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RESEARCH ARTICLE

Model-Based Control with Intermittent Feedback: Bridging the Gap between Continuous and Instantaneous Feedback

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The aim of this paper is to provide a set of results in stability of model-based networked control systems with intermittent feedback, which we intend will serve as a nexus between the study of systems with instantaneous feedback and with continuous feedback.

We apply the concept of Intermittent Feedback to a class of networked control systems known as Model-Based Networked Control Systems (MB-NCS). Model-Based Networked Control Systems use an explicit model of the plant in order to reduce the network traffic while attempting to prevent excessive performance degradation, while Intermittent Feedback consists of the loop remaining closed for some fixed interval, then open for another interval. We begin by introducing the basic architecture for model-based control, then discuss the concept of intermittent feedback, its applications in various fields, and its role as a link between instantaneous and continuous feedback. We then provide our results for the model-based architecture with intermittent feedback. We also address the case with output feedback (through the use of a state observer), providing a full description of the state response of the system, as well as a necessary and sufficient condition for stability in each case. Extensions of our results to cases with nonlinear plants are also presented. Next, we investigate the situation where the update times τ and h are time-varying, first addressing the case where they have upper and lower bounds, then moving on to the case where their distributions are i.i.d or driven by a Markov chain. Finally, we study the case of model-based control with intermittent feedback for discrete-time plants, again providing stability conditions for the basic architecture, the state observer case, and the case with time-varying parameters.

Keywords: control systems; networked control; networked control systems; intermittent feedback; model-based control

1 Introduction

A networked control system (NCS) is a control system in which a data network is used as feedback media. NCS is an important area in control, see for example recent surveys such as (Baillieul and Antsaklis 2007) and (Hespanha 2007) in the recent Special Issue on Networked Control Systems, as well as (Nair and Evans 2000), (Took 2002), and (Walsh 1999). The use of networks as media to interconnect the different components of an industrial system is rapidly increasing. However, the use of NCSs poses some challenges. One of the main problems to be addressed when considering an NCS is the size of the bandwidth required by each subsystem. A particular class of NCSs is model-based networked control systems (MB-NCS), introduced in (Montestruque 2002). The model-based networked control systems approach is based on the concept that it is more important to reduce the number of packets sent over the network than the size of the packets themselves. This is due to the fact that, in the context of networked control systems, packets usually feature small transport time and big overhead, so that data compression provides very limited benefits in the use of network. With this in mind, the key idea then is to use partial information about the plant to our advantage, so as to reduce the

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use of network as much as possible. One of the main advantages of the model-based approach is that it yields a stabilizing controller even if the plant itself is not stable, which is not the case in most other NCS approaches; see also (Montestruque 2003, 2004). Related problems have been studied in (Fridman et al. 2004), which uses an input delay approach to obtain sufficient stability conditions for sampled-data linear systems.

Here we extend this work by taking advantage of the novel concept of intermittent feedback. In the previous work done in MB-NCS, the updates given to the model of the plant state were performed in instantaneous fashion, but with intermittent feedback the system remains in closed loop control mode for more extended intervals. This notion makes sense as it is a good representation of what occurs in both nature and industry. For example, when driving a car, when approaching a curve or hilly terrain, we pay attention to the road for a longer time, which is equivalent to staying in closed-loop mode, and we only reduce our attention -switch to open loop control- when the road is once again straight. It is worth noting that while the application of intermittent feedback to MB-NCS, the concept has been studied in different contexts, in fields such as chemical engineering (Kim 2001), psychology and behavior (Salzberg et al 1971, Schmidt 2005), and robotics (Koay and Bugmann 2004, Ronco and Hill, 1999). While intermittent control is a very intuitive notion, its combination with the MB-NCS architecture allows for obtaining important results and opening new paths in controlling NCSs effectively. For example, by combining intermittent feedback with the model-based architecture, we may gradually improve the parameters of the model -in a way, the system is "learning" or "adapting"- so that as time elapses, the control performance increases and the required use of network decreases.

In addition to its application to the MB-NCS architecture, we hope that the study of intermittent feedback will provide a conceptual bridge between continuous feedback -as we study in classical controls- and instantaneous feedback, as has been studied for example in sampled-data systems. It is worth remarking that the results presented herein converge to those of the instantaneous case when the length of the intervals during which the loop is closed approaches zero, and to those of continuous time control when the loop is closed approaches h.

In the earlier sections, we provide results for the cases where the plant is continuous-time. Full proofs are provided in the appendix of this paper. Some of the results have been presented in previous conference proceedings (Estrada et al. 2006) and (Estrada and Antsaklis 2007, 2008-1, 2008-2), and additional details can found in the technical report (Estrada and Antsaklis, 2008-3).

We then investigate what happens in the case of discrete-time plants as well (Estrada and Antsaklis, 2008). The results presented in the latter sections are a natural extension of the corresponding ones in continuous time, to a case where packets of information are transmitted at discrete intervals. It is important to note that, in the discrete time case, the parameters τ and h, which correspond to how often the loop is closed and for how long the loop is closed each time, are different from the sampling time of the digital plant, since they are tailored after the demands of use of the network, not by the internal clock of the plant. Note also that even when the loop is closed, information is being sent at discrete intervals, typically at a higher rate determined by the internal clock of the plant.

The rest of the paper is organized as follows: in Section 2, we introduce our approach for model-based control with intermittent feedback and study it in detail. We provide a full description of the output, as well as a necessary and sufficient condition for stability of the system. This section deals with the case where the intervals at which the loop is closed and the intervals for which the loop remains closed are both fixed. In Section 3 we extend our results to the case where full information is of the state is not available, and thus we most resort to output feedback, using a state observer. Once again, we provide a full description of the output, as well as necessary and sufficient conditions for stability. We study the case where the plant is nonlinear in Section 4. The case for time-varying updates is presented in Section 5. In Section 6, we present corresponding results for discrete-time plants. Finally, in Section 7, we provide conclusion and discuss future work.

2 Model-Based Control with Intermittent Feedback, Basic Architecture

Let us start by introducing model-based control with intermittent feedback, in its simplest setup.

2.1 *Problem Formulation*

The basic setup for MB-NCS with intermittent feedback is essentially the same as that proposed in the literature for traditional MB-NCS; see references (Montestruque and Antsaklis 2002, 2003, 2004) for more results on MB-NCS.

