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Switching Stabilization and l_2 Gain Performance Controller Synthesis for Discrete-Time Switched Linear Systems

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Abstract— **In this paper, the switching controller synthesis problem for a class of discrete-time switched linear systems is considered. In particular, a state dependent switching law is designed to exponentially stabilize the switched systems, while a finite l_2 induced gain is achieved. Sufficient synthesis conditions are proposed as bilinear matrix inequalities, which are derived based on multiple Lyapunov functions.**

I. Introduction

The stability issues of switched systems, especially switched linear systems, have attracted a lot of attentions recently and the literature on this topic is immense; see for example the survey papers [15], [19], [7], [25], the recent books [16], [26] and the references cited therein. Generally speaking, the literature on switched systems' stability can be divided into two groups. One is on the stability analysis of switched systems under given switching signals (maybe arbitrary, slow switching etc.); the other is on the synthesis of stabilizing switching signals for a given collection of dynamical systems. The stability analysis for switched systems is usually based on Lyapunov's second method, such as the existence of a common Lyapunov function to guarantee stability under arbitrary switching [6], or the existence of multiple Lyapunov functions [2], [29], [19] for stability under certain classes of switching signals. In general, the construction of multiple Lyapunov functions for switched systems is a non-convex problem and very challenging. Usually, quadratic or piecewise-quadratic Lyapunov functions are employed to make the problem numerically tractable, see e.g. [13], [14], [17], [12].

In the switching stabilization literature, early efforts were focused on quadratic stabilization for certain classes of systems. For example, a quadratic stabilization switching law between two linear time invariant (LTI) systems was considered in [28], in which it was shown that the existence of a stable convex combination of the two subsystem matrices implies the existence of a state-dependent switching rule that stabilizes the switched system along with a quadratic Lyapunov function. A generalization to more than two LTI subsystems was suggested in [21] by using a "min-projection strategy". In [10], it was shown that the stable convex combination condition is also necessary for the quadratic stabilizability of two mode switched LTI system. However, it is only sufficient for switched LTI systems with more than two modes. A necessary and sufficient condition for quadratic stabilizability of switched controller systems was derived in [24]. There are extensions of [28] to output-dependent switching and discrete-time case [15], [31]. For robust stabilization, a quadratic stabilizing switching law was designed for polytopic uncertain switched linear systems based on linear matrix inequality (LMI) techniques in [31]. All of these methods guarantee stability by using a common quadratic Lyapunov function, which is conservative in the sense that there are switched systems that can be asymptotically (or exponentially) stabilized without using a common quadratic Lyapunov function. In this paper, multiple Lyapunov functions will be used for the synthesis purpose instead.

There have been some results in the literature that propose constructive synthesis methods to switched systems using multiple Lyapunov functions. For example, in [27], piecewise quadratic Lyapunov functions was employed for two mode switched LTI systems. Exponential stabilization for continuous-time switched LTI systems was considered in [22] also based on piecewise quadratic Lyapunov functions, and the synthesis problem was formulated as a bilinear matrix inequality (BMI) problem. In [12], a probabilistic algorithm was proposed for the synthesis of an asymptotically stabilizing switching law for switched LTI systems along with a piecewise quadratic Lyapunov function. Stabilization for switched nonlinear systems was considered in [8] based on multiple Lyapunov functions. There are also some interesting work on designing the state-feedback or output feedback gains for each subsystem so as to stabilize the switched system under arbitrary switching [5], [9], under given switching signals (e.g. slow switching [3]), or under autonomous switchings due to the partition of the state space [20], [23]. Additionally, exponentially stabilizing switching laws were designed based on solving extended LQR optimal problems in [4].

