K.M. Passino and P.J. Antsaklis, "Inverse Stable Sampled Low Pass Systems," Control Systems Technical Report #47, Dept. of Electrical and Computer Engineering, University of Notre Dame, March 1986. Also in Journal of Control, Vol. 47, No. 6, pp. 1905-1913, June 1988

INT. J. CONTROL, 1988, VOL. 47, NO. 6, 1905–1913

Inverse stable sampled low-pass systems

K. M. PASSINO† and P. J. ANTSAKLIS†

It is shown that for low-pass systems the fractional-order hold and the pulse amplitude modulation signal reconstruction methods produce an inverse stable sampled transfer function $H(z)$ in some cases when zero-order hold fails to do so. Therefore they may be desirable alternatives to the zero-order hold reconstruction in certain control problems. The problem of how well $H(z)$ models the continuous $G(s)$ is also discussed.

1. Introduction

In the study of sampled systems, given a continuous plant $G(s)(y(s) = G(s)u(s))$, a sampled transfer function $H(z)$ must be determined so that if the sampled input $u(kT)$, where T is the sampling period, is applied to $H(z)$, the output $\hat{y}(kT)$ is an acceptable approximation of the sampled output $y(kT)$. It is clear that for a successful digital control design $H(z)$ should accurately model at least those characteristics of $G(s)$ that are important to the control design specifications. We will refer to this accuracy in modelling as the accuracy in the discrete equivalence of $H(z)$ to $G(s)$.

We are interested in using a reconstruction circuit $G_r(s)$ to obtain the sampled transfer function $H(z)$. $H(z)$ in this case represents the reconstruction circuit, preceded by an impulse modulator, followed in cascade by $G(s)$ and a sampler. It follows that

$$H_r(z) = Z\{G_r G(s)\} \tag{1}$$

where $Z\{\ \}$ denotes the z-transform of the corresponding continuous time signal. For the zero-order hold (ZOH) case, $G_r(s) = G_0(s)$ where

$$G_0(s) = \frac{1 - \exp(-Ts)}{s} \tag{2}$$

Consequently,

$$H_0(z) = \frac{Z_0(z)}{P_0(z)} = Z\left\{\frac{(1 - \exp(-Ts))G(s)}{s}\right\} = (1 - z^{-1})Z\left\{\frac{G(s)}{s}\right\} \tag{3}$$

Zeros of $H(z)$ inside the unit disc are very desirable in control design (especially in adaptive systems). Unfortunately, a $G(s)$ with zeros inside the left half s-plane is not necessarily transformed to an $H(z)$ with zeros inside the unit disc when signal reconstruction methods are used. In contrast, the poles p_i of $G(s)$ are transformed as $p_i \rightarrow \exp(p_i T)$, a transformation which maps the left half-plane onto the unit disc; i.e. while a stable $G(s)$ always has a stable discrete equivalent $H(z)$, a minimum phase $G(s)$ does not necessarily imply an inverse stable $H(z)$, thus leading to a more difficult to control pulse transfer function $H(z)$.

This problem was studied by Åström *et al.* (1984) (also Tuschak 1981, Martensson

Received 11 May 1987.

† Department of Electrical and Computer Engineering, University of Notre Dame, Notre Dame, IN 46556, U.S.A.

1982, Keviczky and Kumar 1981) for the case of ZOH reconstruction. Åström *et al.* (1984) showed that given $G(s)$ with m zeros and n poles ($m < n$), m zeros of $H(z)$ go to 1 as $T \to 0$, while the remaining $n - m - 1$ zeros of $H(z)$ go to the zeros of $B_{n-m}(z)$. These $B_n(z)$ polynomials are the numerator polynomials of the ZOH discrete equivalent of $G(z) = 1/s^n$

$$H(z) = \frac{T^n}{n!} \frac{B_n(z)}{(z-1)^n} \tag{4}$$

This implies that in many cases, where T must be chosen relatively small to satisfy other criteria, $H(z)$ will be inverse unstable. Working with low-pass $G(s)$, a common characteristic of systems in control, and ZOH reconstruction, Tuschak (1981) related the magnitude and phase of $G(s)$ to the number of zeros of $H(z)$ outside the unit disc, thus establishing results for ZOH and finite T.