Consider the control of a continuous linear plant where the state sensor is connected to a linear controller/actuator via a network. In this case, the controller uses an explicit model of the plant that approximates the plant dynamics and makes possible the stabilization of the plant even under slow network conditions.

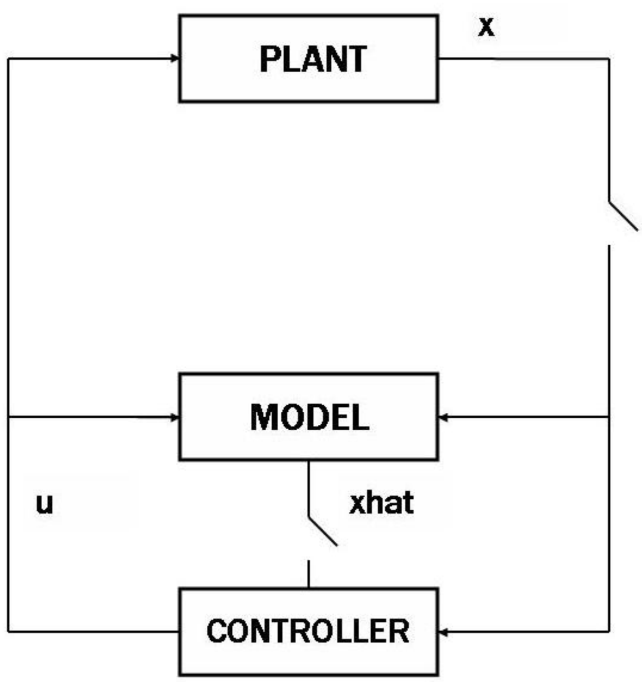

Figure 1. MB-NCS with intermittent feedback - basic architecture

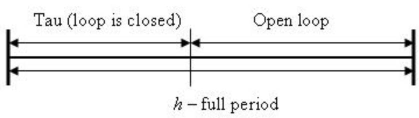

Figure 2. Partition of the time interval into close and open loop intervals

The main idea here is to perform update the model's state every h seconds using the actual state of the plant that is provided by the sensor. The rest of the time the control actions is based on a plant model that is incorporated in the controller/actuator and is running open loop.

As mentioned before, the main difference between model-based networked control systems as have been studied previously, and the case with intermittent feedback, which we are here discussing, is that in the literature, the loop is closed instantaneously, and the rest of the time the system is running open loop. Here, we part from the same basic idea, but the loop will remain closed for intervals of time which are different from zero. Intuitively, we should be able to achieve much better results the longer the loop is closed, as the level of degradation of the information increases the longer the system is running open loop, so intermittent feedback should yield better results than those for traditional MB-NCS.

In dealing with intermittent feedback, we have two key time parameters: how frequently we want to close the loop, which we shall denote by h, and how long we wish the loop to remain closed, which we shall denote by τ. Naturally, in the more general cases both h and τ can be time-varying. For the purposes of this section, however, we will deal only with the case where both h and τ are fixed.

We consider then a system such that the loop is closed periodically, every h seconds, and where each time the loop is closed, it remains so for a time of τ seconds. The loop is closed at times t_k, for $k = 1, 2, ...$ Thus, there are two very clear modes of operation: closed loop and open loop. The system will be operating in closed loop mode for the intervals $[t_k, t_k + \tau)$ and in open loop for the intervals $[t_k + \tau, t_{k+1})$. When the loop is closed, the control decision is based directly on the information of the state of the plant, but we will keep track of the error nonetheless.

The plant is given by $\dot{x} = Ax + Bu$, the plant model by $\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u$, and the controller by $u = K\hat{x}$. The state error is defined as $e = x - \hat{x}$ and represents the difference between plant state and the model state. The modelling error matrices $\tilde{A} = A - \hat{A}$ and $\tilde{B} = B - \hat{B}$ represent the plant and the model. We also define the vector $z = [x^T \; e^T]^T$.

In the next subsection we will derive a complete description of the response of the system.

2.2 *State Response of the System*

We will now proceed to derive the response to prove the above proposition in a direct way. To this effect, let us separately investigate what happens when the system is operating under closed and open loop conditions.

During the open loop case, that is, when $t \in [t_k + \tau, t_{k+1})$, we have that

$$u = K\hat{x} \tag{1}$$

so

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BK \\ 0 & \hat{A} + \hat{B}K \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix} \tag{2}$$

with initial conditions $\hat{x}(t_k + \tau) = x\,(t_k + \tau)$.

Rewriting in terms of x and e, that is, of the vector z :

$$\dot{z}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A + BK & -BK \\ \tilde{A} + \tilde{B}K & \hat{A} - \tilde{B}K \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{3}$$

$$z(t_k + \tau) = \begin{bmatrix} x(t_k + \tau) \\ e(t_k + \tau) \end{bmatrix} = \begin{bmatrix} x(t_k + \tau^-) \\ 0 \end{bmatrix}, \; \forall t \in [t_k + \tau, t_{k+1})$$

Thus, we have

$$\dot{z} = \Lambda_o z, \;\; \text{where} \;\; \Lambda_o = \begin{bmatrix} A + BK & -BK \\ \tilde{A} + \tilde{B}K & \hat{A} - \tilde{B}K \end{bmatrix}, \; \forall t \in [t_k + \tau, t_{k+1}) \tag{4}$$

.

The closed loop case is a simplified version of the case above, as the difference resides in the fact that the error is always zero. Thus, for $t \in [t_k, t_k + \tau)$, we have

$$\dot{z} = \Lambda_c z, \text{ where } \Lambda_c = \begin{bmatrix} A + BK & -BK \\ 0 & 0 \end{bmatrix}, \; t \in [t_k, t_k + \tau) \tag{5}$$

.

From this, it should be quite clear that given an initial condition $z(t = 0) = z_0$, the solution of the trajectory of the vector is given by

$$z(t) = e^{\Lambda_c(t)} z_0 \, , \; t \in [0, \tau). \tag{6}$$

In particular, at time τ, $z(\tau) = e^{\Lambda_c(\tau)} z_0$.