This paper aims at addressing the switching control law synthesis problem for discrete-time switched linear system based on multiple Lyapunov functions. In particular, the exponentially stabilization with bounded l_2 induced gain performance is investigated here. This paper is motivated by [22], where BMI synthesis condition is developed for exponential stabilization of continuous-time switched linear systems. The first part of this current paper can be seen as an extension of [22] to the discrete-time counterpart. However, the extension is nontrivial due to some distinctive features of discrete-time switched systems. First, to guarantee stability, we need to require the piecewise quadratic Lyapunov function not to increase its value at the switching instants. For

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continuous-time case, the switching happens exactly when the state trajectory hits the switching surface. Even without knowledge of the direction that the state trajectories will follow when crossing the switching surface, one still can fulfil the above non-increasing requirement by simply forcing the two quadratic Lyapunov functions' values agree at the switching surface [22]. However, the situation becomes complicated in the discrete-time case. Unlike the continuous-time case, discrete-time switched systems do not have the nice property that the switching occurs exactly on the switching surface. Instead, the switching happens in a region around the switching surface. As a result, we can not simply capture the switching instants for discrete-time switched systems as the time instants when the state trajectories cross the switching surfaces. Therefore, in order to guarantee the non-increasing requirement at the switching instants for the discrete-time case, we need to include more constraints involving state transitions for the discrete-time switched systems around the switching surfaces. This makes the switching stabilization problem for discrete-time switched systems more challenging. This may partially explain why most existing results for the switching control law design are focused on the continuous-time case. In addition, we also studied the switching controller synthesis problem to guarantee that the l_2 induced gain is below certain bound. To the authors' knowledge, most of the existing results on the robust performances of switched systems are usually on the analysis part [30], [11] or on the continuous feedback controllers design [20], while conditions for switching controller synthesis to guarantee robust performance are rare.

The rest of the paper is organized as follows. In Section II, mathematical models for the discrete-time switched linear system are described, and the switching controller synthesis problem is formulated. In Section III, the stabilization problem is investigated based on the MLF theorem. The l_2 induced gain is studied in Section IV, which is based on an extension of the MLF theorem. Sufficient conditions for controller synthesis is proposed in the form of BMIs. Finally, concluding remarks are presented.

Notation: The relation $A > B$ ($A < B$) means that the matrix $A - B$ is positive (negative) definite, similar for $A \geq B$. The superscript T stands for matrix transposition and the notation M^{-1} denotes the inverse matrix of M. The matrix I stands for identity matrix of proper dimension. l_2 is the Lebesgue space consisting of all discrete-time vector-valued function that are square-summable over $\mathbb{Z}^+$. $\|z\|_2$ denotes the l_2 norm of z, which is defined as $\|z\|_2^2 = \sum_0^{+\infty} z^T(t)z(t)$.

II. Problem Formulation and Preliminaries

In this paper, we consider a collection of discrete-time linear systems described by the difference equations

$$\begin{cases} x(t+1) &= A_i x(t) + B_i w(t) \\ z(t) &= C_i x(t) + D_i w(t) \end{cases} \quad (1)$$

where $t \in \mathbb{Z}^+$, the state $x \in \mathbb{R}^n$, disturbance $w \in \mathbb{R}^r$, and output $z \in \mathbb{R}^m$. It is assumed that the disturbance $w(t)$ is with finite l_2 norm. Denote the finite set $I_N = \{1, \cdots, N\}$, which stands for the collection of finite discrete modes. For any subsystem i, the state matrices A_i, B_i, C_i, and D_i are constant matrices of appropriate dimensions.

The problem that we are interested in here is to design a (static state feedback) switching control law, i.e., a map $i(x) : x \mapsto i$, such that the switched system (1) is exponentially stable with bounded l_2 induced gain from w to z. It is assumed that none of the subsystems is stable, since otherwise the problem would be trivial.

First of all, we recall Finsler's Lemma, which has been used previously in the control literature mainly with the purpose of eliminating design variables in matrix inequalities, see e.g. [1].