It is shown here that the zeros $H(z)$ can be moved inside the unit circle if other reconstruction methods $G_r(s)$ are used. Note that the zeros of $H(z)$ can be arbitrarily assigned if a reconstruction method is implemented which assigns n (n the order of $G(s)$) appropriate values to the input in one sampling period (Åström and Wittenmark 1984). Here, however, we are interested in rather typical reconstruction methods that can easily be implemented with 'off-the-shelf' components. In particular, we study the fractional order hold (FROH), and a method commonly used in communication systems, the pulse amplitude modulation (PAM) reconstruction. The accuracy in the discrete equivalence of $H(z)$ to $G(s)$ is also of interest here. Note that an inverse stable $H(z)$ is desirable, but not at the expense of the adequacy of the model $H(z)$ (discrete equivalence of $H(z)$ to $G(s)$). For example, Åström *et al.* (1984) show that under certain assumptions, as $T \to \infty$ in ZOH all the zeros of $H(z)$ go inside the unit disc as desired but $H(z)$ becomes a poor approximation of $G(s)(H(z) = G(0)z^{-1})$. We shall use the frequency response of $G(s)$ and $H(z)$ over $0 \leqslant \omega \leqslant \pi/T$ to discuss this problem. Relation (1) clearly shows the challenge: $H_r(z)$ that must model $G(s)$ only, is actually the z-transform of the product $G_r(s)G(s)$. The reconstruction circuit $G_r(s)$ will, in general, distort the characteristics of $G(s)$ and $H_r(z)$ will only be an approximation.

In § 2, FROH reconstruction is studied. Note that the FROH depends on a parameter β; for $\beta = 0$ it reduces to ZOH while for $\beta = 1$ it becomes a first order hold (FOH) reconstruction. New state-space algorithms for the calculation of the FROH transformation are first developed. A root locus approach is then developed to determine β (if such β exists) so that $H(z)$ can be made inverse stable. Furthermore, it is shown that with certain low-pass assumptions on $G(s)$ the FROH can be used to move zeros inside the unit disc for any T (Theorem 1). The FROH, however, can deteriorate discrete equivalence. The PAM reconstruction is discussed in § 3. PAM depends on the parameter τ ($\leqslant T$); when $\tau = T$, it becomes a ZOH reconstruction. It is shown that τ can be adjusted to reduce the distortion caused by $G_r(s)$ in (1), and that this choice of τ also moves zeros inside the unit circle for any T (Theorem 2). The low-pass assumptions and development are similar to those of FROH.

2. Fractional order hold signal reconstruction

Let

$$\dot{x}(t) = Fx(t) + Gu(t), \quad y(t) = Hx(t) \tag{5}$$

be a minimal realisation of $G(s)$ and consider the FROH signal reconstruction method described by

$$\hat{u}(t) = u(kT) + \beta\left[\frac{u(kT) - u(kT - T)}{T}\right](t - kT), \quad kT \leqslant t < kT + T \tag{6}$$

where the approximation $\hat{u}(t)$ is formed from the samples $u(kT)$ of the signal $u(t)$ with T the sampling period, and β a real number. In a configuration given by (1), the sampled system is

$$\left.\begin{aligned} \begin{bmatrix} x(kT+T) \\ x_1(kT+T) \end{bmatrix} &= \begin{bmatrix} \phi & \beta\Lambda \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x(kT) \\ x_1(kT) \end{bmatrix} + \begin{bmatrix} \Gamma - \beta\Lambda \\ 1 \end{bmatrix} u(kT) \\ y(kT) &= [H \quad 0]\begin{bmatrix} x(kT) \\ x_1(kT) \end{bmatrix} \end{aligned}\right\} \tag{7}$$

where (Franklin and Powell 1980, Passino and Antsaklis 1985)

$$\phi = \exp(FT), \quad \Gamma = \left[\int_0^T \exp(F\eta)\, d\eta\right]G, \quad \Lambda = \left[-\int_0^T (1 - \eta/T)\exp(F\eta)\, d\eta\right]G \tag{8}$$