Once the loop is opened, the open loop behavior takes over, so that

$$z(t) = e^{\Lambda_o(t-\tau)} z(\tau) = e^{\Lambda_o(t-\tau)} e^{\Lambda_c(\tau)} z_0 \, , \; t \in [\tau, t_1). \tag{7}$$

In particular, when the time comes to close the loop again, that is, after time h, then $z\left(t_1^-\right) = e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} z_0$.

Notice, however, that at this instant when we close the loop again, we are also resetting the error to zero, so that we must pre-multiply by $\begin{bmatrix} I \; 0 \\ 0 \; 0 \end{bmatrix}$ before we analyse the closed loop trajectory for the next cycle. Because we wish to always start with an error that is set to zero, we should actually multiply by $\begin{bmatrix} I \; 0 \\ 0 \; 0 \end{bmatrix}$ at the beginning.

So then, after k cycles, going through this analysis yields a solution.

$$z\left(t_k\right) = \left(\begin{bmatrix} I \; 0 \\ 0 \; 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I \; 0 \\ 0 \; 0 \end{bmatrix} \right)^k z_0 = \Sigma^k z_0 \, , \tag{8}$$

where $\Sigma = \begin{bmatrix} I \; 0 \\ 0 \; 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I \; 0 \\ 0 \; 0 \end{bmatrix}$.

The final step is to consider the last (partial) cycle that the system goes through, that is, the time $t \in [t_k, t_{k+1})$. If the system is in closed loop, that is, $t \in [t_k, t_k + \tau)$, then the solution can be achieved merely by pre-multiplying $z\left(t_k\right)$ by $e^{\Lambda_c(t-t_k)}$. In the case of the system being in open loop, that is, $t \in [t_k + \tau, t_{k+1})$, then clearly we must pre-multiply by $e^{\Lambda_o(t-(t_k+\tau))} e^{\Lambda_c(\tau)}$.

The results can thus be summarized in the following proposition.

Proposition 2.1: *The system described above with initial conditions* $z\left(t_0\right) = \begin{bmatrix} x\left(t_0\right) \\ 0 \end{bmatrix} = z_0$ *has the following response:*

$$z\left(t\right) = \begin{cases} e^{\Lambda_c(t-t_k)} \left(\begin{bmatrix} I \; 0 \\ 0 \; 0 \end{bmatrix} \Sigma \begin{bmatrix} I \; 0 \\ 0 \; 0 \end{bmatrix} \right)^k z_0 \; , \quad t \in [t_k, t_k + \tau) \\ e^{\Lambda_o(t-(t_k+\tau))} e^{\Lambda_c(\tau)} \left(\begin{bmatrix} I \; 0 \\ 0 \; 0 \end{bmatrix} \Sigma \begin{bmatrix} I \; 0 \\ 0 \; 0 \end{bmatrix} \right)^k z_0 \; , \quad t \in [t_k + \tau, t_{k+1}) \end{cases} \tag{9}$$

where $\Sigma = e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)}$, $\Lambda_o = \begin{bmatrix} A + BK & -BK \\ \tilde{A} + \tilde{B}K & \hat{A} - \tilde{B}K \end{bmatrix}$, $\Lambda_c = \begin{bmatrix} A + BK & -BK \\ 0 & 0 \end{bmatrix}$, *and* $t_{k+1} - t_k = h$.

In the next subsection we will present a necessary and sufficient condition for the stability of the system.

2.3 *Stability condition*

We now present a necessary and sufficient condition for the stability of the model-based networked control system with intermittent feedback. We use the following definition for global exponential stability. (Antsaklis and Michel, 1997)

Definition 2.2: The equilibrium $z = 0$ of a system described by $\dot{z} = f(t, z)$ with initial condition $z(t_0) = z_0$ is exponentially stable at large (or globally) if there exists $\alpha > 0$ and for any $\beta > 0$, there exists $k(\beta) > 0$ such that the solution

$$\|\phi(t, t_0, z_0)\| \leq k(\beta) \|z_0\| e^{-\alpha(t-t_0)}, \ \forall t \geq t_0 \tag{10}$$

whenever $\|z_0\| < \beta$.

With this definition of stability, we state the following theorem characterizing the necessary and sufficient conditions for the system described in the previous section to have globally exponential stability around the solution $z = 0$. The norm used here is the 2-norm, but any other consistent norm can also be used.

Theorem 2.3: *The system described above is globally exponentially stable around the solution* $z = \begin{bmatrix} x \\ e \end{bmatrix}$ *if and only if the eigenvalues of* $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ *are strictly inside the unit circle, where* $\Sigma = e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)}$.

The proof may be found in the Appendix.

2.4 *Bridging the Gap: Relationship to previous results*

Let us now relate these results to the ones for instantaneous feedback. As explained in (Montestruque and Antsaklis 2002), the main idea is once again to perform the feedback by updating the model's state using the actual state of the plant that is provided by the sensor. The rest of the time the control action is based on a plant model that is incorporated in the controller/actuator and is running open loop for a period of h seconds.

The equations for plant, model, and controller are as follows:

Plant: $\dot{x} = Ax + Bu$

Model: $\dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u$

Controller: $u = k(\hat{x})$

Also, a state error is defined as $e = x - \hat{x}$ and represents the difference between the plant state and the model state. The dynamics of the overall system are captured in the following equation:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} A + BK & -BK \\ \tilde{A} + \tilde{B}K & \hat{A} - \tilde{B}K \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \tag{11}$$
$$\begin{bmatrix} x(t_k) \\ e(t_k) \end{bmatrix} = \begin{bmatrix} x\left(t_k^-\right) \\ 0 \end{bmatrix},$$
$$\forall t \in [t_k, t_{k+1}) \text{ with } t_{k+1} - t_k = h\,.$$

where $\tilde{A}$ and $\tilde{B}$ represent the error matrices $\tilde{A} = A - \hat{A}$ and $\tilde{B} = B - \hat{B}$. Defining $z = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}$, and $\Lambda = \begin{bmatrix} A + BK & -BK \\ \tilde{A} + \tilde{B}K & \hat{A} - \tilde{B}K \end{bmatrix}$, the equation above has the form $\dot{z} = \Lambda z$.

The complete output description of the above system is summarized in the following proposition.