Lemma 1 (Finsler's Lemma): Let $\zeta \in \mathbb{R}^n$, $P = P^T \in \mathbb{R}^{n\times n}$, and $H \in \mathbb{R}^{m\times n}$ such that $rank(H) = r < n$. The following statements are equivalent:

1) $\zeta^T P \zeta < 0$, for all $\zeta \neq 0$, $H\zeta = 0$;
2) $\exists X \in \mathbb{R}^{n\times m}$ such that $P + XH + H^T X^T < 0$.

□

In the Finsler's Lemma, item 1) has a constrained quadratic form in $\mathbb{R}^n$ while item 2) provides an unconstrained quadratic form, where the constraint is taken into account by introducing multiplier X.

III. Stabilization

We first consider the stabilization issues for system (1), and so we assume that $w = 0$, and consider

$$x(t+1) = A_i x(t) \quad (2)$$

Our goal here is to design (state feedback) switching control laws so that the closed-loop switched linear system is exponentially stable to the origin. Notice that for all the subsystems in the form of (2), the origin is the common equilibrium.

To be precise, the exponential stability of the switched system (2) is defined as follows

Definition 1: The origin of the system (2) is exponentially stable if all trajectories satisfy

$$\|x(t)\| \leq \kappa \xi^t \|x_0\| \quad (3)$$

for some $\kappa > 0$ and $0 < \xi < 1$. Here $\|\cdot\|$ stands for standard Euclidian norm in $\mathbb{R}^n$. □

First of all, we recall a well-known approach in switched systems literature to guarantee exponentially stability using multiple Lyapunov functions.

A. Multiple Lyapunov Function Theorem

Since we assume that none of the subsystems, $x(t+1) = A_i x(t)$, is stable, there does not exist a Lyapunov function for the subsystems in a classical sense. However, it is still possible to restrict our concern in certain region of the state space, say $\Omega_i \subset \mathbb{R}^n$, and the abstracted energy of the i-th subsystem could be decreasing along the trajectories inside this region (there is no requirement outside the region Ω_i). This idea is captured by the concept of Lyapunov-like function.

Definition 2 (Lyapunov-like function): By saying that a subsystem has an associated Lyapunov-like function V_i in region Ω_i, we mean that

1) There exist constant scalars $\beta_i \geq \alpha_i > 0$ such that
$$\alpha_i \|x(t)\|^2 \leq V_i(x(t)) \leq \beta_i \|x(t)\|^2$$
hold for any $x(t) \in \Omega_i$;
2) For all $x(t) \in \Omega_i$ and $x(t) \neq 0$,
$$\Delta V_i(x(t)) = V_i(x(t+1)) - V_i(x(t)) < 0.$$

□

The first condition implies positiveness and radially unboundedness for $V_i(x)$ when $x \in \Omega_i$, while the second condition guarantees the decreasing of the abstracted energy, value of function $V_i(x)$, along trajectories of subsystem i inside Ω_i. Notice that it is possible that $x(t) \in \Omega_i$ while $x(t+1) \notin \Omega_i$.

Suppose that all the regions Ω_i cover the whole state space; Then we obtain a set of Lyapunov-like functions. To study the global stability of the switched systems, one needs to concatenate these Lyapunov-like functions together and form a non-traditional Lyapunov function, called multiple Lyapunov function (MLF). MLF is proved to be a powerful tool for studying the stability of switched systems, see e.g. [2], [19], [15], [7]. The basic idea of MLF method can be described as follows. It is known that the MLF's value would decrease when every subsystem is active only in the its corresponding region Ω_i. If we can also restrict the switching signals in such a way that, at every time we enter (switch into) a certain subsystems, its corresponding Lyapunov function value is smaller than its value at the previous entering time, then the switched system is asymptotically stable. In other words, for each subsystem the corresponding Lyapunov function value at every entering instant form a monotonically decreasing sequence. Here, we adopt the idea in a more conservative way to require that at every switching instant the MLF's value is also non-increasing. On one hand this will simplify our controller design, and on the other, it will deduce stronger property, i.e., exponential stability. In summary, we could present this result as a theorem, which is adopted from [7], [22].

