned} v_{i,j}[k+1] &= J_i v_{i,j}[k] \\ x^q[k+1] &= (A+BK)x^q[k] \end{aligned}$$

2. Wait for quantized data, $s[k]$, from encoder.

3. If data received:

$$\begin{aligned} v_{I,J}[k+1] &= \frac{1}{Q} v_{I,J}[k+1] \\ x^q[k+1] &= x^q[k+1] + A x_s^{(I,J)}[k] \end{aligned}$$

where $x_{s[k]}^{(I,J)}$ is defined in (29). Then send ack back to the encoder.

4. Update the time index, $k = k+1$, and return to step 1.

Remark 5 *The above algorithm assumes $\{v_{i,j}[k]\}$ and $x^q[k]$ are ”synchronized’ at the beginning of the k-th time interval. The ”ack” message is introduced to detect the possible packet-dropout from encoder to decoder on the communication channel.*

The proof of Theorem 2 will need the following lemmas. Generally speaking, Lemma 1 is used to prove the fairness of the DBAP quantization policy [7]; Lemma 2 is a matrix theory needed for our derivation.

Lemma 1 *For any i_1, j_1, i_2, j_2, there exists a finite constant r, such that*

$$r^{-1} \leq \frac{\|v_{i_1,j_1}[k]\|_2}{\|v_{i_2,j_2}[k]\|_2} \leq r \tag{30}$$

The following lemma is a standard result from matrix computation theory [24].

Lemma 2 *Let J be an upper bi-diagonal complex $p \times p$ matrix of the form*

$$J = \begin{bmatrix} \lambda & 1 & & 0 \\ 0 & \ddots & \ddots & \\ & & \lambda & 1 \\ 0 & & 0 & \lambda \end{bmatrix} \tag{31}$$

$v = [v^1, \cdots, v^p]^T \in \Re^p$, where $v^i \in \Re$, then [2]

$$\|J^m v\| \sim \binom{m}{p-1}\left(|\lambda|^{m-(p-1)} v^p + o(\frac{1}{m})\right) \tag{32}$$

[2]By $g(m) = o(\frac{1}{m})$ as $m \to \infty$, we mean that $|mg(m)| \leq \sigma$ for all m sufficiently large, where σ is a positive constant.

Proof 3

$$J^m = \begin{bmatrix} \lambda^m & \binom{m}{1}\lambda^{m-1} & \cdots & \binom{m}{p-1}\lambda^{m-(p-1)} \\ 0 & \lambda^m & \cdots & \binom{m}{p-2}\lambda^{n-(p-2)} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^m \end{bmatrix} \tag{33}$$

$$\begin{aligned} J^m v &= \begin{bmatrix} \lambda^m & \binom{m}{1}\lambda^{m-1} & \cdots & \binom{m}{p-1}\lambda^{m-(p-1)} \\ 0 & \lambda^m & \cdots & \binom{m}{p-2}\lambda^{n-(p-2)} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^m \end{bmatrix} \\ &\quad \cdot \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^p \end{bmatrix} \\ &= \binom{m}{p-1}\lambda^{m-(p-1)} \begin{bmatrix} v^p + o(\frac{1}{m}) \\ o(\frac{1}{m}) \\ \vdots \\ o(\frac{1}{m}) \end{bmatrix} \end{aligned}$$

take the norm of both sides, we get (32).

Proof 4 *(Theorem 2) Without loss of generality, let us assume that*

$$|v_{i,j}^{n_i}[0]| = v_c \neq 0 \tag{34}$$

For the time interval $[0,k]$, let us assume $v_{i,j}$ is quantized exact $n_{i,j}$ times, and combine with (32) for sufficient large k,

$$\begin{aligned} \|v_{i,j}[k]\| &= \frac{1}{Q^{n_{i,j}}}\|J_i^k v_{i,j}[0]\| \\ &\sim \frac{1}{Q^{n_{i,j}}}\binom{k}{n_i-1}\left(|\lambda_i|^{k-(n_i-1)} v_{i,j}^{n_i}[0] + o(\frac{1}{k})\right) \end{aligned}$$

from our previous lemma, $\|v_{i,j}[k]\|$ are balanced, such that, as k goes to infinity,