Proposition 2.4: *The system described in with initial conditions* $z(t_0) = \begin{bmatrix} x(t_0) \\ 0 \end{bmatrix} = z_0$ *has the following response:*

$$z(t) = e^{\Lambda(t-t_k)} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda h} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 \,,$$
$$t \in [t_k, t_{k+1}) \text{ with } t_{k+1} - t_k = h \,.$$

The reference (Montestruque and Antsaklis 2002) also provides a necessary and sufficient condition for stability, which is presented here for completeness.

Theorem 2.5 : *The system described above is globally exponentially stable around the solution* $z = \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *if and only if the eigenvalues of* $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda h} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ *are inside the unit circle.*

Please see (Montestruque and Antsaklis 2002) for the corresponding proofs.

If we compare the above results to the ones obtained in the previous section, we observe that our results for intermittent feedback converge to those of the instantaneous case when the length of the intervals during which the loop is closed approaches zero. Furthermore, when τ, the time that the loop is closed, approaches h, our results converge to those for continuous time control. Numerical examples are presented in (Estrada and Antsaklis, 2006) which illustrate how the behavior of the stability margins ranges between those of the instantaneous case and those of the continuous time case depending on the percentage of time that the loop is closed.

3 Model-Based Control with Intermittent Feedback, Observer Case

In the previous section we considered plants where the full vector of the state was available at the output. When the state is not directly measurable, we must resort to a state observer. In this section we extend our results to this situation.

3.1 *Problem formulation*

As in the architecture used in (Montestruque and Antsaklis, 2002) for instantaneous model-based feedback, we assume that the state observer is collocated with the sensor. We use the plant model to design the state observer. Our configuration is based on the analogous setup for model-based control with output feedback, proposed by Montestruque.

In (Montestruque and Antsaklis, 2002) it is justified that the sensor carry the computational load of an observer by the fact that, typically, sensors that can be connected to a network have an embedded processor (usually in charge of performing the sampling, filtering, etc.) inside. The observer has as inputs the output and input of the plant. In the implementation, in order to acquire the input, which is at the other side of the communication link, the observer can have a version of the model and controller, and knowledge of the update times τ and h. The controller and the observer are also synchronized.

The observer has the form of a standard state observer with gain L. It makes use of the plant model.

In summary, the system equations are the following:

Plant: $\dot{x} = Ax + Bu,\ y = Cx + Du$
Model: $\widehat{\dot{x}} = \hat{A}\hat{x} + \hat{B}u,\ y = \hat{C}x + \hat{D}u$
Controller: $u = K\hat{x}$

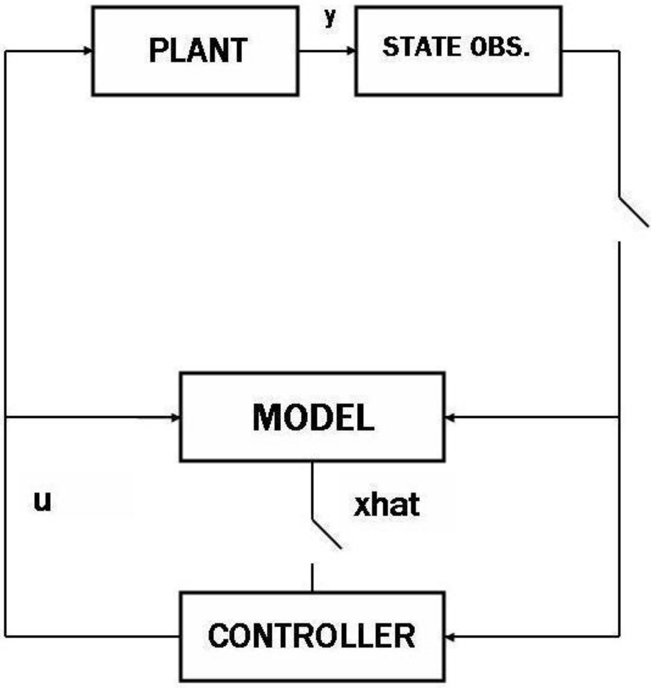

Figure 3. MB-NCS with intermittent feedback - state observer

Observer: $\dot{\bar{x}} = (\hat{A} - L\hat{C})\bar{x} + \begin{bmatrix} \hat{B} - L\hat{D} & L \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$
Controller model state: $\hat{x}$
Observer's estimate: $\bar{x}$
When loop is closed: $e = 0$

Error matrices: $\tilde{A} = A - \hat{A},\ \tilde{B} = B - \hat{B},\ \tilde{C} = C - \hat{C},\ \tilde{D} = D - \hat{D}$

We will derive the state response of the system in the following subsection.

3.2 *State response of the system*

To find the state response of the system, we proceed in the same fashion as we did before.
During open loop case, that is, when $t \in [t_k + \tau, t_{k+1})$, we have that

$$u = K\hat{x} \tag{12}$$

so

$$\begin{aligned} \dot{x} &= Ax + BK\hat{x} \\ \dot{\hat{x}} &= (\hat{A} + \hat{B}K)\hat{x} \end{aligned} \tag{13}$$

and

$$\begin{aligned} \dot{\bar{x}} &= (\hat{A} - L\hat{C})\bar{x} + \begin{bmatrix} \hat{B} - L\hat{D} & L \end{bmatrix} \begin{bmatrix} K\hat{x} \\ Cx + DK\hat{x} \end{bmatrix} \\ &= \begin{bmatrix} LC & \hat{B}K + L\tilde{D}K & \hat{A} - LC \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \\ \bar{x} \end{bmatrix} \end{aligned} \tag{14}$$

We define $z = \begin{bmatrix} x \\ \bar{x} \\ e \end{bmatrix}$ with initial condition $\hat{x}(t_k) = \bar{x}(t_k)$.

Thus,

$$\dot{z} = \Lambda_o z \tag{15}$$

where $\Lambda_o = \begin{bmatrix} A & BK & -BK \\ LC & \hat{A} - L\hat{C} + \hat{B}K + L\tilde{D}K & -\hat{B}K - L\tilde{D}K \\ LC & L\tilde{D}K - L\hat{C} & A - L\tilde{D}K \end{bmatrix}$

and

$$z(t_k + \tau) = \begin{bmatrix} x(t_k+\tau) \\ \bar{x}(t_k+\tau) \\ e(t_k+\tau) \end{bmatrix} = \begin{bmatrix} x(t_k+\tau)^- \\ \bar{x}(t_k+\tau)^- \\ 0 \end{bmatrix}$$

Similarly, for the closed loop case, that is, when $t \in [t_k, t_k + \tau)$, we have

$$\dot{z} = \Lambda_c z \tag{16}$$

where $\Lambda_c = \begin{bmatrix} A & BK & -BK \\ LC & \hat{A} - L\hat{C} + \hat{B}K + L\tilde{D}K & -\hat{B}K - L\tilde{D}K \\ 0 & 0 & 0 \end{bmatrix}$ because the error is always zero.