Theorem 1: Suppose that each subsystem has an associated Lyapunov-like function V_i in its active region Ω_i, each with equilibrium point $x = 0$. Also, suppose that $\bigcup_i \Omega_i = \mathbb{R}^n$. Let $s(t)$ be a class of switching sequences such that $s(t)$ can take value i only if $x(t) \in \Omega_i$, and in addition
$$V_j(x(t_{i,j})) \leq V_i(x(t_{i,j}))$$
where $t_{i,j}$ denotes the time that the subsystems j is switched in from subsystem i, i.e., $x(t_{i,j}-1) \in \Omega_i$ while $x(t_{i,j}) \in \Omega_j$. Then, the switched linear system (2) is exponentially stable under the switching signals $s(t)$. □

In the sequel, we will restrict our attention to quadratic Lyapunov-like functions and corresponding multiple quadratic Lyapunov function. Before that, we need to represent partitions of the state space.

B. Partition of the state space

The purpose to dividing the whole state space $\mathbb{R}^n$ into pieces, denoted by Ω_i, is to facilitate the identification of a Lyapunov-like function for one of these subsystems. After successfully obtaining these Lyapunov-like functions associated within each region Ω_i, one may patch them together via following the conditions in the above MLF theorem so as to guarantee global stability.

For this purpose, it is necessary to require that all these regions Ω_i cover the whole state space, i.e.,

- Covering Property:
$$\Omega_1 \cup \cdots \cup \Omega_N = \mathbb{R}^n;$$

This condition merely says that there are no regions in the state space where none of the subsystems is activated.

Since we will restrict our attention to quadratic Lyapunov-like function for its merit of computational efficiency, we will consider regions given (or approximated) by quadratic forms
$$\Omega_i = \{x \in \mathbb{R}^n |\ x^T Q_i x \geq 0\},$$
where $Q_i \in \mathbb{R}^{n\times n}$ are symmetric matrices, and $i \in \{1, \cdots, N\}$.

The following lemma gives a sufficient condition for the covering property.

Lemma 2: [22] If for every $x \in \mathbb{R}^n$
$$\sum_{i=1}^{N} \theta_i x^T Q_i x \geq 0 \tag{4}$$
where $\theta_i \geq 0$, $i \in I_N$, then $\bigcup_{i=1}^N \Omega_i = \mathbb{R}^n$. □

Consider the *largest region function strategy*, i.e.,
$$i(x) = \arg\left(\max_{i\in I_N} x^T Q_i x\right) \tag{5}$$

This is due to the selection of subsystems (at state x) corresponding to the largest value of the region function $x^T Q_i x$. This switching strategy was previously introduced in [22] for continuous-time switched linear systems.

C. Quadratic Lyapunov-like Functions

In this subsection, we aim to find conditions expressed as LMIs for the existence of quadratic Lyapunov-like function in the form of $V_i(x) = x^T P_i x$ assigned to each region Ω_i. Dy definition, a Lyapunov-like function $V_i(x) = x^T P_i x$ needs to satisfy the following two conditions:

1) Condition 1: There exist constant scalars $\beta_i \geq \alpha_i > 0$ such that
$$\alpha_i \|x(t)\|^2 \leq V_i(x(t)) \leq \beta_i \|x(t)\|^2$$
holds for any $x(t) \in \Omega_i$.