This represents a more general transformation technique than described by Makhlouf (1971) and efficient numerical techniques have been developed for its solution by Passino (1984). Using the fact that the zero polynomial $Z_\beta(z)$ of (8) is the determinant of its Rosenbrock system matrix (Rosenbrock 1970)

$$H_\beta(z) = \frac{Z_\beta(z)}{P_\beta(z)} = \frac{\beta(z-1)Z_{\beta 1}(z) + zZ_0(z)}{zP_0(z)} \tag{9}$$

where

$$Z_{\beta 1}(z) = -\begin{vmatrix} zI - \phi & \Lambda \\ -H & 0 \end{vmatrix} \tag{10}$$

In the following, the zero polynomial of the FROH discrete equivalent of $G(s) = 1/s^n$ is determined explicitly to compare to ZOH. Consider (5) in controllable companion form for $G(s) = 1/s^n$. The expressions for ϕ, Γ and Λ of (8) are

$$\phi = \begin{bmatrix} 1 & T & \dots & T^{n-1}/(n-1)! \\ 0 & 1 & & \vdots \\ \vdots & \vdots & & T \\ 0 & 0 & \dots & 1 \end{bmatrix} \qquad \begin{aligned} \Gamma &= [T^n/n!, \dots, T]^t \\ \Lambda &= -[T^n/(n+1)!, \dots, T/2!]^t \end{aligned} \tag{11}$$

The zero polynomial $Z_\beta(z)$ has been determined by Passino (1984) using the fact that the matrix in (10), the determinant of which must be calculated, is in the Hessenberg form (Franklin 1968). In particular it has been shown that

$$Z_\beta(z) = \frac{T^n}{n!}(z - \beta)B_n(z) + \frac{\beta T^n}{(n+1)!}B_{n+1}(z), \quad Z_0(z) = \frac{T^n}{n!}B_n(z) \tag{12}$$

where $B_n(z)$ are the polynomials in (4). To produce this result a novel recursive relation was introduced (Passino 1984), namely

$$B_n(z) = \sum_{k=0}^{n-1} \binom{n}{k} (z-1)^{n-k-1} B_k(z), \quad B_0(z) \triangleq 1 \tag{13}$$

In view of (9), (12), it is clear that a *root-locus approach* can be used to determine the range of β for which the zeros of $Z_\beta(z)$ lie inside the unit disc. If such β exists, the corresponding FROH reconstruction will give an $H_\beta(z)$ that is inverse stable.

If $G(s)$ is a low-pass system then the exact number of zeros of $H(z)$ outside the unit disc can be determined in terms of $G(s)$. In particular the following theorem is proved in the Appendix.

Theorem 1

Assume that $G(s)$ is a low-pass system satisfying conditions (A 10), and that FROH is implemented. Let μ and ν denote the number of poles and zeros of $H_\beta(z)$ outside the unit circle. Then as $\beta \to -1$ with T fixed and finite

$$\nu = \mu + \lambda - 1, \quad -(\lambda+1)\pi < \arg[G(j\pi/T)] \leqslant -\lambda\pi \tag{14}$$

This represents an improvement over the ZOH case ($\nu = \mu + \lambda$). Notice that μ is known (due to the $\exp(-p_i T)$ transformation of the poles) and λ is determined by the phase of $G(s)$ via (14). Hence determining the number of zeros outside the unit circle can be accomplished by drawing a Bode plot of the phase of the plant and using (14). Also note that by varying π/T on the Bode plot one can determine the range of T for stable zeros ($\nu = 0$).

The FROH reconstruction was applied to several transfer functions and some of the results are reported below. First consider the case for $G(s)$ with $T \to 0$; i.e. $G(s) = 1/s^n$ (see discussion above). Using root locus it can be shown that if $n = 1$, all zeros are inside the unit disk for $\beta < 4$. $H_\beta(z)$ is also inverse stable when $n = 2$ for $-1 < \beta < 0$ (see discussion in the Appendix); for example, for $\beta = -0{\cdot}3$ the zeros are at $-0{\cdot}666$ and $-0{\cdot}333$. However, if $n = 3$ or higher there is no β that will bring all of the zeros inside the unit circle. Compared to the ZOH where $H_0(z)$ is inverse unstable for $n \geqslant 2$ (Åström *et al.* 1984), the FROH reconstruction offers some advantages when n is small; in other words, if T must be chosen small to satisfy other criteria, the FROH reconstruction will produce inverse stable $H_\beta(z)$ for a wider class of plants $G(s)$.