From this, it should be quite clear that given an initial condition $z(t=0) = z_0$, then after a certain time $t \in [0, \tau)$, the solution of the trajectory of the vector is

$$z(t) = e^{\Lambda_c(t)} z_0, \;\; t \in [0, \tau) \tag{17}$$

In particular,

$$z(\tau) = e^{\Lambda_c(\tau)} z_0 \tag{18}$$

Once the loop is opened

$$z(t) = e^{\Lambda_o(t-\tau)} z(\tau) = e^{\Lambda_o(t-\tau)} e^{\Lambda_c(\tau)} z_0 \;, \quad t \in [\tau, t_1) \tag{19}$$

We close the loop again at $t = h$.

$$z\left(t_1^-\right) = e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} z_0 \tag{20}$$

But we must reset the error to zero, so we pre- and post-multiply by $\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

After going through k cycles, we find that

$$z(t_k) = \Sigma^k z_0 \tag{21}$$

where $\Sigma = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Taking into account the last (partial) cycle,

$$z(t) = \begin{cases} e^{\Lambda_c(t-t_k)}\Sigma^k z_0 \ , \ t \in [t_k, t_k+\tau) \\ e^{\Lambda_o(t-(t_k+\tau))}e^{\Lambda_c(\tau)}\Sigma^k z_0 \ , \ t \in [t_k+\tau, t_{k+1}) \end{cases} \tag{22}$$

where $\Sigma = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)}e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and Λ_o, Λ_c as before.

We summarize the result in this proposition.

Proposition 3.1: *The system described above has a state response:*

$$z(t) = \begin{cases} e^{\Lambda_c(t-t_k)}\Sigma^k z_0 \ , \ t \in [t_k, t_k+\tau) \\ e^{\Lambda_o(t-(t_k+\tau))}e^{\Lambda_c(\tau)}\Sigma^k z_0 \ , \ t \in [t_k+\tau, t_{k+1}) \end{cases} \tag{23}$$

where $\Sigma = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)}e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *and* $\Lambda_o = \begin{bmatrix} A & BK & -BK \\ LC & \hat{A} - L\hat{C} + \hat{B}K + L\tilde{D}K & -\hat{B}K - L\tilde{D}K \\ LC & L\tilde{D}K - L\hat{C} & A - L\tilde{D}K \end{bmatrix}$,

$$\Lambda_c = \begin{bmatrix} A & BK & -BK \\ LC & \hat{A} - L\hat{C} + \hat{B}K + L\tilde{D}K & -\hat{B}K - L\tilde{D}K \\ 0 & 0 & 0 \end{bmatrix}.$$

In the next subsection, we obtain a necessary and sufficient condition for stability.

3.3 *Stability condition*

As before, we provide a necessary and sufficient condition for stability.

Theorem 3.2: *The system described above is globally exponentially stable around the solution* $z = \begin{bmatrix} x \\ \bar{x} \\ e \end{bmatrix} = \mathbf{0}$ *if and only if the eigenvalues of* Σ *are strictly inside the unit circle, where where* $\Sigma = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)}e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$, *and* Λ_o, Λ_c *as before.*

4 Nonlinear Systems

In the previous sections we have restricted our study to the cases where the plant is linear. Let us now lift this restriction and seek to find the corresponding stability properties for nonlinear plants with intermittent feedback.

The setup and procedure that follows closely mirrors that proposed by (Montestruque and Antsaklis, 2003) for traditional MB-NCS. The sufficient conditions obtained relate the stability of the nonlinear MB-NCS with the value of a function that depends on the Lipschitz constants of the plant and model as well as the stability properties of the compensated non-networked model. The results are obtained by studying the worst-case behavior of the norm of the plant state and the error, thus leading to conservative results.

4.1 *Stability of a class of nonlinear MB-NCS*

Let the plant be given by:

$$\dot{x} = f(x) + g(u) \tag{24}$$

We use a model on the actuator side of the plant to estimate the actual state of the plant. The controller will be assumed to be a nonlinear state feedback controller. The control signal u is generated by taking into account the plant model state . The plant state sensor will send through the network the real value of the plant state to the model (that is, the loop will be closed) every h seconds, and the loop will remain closed for τ seconds during each cycle. During these times, the state of the model is set to be the same as that of the plant. We will assume the plant model dynamics are given by:

$$\dot{\hat{x}} = \hat{f}(x) + \hat{g}(u) \tag{25}$$

And the controller has the following form:

$$u = h(\hat{x}) \tag{26}$$

We define as the error between the plant state and the plant model state, $e = x - \hat{x}$. Combining the above, we obtain:

$$\begin{aligned} \dot{x} &= f(x) + g(h(\hat{x})) = f(x) + m(\hat{x}) \\ \dot{\hat{x}} &= \hat{f}(x) + \hat{g}(h(\hat{x})) = f(x) + \hat{m}(\hat{x}) \end{aligned} \tag{27}$$

Assume also that the plant model dynamics differ from the actual plant dynamics in an additive fashion:

$$\begin{aligned} \hat{f}(\zeta) &= f(\zeta) + \delta_f(\zeta) \\ \hat{m}(\zeta) &= m(\zeta) + \delta_m(\zeta) \end{aligned} \tag{28}$$

Thus:

$$\begin{aligned} \dot{x} &= f(x) + m(\hat{x}) \\ \dot{\hat{x}} &= f(x) + \hat{m}(\hat{x}) + \delta_f(\hat{x}) + \delta_m(\hat{x}) \end{aligned} \tag{29}$$

Assume that f and δ satisfy the following local Lipschitz conditions for with $x, y \in B_L$, a ball centered on the origin:

$$\begin{aligned} \|f(x) - f(y)\| &\le K_f \|x - y\| \\ \|\delta(x) - \delta(y)\| &\le K_\delta \|x - y\| \end{aligned} \tag{30}$$

It is to be noted that if the plant model is accurate the Lipschitz constant K_δ will be small.