Considering quadratic Lyapunov-like function candidate $V_i(x(t)) = x(t)^T P_i x(t)$, we obtain that
$$\alpha_i x(t)^T I x(t) \leq x(t)^T P_i x(t) \leq \beta_i x(t)^T I x(t),$$
holds for $x(t)^T Q_i x(t) \geq 0$. That is
$$\begin{cases} x(t)^T(\alpha_i I - P_i)x(t) \leq 0 \\ x(t)^T(P_i - \beta_i I)x(t) \leq 0 \end{cases}$$

holds for $x(t)^T Q_i x(t) \geq 0$. Applying the $\mathcal{S}$-procedure [1], the above constrained inequalities follow from the LMIs

$$\begin{cases} \alpha_i I - P_i + \eta_i Q_i \leq 0 \\ P_i - \beta_i I + \rho_i Q_i \leq 0 \end{cases}$$

where $\eta_i \geq 0$ and $\rho_i \geq 0$ are unknown scalars.

2) Condition 2: For all $x(t) \in \Omega_i$, $x(t) \neq 0$,

$$\Delta V_i(x(t)) = V_i(x(t+1)) - V_i(x(t)) < 0,$$

where $x(t+1) = A_i x(t)$.

This is equivalent to

$$x(t)^T [A_i^T P_i A_i - P_i] x(t) < 0 \tag{6}$$

for $x(t) \in \Omega_i$.

Applying Finsler's Lemma, with

$$P = \begin{bmatrix} -P_i & 0 \\ 0 & P_i \end{bmatrix}, \ \zeta = \begin{bmatrix} x(t) \\ x(t+1) \end{bmatrix},$$

$X = \begin{bmatrix} F_i \\ G_i \end{bmatrix}$, and $H = \begin{bmatrix} A_i & -I \end{bmatrix}$, then (6) is equivalent to

$$\zeta^T \begin{bmatrix} A_i^T F_i^T + F_i A_i - P_i & A_i^T G_i^T - F_i \\ G_i A_i - F_i^T & P_i - G_i - G_i^T \end{bmatrix} \zeta < 0$$

for $\zeta^T \begin{bmatrix} Q_i & 0 \\ 0 & 0 \end{bmatrix} \zeta \geq 0$. Here F_i, $G_i \in \mathbb{R}^{n \times n}$ are unknown matrices.

Applying the $\mathcal{S}$-procedure, the above constrained stability condition is implied by the following unconstrained condition for unknown matrices $P_i = P_i^T$, $Q_i = Q_i^T$, F_i, $G_i \in \mathbb{R}^{n \times n}$, and scalars $\mu_i \geq 0$,

$$\begin{bmatrix} A_i^T F_i^T + F_i A_i - P_i + \mu_i Q_i & A_i^T G_i^T - F_i \\ G_i A_i - F_i^T & P_i - G_i - G_i^T \end{bmatrix} < 0$$

Combining the above two conditions, we introduce methods to find quadratic Lyapunov-like functions for each subsystem within certain regions in the state space, which guarantee that the abstract energy of the subsystem is decreasing while staying within these regions. The next step is to properly patch these quadratic Lyapunov-like functions together, so as to obtain a global piecewise quadratic Lyapunov function to guarantee the decreasing of the abstract energy for the whole switched system. This is done in the next subsection based on the MLF theorem.

D. Switching Condition

Following Theorem 1, in order to guarantee exponential stability we also need to make sure that

1) Subsystem i is active only when $x(t) \in \Omega_i$,
2) When switching occurs, it is required to guarantee that Lyapunov function value is not increasing.

To verify the first condition, suppose that the covering condition (4) holds, i.e., $\sum_{i=1}^{N} \theta_i x^T Q_i x \geq 0$ for some $\theta_i \geq 0$, $i \in I_N$. Then, based on the largest region function strategy, namely,

$$i(x) = \arg\left(\max_{i \in I_N} x^T Q_i x\right),$$

the state x with current active mode i satisfies $x^T Q_i x \geq 0$. This implies $x \in \Omega_i$. So the first condition holds for the largest region function strategy.

Secondly, assume that a switching, $i \to j$, occurs at time instant t, i.e., $x(t) \in \Omega_j$ while $x(t-1) \in \Omega_i$ for $i \neq j \in I_N$, it is required that $V_j(x(t)) \leq V_i(x(t))$.