Consider now finite T and low-pass $G(s)$. In view of Theorem 1 FROH can make $H(z)$ inverse stable where ZOH cannot. Using FROH reconstruction the minimum T for stable zeros can be reduced compared to ZOH. With $G(s) = 1/(s+1)^3$ it was shown (Åström *et al.* 1984) that the ZOH equivalent $H_0(z)$ has a zero outside the unit disc if $0 \leqslant T < T_{\min}$, $T_{\min} = 1{\cdot}8399$. Using FROH with $T = 1{\cdot}5$ ($< T_{\min}$) and $\beta = -0{\cdot}5$ the zeros are at $-0{\cdot}117$, $-0{\cdot}589 \pm j0{\cdot}274$. When $T = 1$ and ZOH is used the zeros are at 0, $-0{\cdot}124$, $-1{\cdot}8$ while if FRPH is used with $\beta = -0{\cdot}6$ the zeros are at $-0{\cdot}19$, $-0{\cdot}769 \pm j0{\cdot}216$ and with $\beta = -0{\cdot}8$ the zeros are at $-0{\cdot}18$, $-0{\cdot}736 \pm j0{\cdot}666$.

Analysis in the frequency domain can give further insight into our problem. $G_0(s)$ (in (2)) has magnitude and phase given by

$$|G_0(j\omega)| = T|\mathrm{sinc}\,(\omega T/2)|, \quad \text{phase}\,[G_0(j\omega)] = -\omega T/2 \tag{15}$$

$H_0(z)$, which should match $G(s)$ over the range $-\pi/T \leqslant \omega \leqslant \pi/T$, is determined from $G_0 G$. Notice that $H_0(\exp(j\omega T))$ will only approximate $G(j\omega)$ with the approximation becoming worse as ω approaches π/T. For ZOH the poles (p_i) of $G(s)$ transform as

$\exp(-p_i T)$. Therefore, the zeros and the gain in $H_0(z)$ must be so that the discrete equivalence between $H_0(z)$ and $G_0 G(s)$ required by (3) is attained. This often requires (Passino 1984) the addition of small amounts of phase lead; the zeros of $H_0(z)$ to produce this small amount of lead must be outside the unit circle. In the FROH transformation the parameter β can be chosen to give phase lead (see the Appendix); consequently it can move zeros inside the unit disc. However, some choices of β to obtain inverse stable $H_\beta(z)$ can destroy the accuracy of discrete equivalence between $H_\beta(z)$ and $G(s)$ over an important frequency range, e.g. the $1/(s+1)^3$ example above (Passino and Antsaklis 1985).

3. Pulse amplitude modulation reconstruction

Consider the PAM signal reconstruction method described by

$$\hat{u}(t) = \begin{cases} u(kT) & kT \leqslant t < kT + \tau \\ 0 & kT + \tau \leqslant t < kT + T \end{cases} \tag{16}$$

The continuous transfer function $G_p(s)$ of the PAM reconstruction is

$$G_p(s) = \frac{1 - \exp(-\tau s)}{s}, \quad 0 < \tau \leqslant T \tag{17}$$

and we define the PAM discrete equivalent of $G(s)$ according to (1) as $H_p(z)$.

It is of interest to determine what happens to $H_p(z)$ as $\tau \to 0$ (T fixed). We shall assume that the PAM circuit also includes a normalization gain $1/\tau$. For the practical cases to be considered, this gain will not become excessively large. The transfer function (17) now becomes

$$G_p(s) = \left(\frac{1}{\tau s}\right)[1 - \exp(-\tau s)] = \left[1 - \frac{(\tau s)}{2!} + \frac{(\tau s)^3}{3!} - \dots\right] \tag{18}$$

We are interested in frequencies satisfying $-\pi/T \leqslant \omega \leqslant \pi/T$. For $\tau \to 0$, $\tau s \to 0$ and $G_p(s) \to 1$, and in the limit the PAM discrete equivalent is $Z\{G(s)\}$. This relation implies that evaluating the zeros of $H_p(z)$ when $\tau \to 0$ is quite straightforward. $Z\{G(s)\}$ can be either found in transform tables directly or after using partial fractions.