Assume that the non-networked compensated plant model is exponentially stable when $\hat{x}(t_0) \in B_S$, $\hat{x}(t) \in B_\tau$, for $t \in [t_0, t_0 + \tau)$ with B_S and B_τ balls centered on the origin.

$$\|\hat{x}(t)\| \le \alpha \|\hat{x}(t_0)\| e^{-\beta(t-t_0)} \text{ with } \alpha, \beta > 0\,. \tag{31}$$

Theorem 4.1 : *The nonlinear MB-NCS with dynamics described above, and that satisfies the Lipschitz conditions described and with exponentially stable compensated plant model is asymptotically stable if:*

$$\left(1 - \alpha \left(e^{-\beta(h-\tau)} + \left(e^{K_f(h-\tau)} - e^{-\beta(h-\tau)}\right)\left(\frac{K_\delta}{K_f + \delta}\right)\right)\right) > 0 \tag{32}$$

4.2 *Stability for a more general class of non-linear MB-NCS*

We now extend the results to a nonlinear system whose plant dynamics are given by

$$\dot{x} = f(x) + g(x, u)\,. \tag{33}$$

As above, we will follow the procedure used in (Montestruque and Antsaklis, 2003).
The model and controller are given by

$$\begin{aligned} \dot{\hat{x}} &= \hat{f}(\hat{x}) + \hat{g}(\hat{x}, u) \\ u &= k(\hat{x}) \end{aligned} \tag{34}$$

Substituting, we get:

$$\begin{aligned} \dot{x} &= f(x) + g(x, k(\hat{x})) = f(x) + m(x, \hat{x}) \\ \dot{\hat{x}} &= \hat{f}(\hat{x}) + \hat{g}(\hat{x}, k(\hat{x})) = \hat{f}(\hat{x}) + \hat{m}(\hat{x}, \hat{x}) \end{aligned} \tag{35}$$

Again, let us assume that the uncertainty between the plant and the model is of the additive type:

$$\begin{aligned} \hat{f}(\zeta) &= f(\zeta) + \delta_f(\zeta) \\ \hat{m}(\zeta) &= m(\zeta, \zeta) + \delta_m(\zeta) \end{aligned} \tag{36}$$

So, the error dynamics between the plant and the model are:

$$e = f(x) - f(\hat{x}) - \delta_f(\hat{x}) + m(x, \hat{x}) - m(\hat{x}, \hat{x}) - \delta_m(\hat{x}) \tag{37}$$

Assume also that the Lipschitz conditions hold:

$$\|f(x) - f(y)\| \le K_f \|x - y\| \tag{38}$$

$$\|m(x, s) - m(y, s)\| \le K_m(s) \|x - y\| \tag{39}$$

$$\|\delta_f(x) - \delta_f(y)\| \le K_{\delta_f} \|x - y\| \tag{40}$$

$$\|\delta_m(x) - \delta_m(y)\| \le K_{\delta_m} \|x - y\| \tag{41}$$

Define also $K_{m,\max} = \max_{s \in B_S} (K_m(s))$ for B_S, where B_S is a ball centered at the origin. Assume as well that the non-networked compensated plant model is exponentially stable when $\hat{x}(t_0) \in B_S$, $\hat{x}(t) \in B_\tau$, for $t \in [t_0, t_0 + \tau)$ with B_S and B_τ balls centered on the origin.

$$\|\hat{x}(t)\| \leq \alpha \|\hat{x}(t_0)\| e^{-\beta(t-t_0)} \text{ with } \alpha, \beta > 0\,. \tag{42}$$

The following theorem states a sufficient condition for stability.

Theorem 4.2: *The nonlinear system with dynamics described above and that satisfies the Lipschitz conditions described and with exponentially stable compensated plant model satisfying the above is asymptotically stable if:*

$$\left(1 - \alpha\left(e^{-\beta(h-\tau)} + \left(e^{(K_f+K_{m,\max})(h-\tau)} - e^{-\beta(h-\tau)}\right)\left(\frac{K_{\delta_f} + K_{\delta_m}}{K_f + K_{m,\max} + \beta}\right)\right)\right) > 0 \tag{43}$$

5 Stability of MB-NCS with Intermittent Feedback and time-varying updates

Until now we have only considered the case where the parameters τ and h are constant. Let us now take a closer look at what happens when these parameters vary with time. The definitions for Lyapunov stability and mean square stability used throughout this section are the same as those in (Montestruque and Antsaklis, 2004).

5.1 *Lyapunov stability with bounded intervals*

We shall first analyse the case where the parameters are time-varying, but their probability distributions are unknown. Let the plant, model, and controller have the same dynamics as described in Section 2. The following result describes the state response of the system. The derivation of this result is analogous to that for constant τ and h.

Proposition 5.1: *The system described above with initial conditions* $z = \begin{bmatrix} x(t_0) \\ 0 \end{bmatrix} = z_0$ *has the following response:*

$$z(t) = \begin{cases} e^{\Lambda_o(t-t_k)} \left(\prod_{j=1}^{k} M(j)\right) z_0 \ , \quad t \in [t_k, t_k + \tau) \\ e^{\Lambda_o(t-(t_k+\tau))} e^{\Lambda_c(\tau)} \left(\prod_{j=1}^{k} M(j)\right) z_0, \\ \qquad t \in [t_k + \tau, t_{k+1}) \end{cases}$$

where $M(j) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h(j)-\tau(j))} e^{\Lambda_c(\tau(j))} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\Lambda_o = \begin{bmatrix} A + BK & -BK \\ \tilde{A} + \tilde{B}K & \hat{A} - \tilde{B}K \end{bmatrix}$, $\Lambda_c = \begin{bmatrix} A + BK & -BK \\ 0 & 0 \end{bmatrix}$, $t_{k+1} - t_k = h(k)$, *and* $\tau(j) < h(j)$.

The proof is presented in the appendix.

We now present a condition for Lyapunov stability of this system.

Theorem 5.2: *The system described above is Lyapunov asymptotically stable for* $h \in [h_{\min}, h_{\max}]$ *and* $\tau \in [\tau_{\min}, \tau_{\max}]$ *(with* $\tau_{\max} < h_{\min}$*) if there exists a symmetric positive definite matrix* X *such that* $Q = X - MXM^T$ *is positive definite for all* $h \in [h_{\min}, h_{\max}]$ *and* $\tau \in [\tau_{\min}, \tau_{\max}]$*, where* $M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

Again, the proof is presented in the appendix.