This means that

$$x(t-1)^T [A_i^T P_j A_i - P_i] x(t-1) \leq 0 \tag{8}$$

and $x(t-1) \in \Omega_i$, $x(t) = A_i x(t-1) \in \Omega_j$.

Because the above inequality is non-strict, the Finsler's Lemma can not be directly applied. However, it is possible to obtain a similar relation for non-strict case. In fact,

$$\exists X : \quad P + XH + H^T X^T \leq 0$$

implies $\zeta^T P \zeta \leq 0$, for all $\zeta \neq 0$, $H\zeta = 0$. This can be seen by left multiplying ζ^T and right multiplying ζ to $P + XH + H^T X^T \leq 0$ and using $H\zeta = 0$.

Therefore, with

$$P = \begin{bmatrix} -P_i & 0 \\ 0 & P_j \end{bmatrix}, \ \zeta = \begin{bmatrix} x(t-1) \\ x(t) \end{bmatrix},$$

$X = \begin{bmatrix} F_{ij} \\ G_{ij} \end{bmatrix}$, and $H = \begin{bmatrix} A_i & -I \end{bmatrix}$, (8) is implied by

$$\zeta^T \begin{bmatrix} A_i^T F_{ij}^T + F_{ij} A_i - P_i & A_i^T G_{ij}^T - F_{ij} \\ G_{ij} A_i - F_{ij}^T & P_j - G_{ij} - G_{ij}^T \end{bmatrix} \zeta \leq 0$$

for $\zeta^T \begin{bmatrix} Q_i & 0 \\ 0 & Q_j \end{bmatrix} \zeta \geq 0$. Here F_{ij}, $G_{ij} \in \mathbb{R}^{n \times n}$ are unknown matrices.

Applying the $\mathcal{S}$-procedure, the above constrained stability condition is implied by the following: there exist unknown matrices $P_i = P_i^T$, $Q_i = Q_i^T$, F_{ij}, $G_{ij} \in \mathbb{R}^{n \times n}$, and scalars $\mu_{ij} \geq 0$, such that the matrix

$$\begin{bmatrix} A_i^T F_{ij}^T + F_{ij} A_i - P_i + \mu_{ij} Q_i & A_i^T G_{ij}^T - F_{ij} \\ G_{ij} A_i - F_{ij}^T & P_j - G_{ij} - G_{ij}^T + \mu_{ij} Q_j \end{bmatrix}$$

is negative semi-definite.

E. Synthesis Condition

In summary, the above discussion can be presented as the following sufficient condition for the discrete-time linear system (2) to be exponentially stabilized.

Theorem 2: If there exist matrices P_i ($P_i = P_i^T$), Q_i ($Q_i = Q_i^T$), F_i, G_i, F_{ij}, Q_{ij}, and scalars $\nu > 0$, $\alpha_i > 0$, $\beta_i > 0$, $\eta_i \geq 0$, $\rho_i \geq 0$, $\mu_i \geq 0$, $\mu_{ij} \geq 0$, $\theta_i \geq 0$, solving the optimization problem (9) for all $i, j \in \{1, \cdots, N\}$, $i \neq j$, then the largest region function strategy implies that the origin of the switched linear system (2) is exponentially stable with decay rate $\xi = \sqrt{1 - \nu}$. □