From (18) we see that the choice of τ small gives an accurate discrete equivalent $H_p(z)$. Furthermore, in view of the fact that the ZOH (3) involves $Z\{G(s)/s\}$, the zero properties of $H_p(z)$ as $\tau \to 0$ will, in many cases, improve because they involve $Z\{G(s)\}$, a lower relative degree transfer function. Comparing to (4) we see that zero properties are improved by moving the zeros from the zeros of B_n to B_{n-1}. This implies that for $n = 2$, the inverse unstable ZOH equivalent $H_0(z)$ becomes inverse stable if PAM with $\tau \to 0$ is used. The analogue to Theorem 1 for PAM is also proven in the Appendix.

Theorem 2

Assume $G(s)$ is a low-pass system satisfying conditions (A 10), and that PAM is implemented. Let μ and ν denote the number of poles and zeros of $H_p(z)$ outside the unit circle. Then as $\tau/T \to 0$ with T fixed and finite

$$\nu = \mu + \lambda - 1, \quad -(\lambda + 1)\pi < \arg[G(j\pi/T)] < -\lambda\pi \tag{19}$$

Remark

A similar result to (19) can be shown for

$$G_r(s) = G_{SH}(s) = T_0[1 - \exp(-T/T_0)\exp(-Ts)](1 + T_0 s)^{-1}$$

with $T_0 > 0$ if $T_0 \to 0$; which is just a sample and hold circuit with time constant T_0.

Similar to FROH, this represents an improvement over the ZOH case. As shown in the examples below, τ/T need not approach zero. In applications of PAM to communication, $\tau/T = 1/16$ is quite common. Again, if the proper low-pass conditions (A 10) are met, one can use a Bode phase plot and (20) above, to find the proper value of π/T and thus λ to obtain no inverse unstable zeros ($v = 0$). Alternatively, the above also suggests the following procedure: given $G(s)$, determine the zeros of the ZOH equivalent $H_0(z)(\tau = T)$. If improvement is desirable, determine the zeros of $Z\{G(s)\}(\tau \to 0)$. Then choosing τ between 0 and T one could achieve good zero properties with acceptable τ.

In view of Theorem 2, PAM can make $H(z)$ inverse stable when ZOH cannot. As an example of this, again consider $G(s) = 1/(s+1)^3$. When $T = 0{\cdot}5$ the zeros of the ZOH equivalent $H_0(z)$ are outside the unit disc, namely at $-2{\cdot}58$ and $-0{\cdot}183$ ($T_{\min} = 1{\cdot}8399$). Using PAM reconstruction with $\tau = 0{\cdot}1$, the zeros of $H_p(z)$ are inside the unit disc at $-0{\cdot}873$ and $-0{\cdot}007106$ with poles at the same locations as for the ZOH. When $\tau = T/16 = 0{\cdot}03125$ the zeros are at $-0{\cdot}68444$ and $-0{\cdot}7516 \times 10^{-3}$. In short we see that with PAM one can get an inverse stable discrete equivalent for a larger range of T (e.g. $T < T_{\min}$), and a more accurate discrete equivalent $H_p(z)$. For more discussion on this see Passino and Antsaklis (1985).

4. Concluding remarks

Motivated by the work of Åström *et al.* (1984) and Tuschak (1981), two alternatives to the ZOH reconstruction method, the FROH and the PAM, were studied. It was shown that they can be used to move the zeros of $H(z)$ inside the unit circle when the plant is of low-pass character. $H(z)$ should also be a good model of the continuous plant $G(s)$, the desired accuracy of course depending on the particular application. Although both the FROH and PAM can move zeros inside the unit circle, FROH may impair discrete equivalence, but PAM always improves it compared with ZOH.

ACKNOWLEDGMENTS

This work was partially supported by the National Science Foundation under Grant ECS 84-05714.

Appendix

Proof of Theorems 1 and 2

Based on methods developed for ZOH (Tuschak 1981) the number of unstable zeros of $H(z)$ is determined in terms of the magnitude and phase of $G(s)$. Let

$$G(s) = K\frac{(1+sb_1)(1+sb_2)\dots(1+sb_m)}{(1+sa_1)(1+sa_2)\dots(1+sa_n)} \tag{A 1}$$

with $m < n$. Then, in view of (1)

$$H_r(z) = Z\{G_r G(s)\} = K^* \frac{(z-\sigma_1)(z-\sigma_2)\ldots(z-\sigma_{q-1})}{(z-z_1(z-z_2)\ldots(z-z_q)} \tag{A 2}$$

where q = number of poles of $H_r(z)$, depends on both $G(s)$ and $G_r(s)$. Using sampled data analysis (Franklin and Powell 1980)

$$H_r^*(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_r(j\omega + jk2\pi/T)G(j\omega + jk2\pi/T) \tag{A 3}$$