5.2 *Mean square stability of continuous MB-NCS with IF with i.i.d update times*

Now, let us consider the case where τ is constant, but $h(k)$ are independent identically distributed with probability distribution $F(h)$. This corresponds to the situation where we might not know how frequently we can access the network, but when we do obtain access to it, we continue to have access to it for a fixed amount of time, so as to, for example, complete a given task or transmit a certain set of packets. We present a stability condition for this case:

Theorem 5.3: *The system described above with update times $h(j)$ independent identically distributed random variable with probability distribution $F(h)$ is globally mean square asymptotically stable around the solution $z = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if $K = E\left[\left(e^{\bar{\sigma}(\Lambda_o)(h-\tau)}\right)^2\right] < \infty$ and the maximum singular value of the expected value $M^T M$, $\left\|E\left[M^T M\right]\right\| = \bar{\sigma}\left(E\left[M^T M\right]\right)$ is strictly less than one, where $M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.*

The proof may be found in the Appendix.

5.3 *Mean square stability of continuous MB-NCS with IF with Markov chain-driven update times*

We now consider the situation where the parameter h is driven by a Markov chain and provide a stability condition.

Theorem 5.4: *The system described above with update times $h(k) = h_{\omega_k} \neq \infty$ driven by a finite state Markov chain $\{\omega_k\}$ with state space $\{1, 2, ..., N\}$ and transition probability matrix Γ with elements $p_{i,j}$ is globally mean square asymptotically stable around the solution $z = \left[x^T \; e^T\right]^T = \mathbf{0}$ if there exist positive definite matrices $P(1), P(2), ..., P(N)$ such that*

$$\left(\sum_{j=1}^{N} p_{i,j}\left(H(i)^T P(j) H(i)\right) - P(i)\right) < 0 \;\forall i, j \in 1, ..., N$$

with $H(i) = e^{\Lambda_o(h_i-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

The proof follows that in (Montestruque and Antsaklis, 2004) for the case of instantaneous feedback.

6 Discrete-time plants

6.1 *Problem Formulation*

The basic setup for discrete-time MB-NCS with intermittent feedback is essentially the same as that for continuous time; see also (Estrada and Antsaklis, 2008-2). We make the same assumptions as in (Montestruque and Antsaklis, 2004) for the instantaneous feedback case, where both the sensor and actuator sides are synchronized and updates occur at the same instants of time.

Consider the control of a discrete linear plant where the state sensor is connected to a linear controller/actuator via a network. In this case, the controller uses an explicit model of the plant that approximates the plant dynamics and makes possible the stabilization of the plant even under slow network conditions.

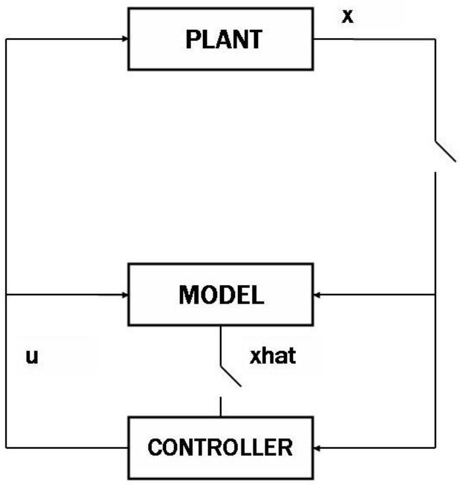

Figure 4. Basic MB-NCS architecture

In dealing with intermittent feedback, we have two key time parameters: how frequently we want to close the loop, which we shall denote by h, and how long we wish the loop to remain closed, which we shall denote by τ. Naturally, in the more general cases both h and τ can be time-varying. Unlike the continuous time formulation, h and τ are both integers here, as they represent the number of ticks of the clock in the corresponding interval.

We consider then a system such that the loop is closed periodically, every h ticks of the clock, and where each time the loop is closed, it remains so for a time of $\tau < n$ ticks of the clock. The loop is closed at times n_k, for $k = 1, 2, ...$ The system will be operating in closed loop mode for the intervals $[n_k, n_k + \tau)$ and in open loop for the intervals $[n_k + \tau, n_{k+1})$, with $n_{k+1} - n_k = h$. When the loop is closed, the control decision is based directly on the information of the state of the plant, but we will keep track of the error nonetheless.

As mentioned in the introduction, it is important to note that the parameters τ and h are different from the sampling time of the digital plant, since they are tailored after the demands of use of the network, not by the internal clock of the plant. It is also important to keep in mind that even when the loop is "closed", information is being sent at discrete intervals, the duration of which is determined by the internal clock of the plant.

The plant is given by $x(n+1) = Ax(n) + Bu(n)$, the plant model by $\hat{x}(n+1) = \hat{A}\hat{x}(n) + \hat{B}u(n)$, and the controller by $u(n) = K\hat{x}(n)$. The state error is defined as $e(n) = x(n) - \hat{x}(n)$ and represents the difference between plant state and the model state. The modelling error matrices $\tilde{A} = A - \hat{A}$ and $\tilde{B} = B - \hat{B}$ represent the plant and the model. We also define the vector $z = [x^T \; e^T]^T$.

In the next section we will derive a complete description of the response of the system as well as a necessary and sufficient condition for stability.

6.2 *State Response of the System and Stability Condition*

We will now proceed to derive the response to prove the above proposition. The approach is similar to that we used in (Estrada and Antsaklis, 2008-1) for the continuous time case. To this effect, let us separately investigate what happens when the system is operating under closed and open loop conditions.

6.2.1 State response of the system

During the open loop case, that is, when $n \in [n_k + \tau, n_{k+1})$, we have that

$$u(n) = K\hat{x}(n) \tag{44}$$

so

$$\begin{bmatrix} x(n+1) \\ \hat{x}(n+1) \end{bmatrix} = \begin{bmatrix} A & BK \\ 0 & \hat{A}+\hat{B}K \end{bmatrix} \begin{bmatrix} x(n) \\ \hat{x}(n) \end{bmatrix} \tag{45}$$

with initial conditions $\hat{x}(n_k+\tau) = x(n_k+\tau)$.