Some remarks are in order. First, similar to its continuous-time counterpart, the optimization problem above is a Bilinear Matrix Inequality (BMI) problem, due to the product of unknown scalars and matrices. BMI problems are NP-hard, and not computationally efficient. However, practical algorithms for optimization problems over BMIs exist

$$
\max \nu
$$
$$
s.t. \left\{ \begin{array}{l} \alpha_i I + \eta_i Q_i \le P_i \le \beta_i I - \rho_i Q_i \\ \begin{bmatrix} A_i^T F_i^T + F_i A_i - P_i + \mu_i Q_i + \nu I & A_i^T G_i^T - F_i \\ G_i A_i - F_i^T & P_i - G_i - G_i^T \end{bmatrix} \le 0, \\ \begin{bmatrix} A_i^T F_{ij}^T + F_{ij} A_i - P_i + \mu_{ij} Q_i & A_i^T G_{ij}^T - F_{ij} \\ G_{ij} A_i - F_{ij}^T & P_j - G_{ij} - G_{ij}^T + \mu_{ij} Q_j \end{bmatrix} \le 0 \\ \theta_1 Q_1 + \cdots + \theta_N Q_N \ge 0 \end{array} \right. \tag{9}
$$

and typically involve approximations, heuristics, branch-and-bound, or local search. As suggested in [22] for continuous-time case, one possible way to compute the BMI problem is to grid up the unknown scalars, and then solve a set of LMIs for fixed values of these parameters. It is argued that the gridding of the unknown scalars can be made quite sparsely [22].

It can be shown that the introduction of multiplier matrices, like F_i, G_i etc., gives a lot of flexility, and many known stability conditions in the literature can be reduced to a special selection of these multiplier matrices, see e.g. [9]. In addition, these multiplier matrices would make the co-design of continuous feedback controllers and switching laws possible, which will be explored in future work.

IV. Performance

Consider the discrete-time systems (1) with l_2-norm bounded disturbance w. The goal of this section is to guarantee the l_2 induced gain from disturbance w to output z is below certain desirable bound.

Here the l_2 induced gain is defined in a standard way, i.e., the l_2 gain of the system is the quantity,

$$
\sup_{\|w\|_2 \neq 0} \frac{\|z\|_2}{\|w\|_2},
$$

where the $\sup$ is taken over all nonzero trajectories of the system. To consider l_2 gain performance, we first extend Theorem 1.

Proposition 1: Suppose each subsystem has an associated Lyapunov-like function V_i in its active region Ω_i with finite l_2 gain performance, each with equilibrium point $x = 0$. This means that

1) There exist constant scalars $\beta_i \ge \alpha_i > 0$ such that

$$
\alpha_i \|x(t)\|^2 \le V_i(x(t)) \le \beta_i \|x(t)\|^2
$$

hold for any $x(t) \in \Omega_i$;

2) For all $x(t) \in \Omega_i$ and $x(t) \neq 0$,

$$
\Delta V_i(x(t)) + z(t)^T z(t) - \gamma_i^2 w(t)^T w(t) < 0.
$$

Also, suppose that $\bigcup_i \Omega_i = \mathbb{R}^n$. Let $s(t)$ be a class of piecewise-constant switching sequences such that $s(t)$ can take value i only if $x(t) \in \Omega_i$, and in addition

$$
V_j(x(t_{i,j})) \le V_i(x(t_{i,j}))
$$

where $t_{i,j}$ denotes the time that the subsystems j is switched from subsystem i, i.e., $x(t_{i,j} - 1) \in \Omega_i$ while $x(t_{i,j}) \in \Omega_j$.

Then, the switched linear system (2) is exponentially stable under the switching signals $s(t)$, and has l_2 induced gain less than γ, where $\gamma = \max_i \gamma_i$. □

In a parallel development to Section III, we consider piecewise quadratic Lyapunov functions and derive corresponding matrix inequalities.