In view of the equality of phase of $H_r^*(j\omega)$ and $G_r G(j\omega)$, an exact relationship between the magnitude and phase of a low-pass $G(s)$ and the number of unstable zeros of $H_r(z)$ can be found (Tuschak 1981). Consider the phase at $\omega = 0$ and π/T

$$\alpha' := \arg\,[G_r G(0)] = \arg\,[K] \tag{A 4}$$

$$\alpha := \arg\,[G_r G(j\pi/T)] = \arg\,[K] + \arg\,[H_{rn}^*(j\pi/T)] \tag{A 5}$$

where $H_{rn}^*(j\omega) := (1/K)H_r^*(j\omega)$ of (A 3). Note that α and α' are equal to the phase of $H_r(z)$ at $z = 1$ and $z = -1$, respectively. Therefore

$$\alpha' = \arg\,[H_r(1)] = (q - 1 - \nu_{\text{LI}} - \nu_{\text{RI}} - \nu_{\text{LO}})\pi - (q - \mu_{\text{LI}} - \mu_{\text{RI}} - \mu_{\text{LO}})\pi + \arg\,[K^*] \tag{A 6}$$

$$\alpha = \arg\,[H_r(-1)] = (q - 1 - \nu_{\text{LO}})\pi - (q - \mu_{\text{LO}})\pi + \arg\,[K^*] \tag{A 7}$$

where ν_{LO} = number of zeros in the Left half z-plane Outside the unit circle, μ_{RI} = number of poles in the Right half z-plane Inside the unit circle, etc. In view of (A 4)–(A 7)

$$\arg\,[H_{rn}^*(j\pi/T)] = (\mu - \nu - 1)\pi \tag{A 8}$$

where $\mu = \mu_{\text{LO}} + \mu_{\text{RO}}$ and $\nu = \nu_{\text{LO}} + \nu_{\text{RO}}$.

Consider now the primary ($k = 0$) and first components ($k = -1$) of the spectrum of $H_{rn}^*(j\omega)$

$$H_{rn}^*(j\omega) = \frac{1}{T}\left[G_r(j\omega)G_n(j\omega) + G_r\left(\frac{j\omega - j2\pi}{T}\right)G_n\left(\frac{j\omega - j2\pi}{T}\right) + \varepsilon\right] \tag{A 9}$$

where $G_n(s) = (1/K)G(s)$ and ε denotes the remaining terms. If $G_n(s)$ and $G_r(s)$ are low-pass systems then

$$\varepsilon \approx 0, \quad \left|\frac{G(j\omega - j2\pi/T}{G(j\omega)}\right| \leqslant 1 \tag{A 10}$$

Fractional-order hold reconstruction

For this reconstruction method

$$G_r(s) = G_\beta(s) = (1 - \beta\exp(-Ts))\frac{1 - \exp(-sT)}{s} + \frac{\beta}{Ts^2}(1 - \exp(-Ts))^2 \tag{A 11}$$

In view of (A 2) and (9)

$$q = n + 1 \tag{A 12}$$

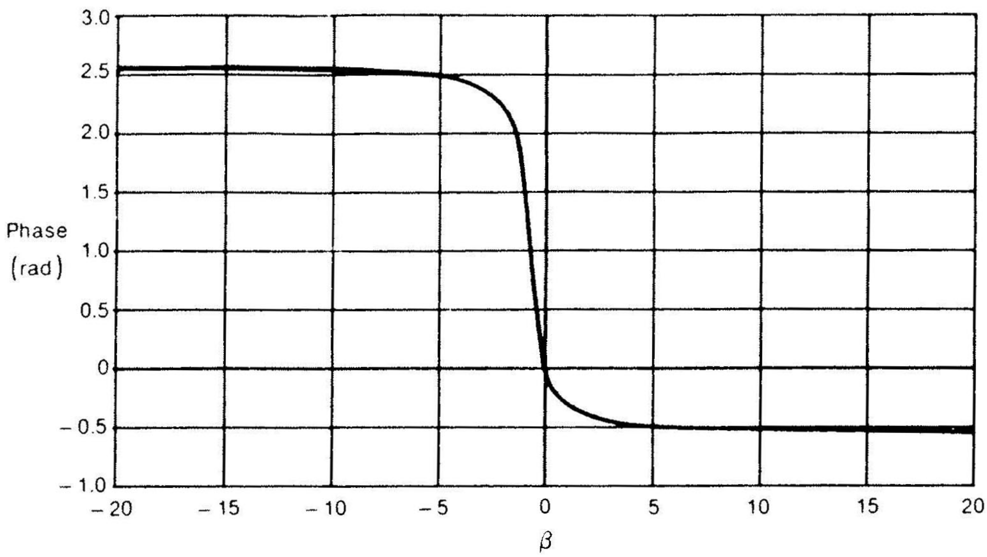
Using (2), (A 3), (9), (10), (11)