$$r^{-1} \leq \frac{\frac{1}{Q^{n_{i_1,j_1}}}\binom{k}{n_{i_1}-1}|\lambda_{i_1}|^{k-(n_{i_1}-1)} v_{i_1,j_1}^{n_{i_1}}[0]}{\frac{1}{Q^{n_{i,j}}}\binom{k}{n_i-1}|\lambda_i|^{k-(n_i-1)} v_{i,j}^{n_i}[0]} \leq r \tag{35}$$

i. e

$$r^{-1} \leq \frac{Q^{n_{i,j}}\binom{k}{n_{i_1}-1}|\lambda_{i_1}|^{k-(n_{i_1}-1)}}{Q^{n_{i_1,j_1}}\binom{k}{n_i-1}|\lambda_i|^{k-(n_i-1)}} \leq r \tag{36}$$

$$k^{-n_{i_1}+1}\left(\frac{\lambda_i}{\lambda_{i_1}}\right)^k \leq \frac{Q^{n_{i,j}}}{Q^{n_{i_1,j_1}}} \leq k^{n_i-1}\left(\frac{\lambda_i}{\lambda_{i_1}}\right)^k \tag{37}$$

multiply over all $(i_1, j_1) \neq (i,j)$, we have

$$k^{-n(\max n_i - 1)}\frac{\lambda_i^{nk}}{|\det A|^k} \leq \frac{Q^{nn_{i,j}}}{Q^k} \leq k^{n(\max n_i - 1)}\frac{\lambda_i^{nk}}{|\det A|^k} \tag{38}$$

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$$\begin{cases} nn_{i,j}-k & \geq & -n(\max n_i-1)\log_Q k+k\log_Q \frac{\lambda_i^n}{|\det A|^k} \\ nn_{i,j}-k & \leq & n(\max n_i-1)\log_Q k+k\log_Q \frac{\lambda_i^n}{|\det A|^k} \end{cases}$$

i.e [3] ,

$$n_{i,j} \sim \frac{k}{n}+\frac{k}{n}\log_Q \lambda_i^n|\det A|+O((\max n_i-1)\log_Q k) \quad (39)$$

take the limit,

$$\begin{aligned} \tau_{i,j} &= \lim_{k\to\infty}\frac{n_{i,j}}{k} \\ &= \frac{1}{n}+\frac{1}{n}\log_Q \frac{\lambda_i^n}{|\det A|} \\ &= \tau_{i,j}^* \end{aligned} \quad (40)$$

thus we have proved $R_{i,j}=\tau_{i,j}^* R$.

$$\begin{aligned} \|v_{i,j}[k]\| &= \frac{1}{Q^{n_{i,j}}}\|J_i^k v_{i,j}[0]\| \\ &\sim \frac{1}{Q^{n_{i,j}}}\binom{k}{n_i-1}\Big(|\lambda_i|^{k-(n_i-1)}v_{i,j}^{n_i}[0] \\ &\quad +o(\frac{1}{k})\Big) \\ &\leq \frac{1}{Q^{n_{i,j}}}\binom{k}{\max n_i-1}\Big(|\lambda_i|^{k-(n_i-1)}v_{i,j}^{n_i}[0] \\ &\quad +o(\frac{1}{k})\Big) \\ &\overset{(a)}{\leq} \kappa\Big[\frac{|\det A|}{Q}\Big]^{\frac{k}{n}}k^{\max n_i-1}\|v_{i,j}[0]\| \\ &\overset{(b)}{\leq} \kappa\Big[\frac{|\det A|}{Q}\Big]^{\frac{k}{n}}k^{\max n_i-1}d_{max}(U[0]) \end{aligned}$$

where (a) *follows from (39), and* κ *is some constant, and* (b) *follows since* $U[k]$ *is a parallelogram with sides* $v_{i,j}[k]$. *The triangle inequality implies* $d_{max}(U[k])$

$$\begin{aligned} d_{max}(U[k]) &\leq \sum_{i=1}^{p}\sum_{j=1}^{n_i}\|v_{i,j}[k]\| \\ &\leq n\kappa\Big[\frac{|\det A|}{Q}\Big]^{\frac{k}{n}}k^{\max n_i-1}d_{max}(U[0]) \end{aligned}$$

Since $\|e[k]\|\leq\|d_{max}(U[k])\|$, *and according to our definition (1), it is straightforward to prove that the asymptotic convergence rate for* $\|e[k]\|$ *is* r_{asmp}^*.

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