The condition that for all $x(t) \in \Omega_i$ and $x(t) \neq 0$,

$$
\Delta V_i(x(t)) + z(t)^T z(t) - \gamma_i^2 w(t)^T w(t) < 0,
$$

means that

$$
x^T(t)[A_i^T P_i A_i - P_i]x(t) + z(t)^T z(t) - \gamma_i^2 w(t)^T w(t) < 0,
$$

for $x(t) \in \Omega_i$, and $z(t) = C_i x(t) + D_i w(t)$, $x(t+1) = A_i x(t) + B_i w(t)$. This can be transformed into a matrix inequality based on Finsler's Lemma, with

$$
P = \begin{bmatrix} -P_i & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}, \ \zeta = \begin{bmatrix} x(t) \\ x(t+1) \\ z(t) \\ w(t) \end{bmatrix},
$$

$$
X = \begin{bmatrix} F_{1i} & F_{2i} \\ G_{1i} & G_{2i} \\ H_{1i} & H_{2i} \\ J_{1i} & J_{2i} \end{bmatrix}, \ H = \begin{bmatrix} A_i & -I & 0 & B_i \\ C_i & 0 & -I & D_i \end{bmatrix}.
$$

Analogously, we can obtain the following sufficient conditions for the discrete-time switched linear system (1) to be stabilized exponentially with l_2 gain less than γ.

Theorem 3: If there exist matrices P_i ($P_i = P_i^T$), Q_i ($Q_i = Q_i^T$), F_{1i}, G_{1i}, H_{1i}, J_{1i}, F_{2i}, G_{2i}, H_{2i}, J_{2i}, F_{ij}, Q_{ij}, and scalars $\alpha_i > 0$, $\beta_i > 0$, $\eta_i \ge 0$, $\rho_i \ge 0$, $\nu > 0$, $\mu_i \ge 0$, $\mu_{ij} \ge 0$, $\theta_i \ge 0$, solving the optimization problem (10) for all $i, j \in \{1, \cdots, N\}$ $i \neq j$, then the linear system (1) can be exponentially stabilized with l_2 gain less than γ by the largest region function strategy. □

V. Concluding Remarks

In this paper, a switching control law, based on static state feedback, is synthesized for a class of discrete-time switched linear systems. The exponential stability and l_2 induced gain performance are investigated based on multiple quadratic Lyapunov-like functions. Sufficient synthesis conditions are proposed as an optimization problem with bilinear matrix inequality constraints, which can be dealt with as LMIs provided that certain associated parameters are selected in advance.

$$\min \ \gamma$$
$$s.t. \begin{cases} \alpha_i I + \eta_i Q_i \le P_i \le \beta_i I - \rho_i Q_i \\ \Lambda_i + U_i + U_i^T < 0 \\ \begin{bmatrix} A_i^T F_{ij}^T + F_{ij} A_i - P_i + \mu_{ij} Q_i & A_i^T G_{ij}^T - F_{ij} \\ G_{ij} A_i - F_{ij}^T & P_j - G_{ij} - G_{ij}^T + \mu_{ij} Q_j \end{bmatrix} \le 0 \\ \theta_1 Q_1 + \cdots + \theta_N Q_N \ge 0 \end{cases} \tag{10}$$

where $$\Lambda_i = \begin{bmatrix} -P_i + \mu_i Q_i & 0 & 0 & 0 \\ 0 & P_i & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -\gamma^2 I \end{bmatrix}, \ U_i = \begin{bmatrix} F_{1i}A_i + F_{2i}C_i & -F_{1i} & -F_{2i} & F_{1i}B_i + F_{2i}D_i \\ G_{1i}A_i + G_{2i}C_i & -G_{1i} & -G_{2i} & G_{1i}B_i + G_{2i}D_i \\ H_{1i}A_i + H_{2i}C_i & -H_{1i} & -H_{2i} & H_{1i}B_i + H_{2i}D_i \\ J_{1i}A_i + J_{2i}C_i & -J_{1i} & -J_{2i} & J_{1i}B_i + J_{2i}D_i \end{bmatrix}.$$

The contributions of the paper are twofold. First, switching control law synthesis methods based on multiple Lyapunov functions were extended to discrete-time switched systems. These method have distinct features different from the continuous-time case. Secondly, the MLF theorem was extended to guarantee l_2 induced gain performance, and this result was applied for switching control law synthesis as well.

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