$$H_\beta^*(j\omega) = g_1 G(j\omega)\{1 + \beta[g_1 - \exp(-j\omega T)]\} + g_2 G\left(\frac{j\omega - j2\pi}{T}\right)\{1 + \beta[1 - \exp(-j\omega T)]\} \quad \text{(A 13)}$$

where $g_1 = [1 - \exp(-j\omega T)]/j\omega T$ and $g_2 = [1 - \exp(-j\omega T)]/T(j\omega - j^{2\pi}/T)$. Also

$$H_\beta^*\left(\frac{j\pi}{T}\right) = \frac{2}{\pi}\exp\left(\frac{-j\pi}{2}\right)\zeta G\left(\frac{j\pi}{T}\right)\left[1 - \frac{\bar{\zeta}\bar{G}(j\pi/T)}{\zeta G(j\pi/T)}\right] \quad \text{(A 14)}$$

where $\zeta = (1 + \beta - j2\beta/\pi)$, and $\bar{\zeta}$, $\bar{G}$ indicate conjugates. Plotting arg $[\zeta]$ vs. β for $G(s)$ satisfying (A 10) yields the Figure.



Phase *versus* beta.

Let

$$\arg[G(j\pi/T)] = -(\lambda\pi + \theta), \quad \lambda \text{ integer} \quad \text{(A 15)}$$

Since

$$\arg[1 + \exp(2j\theta)] = \theta, \quad \arg[\zeta]|_{\beta \to -1} = \frac{\pi}{2} \quad \text{(A 16)}$$

(A 14) simplifies to

$$\arg\left[H_\beta^*\left(\frac{j\pi}{T}\right)\right]\Bigg|_{\beta \to -1} = -\lambda\pi \quad \text{(A 17)}$$

Equating (A 8) and (A 17) yields

$$\nu = \mu + \lambda - 1 \quad \text{(A 18)}$$

This proves Theorem 1 where μ, ν are the number of unstable poles, zeros of $H_\beta(z)$, respectively; λ is found using (A 15).

For the ZOH case this equation is $\nu = \mu + \lambda$ (Tuschak 1981); this implies that using FROH and varying β from $\beta = 0$ (ZOH) to $\beta = -1$ one additional zero can be brought inside the unit circle for fixed and finite T. Notice that the choice of $-1 < \beta < 0$ yields a FROH that contributes the necessary phase lead to move zeros inside the unit circle (see the Figure and discussion at the end of § 2); for $\beta > 0$ FROH contributes lag and the zero properties of $H_\beta(z)$ are not necessarily improved. As an illustration consider $\beta - 1$ (FOH); proceeding as above

$$v = \mu + \lambda \tag{A 19}$$

This shows that if a FOH is used on a low-pass $G(s)$ it will not change the number of zeros of the discrete equivalent outside the unit circle compared with ZOH. Examples to illustrate this are given by Passino (1984, Table 3.2).

Pulse amplitude modulation reconstruction

Similar to FROH, here $G_r(s)$ is given by (18), $q = n$, and (A 3), (9), (10). Then

$$H_p^*(j\omega) = \frac{1}{T} \frac{1 - \exp(-j\omega\tau)}{j\omega\tau} \left[1 + \frac{G(j\omega - j2\pi/T)}{G(j\omega)} \right] G(j\omega) \tag{A 20}$$

For $\tau/T \to 0$ with T fixed and finite we get (A 18) for PAM, which proves Theorem 2. □

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