Rewriting in terms of x and e, that is, of the vector z :

$$z(n+1) = \begin{bmatrix} x(n+1) \\ e(n+1) \end{bmatrix} = \tag{46}$$

$$\begin{bmatrix} A+BK & -BK \\ \tilde{A}+\tilde{B}K & \hat{A}-\tilde{B}K \end{bmatrix} \begin{bmatrix} x(n) \\ e(n) \end{bmatrix}$$

$$z(n_k+\tau) = \begin{bmatrix} x(n_k+\tau) \\ e(n_k+\tau) \end{bmatrix} = \begin{bmatrix} x(n_k+\tau^-) \\ 0 \end{bmatrix},$$

$$\forall n \in [n_k+\tau, n_{k+1}) \tag{47}$$

Thus, we have

$$z(n+1) = \Lambda_{Do} z(n), \text{ where } \Lambda_{Do} = \begin{bmatrix} A+BK & -BK \\ \tilde{A}+\tilde{B}K & \hat{A}-\tilde{B}K \end{bmatrix}, \tag{48}$$

$$\forall n \in [n_k+\tau, n_{k+1})$$

.

The closed loop case is a simplified version of the case above, as the difference resides in the fact that the error is always zero. Thus, for $n \in [n_k, n_k+\tau)$, we have

$$z(n+1) = \Lambda_{Dc} z(n), \text{ where } \Lambda_{Dc} = \begin{bmatrix} A+BK & -BK \\ 0 & 0 \end{bmatrix}, \tag{49}$$

$$n \in [n_k, n_k+\tau)$$

. This should be clear in that the error is always zero, while the state progresses in the same way as before.

From this, it should be quite clear that given an initial condition $z(n=0) = z_0$, then after a certain time $n \in [0,\tau)$, the solution of the trajectory of the vector is given by

$$z(n) = \Lambda_{Dc}^{n} z_0\,,\ n \in [0,\tau). \tag{50}$$

In particular, at time τ, $z(\tau) = \Lambda_{Dc}^{\tau} z_0$.

Once the loop is opened, the open loop behavior takes over, so that

$$z(n) = \Lambda_{Do}^{(n-\tau)} z(\tau) = \Lambda_{Do}^{(n-\tau)} \Lambda_{Dc}^{\tau} z_0\,,\ n \in [\tau, n_1). \tag{51}$$

In particular, when the time comes to close the loop again, that is, after time h, then $z(n_1) = \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} z_0$.

Notice, however, that at this instant when we close the loop again, we are also resetting the error to zero, so that we must pre-multiply by $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ before we analyse the closed loop trajectory for the next cycle. Because we wish to always start with an error that is set to zero, we should actually multiply by $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ at the beginning.

So then, after k cycles, going through this analysis yields a solution.

$$z(t_k) = \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0$$
$$= \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 , \tag{52}$$

where $\Sigma = \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau}$.

The final step is to consider the last (partial) cycle that the system goes through, that is, the time $n \in [n_k, n_{k+1})$. If the system is in closed loop, that is, $n \in [n_k, n_k + \tau)$, then the solution can be achieved merely by pre-multiplying $z(n_k)$ by $\Lambda_{Dc}^{(n-n_k)}$. In the case of the system being in open loop, that is, $n \in [n_k + \tau, n_{k+1})$, then clearly we must pre-multiply by $\Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{\tau}$.

The results can thus be summarized in the following proposition.

Proposition 6.1: *The system described by (48) and (49) with initial conditions* $z(n_0) = \begin{bmatrix} x(n_0) \\ 0 \end{bmatrix} = z_0$ *has the following response:*

$$z(n) = \begin{cases} \Lambda_{Dc}^{(n-n_k)} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 , \\ \quad n \in [n_k, n_k + \tau) \\ \Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{\tau} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 , \\ \quad n \in [n_k + \tau, n_{k+1}) \end{cases} \tag{53}$$

where $\Sigma = \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau}$, $\Lambda_{Do} = \begin{bmatrix} A + BK & -BK \\ \tilde{A} + \tilde{B}K & \hat{A} - \tilde{B}K \end{bmatrix}$, $\Lambda_{Dc} = \begin{bmatrix} A + BK & -BK \\ 0 & 0 \end{bmatrix}$, *and* $n_{k+1} - n_k = h$.

6.2.2 Stability Condition

We will present a necessary and sufficient condition for the stability of the system.

Theorem 6.2: *The system described above is globally exponentially stable around the solution* $z = \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *if and only if the eigenvalues of* $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ *are strictly inside the unit circle, where* $\Sigma = \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau}$.

Extensions of these results to the case with use of state observer are also available. See (Estrada and Antsaklis, 2008-2).

6.3 *Time-Varying Results*

Until now we have only considered the case where the parameters τ and h are constant. Let us now take a closer look at what happens when these parameters vary with time. The definitions for Lyapunov stability and mean square stability used throughout this section are the same as those in (Montestruque and Antsaklis, 2004).

6.3.1 Lyapunov stability with bounded intervals

We shall first analyse the case where the parameters are time-varying, but their probability distributions are unknown. The following result describes the state response of the system. The derivation of this result is analogous to that for constant τ and h and is included for the sake of completeness.

Proposition 6.3: *The system described in (48) and (49) with initial conditions* $z = \begin{bmatrix} x(n_0) \\ 0 \end{bmatrix} = z_0$ *has the following response:*

$$z(n) = \begin{cases} \Lambda_{Dc}^{(n-n_k)} \left(\prod_{j=1}^{k} M(j) \right) z_0 \ , \ n \in [n_k, n_k + \tau_k) \\ \Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{\tau_k} \left(\prod_{j=1}^{k} M(j) \right) z_0 \ , \ n \in [n_k + \tau_k, n_{k+1}) \end{cases}$$

where $M(j) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)(j)} \Lambda_{Dc}^{\tau(j)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\Lambda_{Do} = \begin{bmatrix} A + BK & -BK \\ \tilde{A} + \tilde{B}K & \hat{A} - \tilde{B}K \end{bmatrix}$, $\Lambda_{Dc} = \begin{bmatrix} A + BK & -BK \\ 0 & 0 \end{bmatrix}$, $n_{k+1} - n_k = h(k)$, *and* $\tau(j) < h(j)$.

We now present a condition for Lyapunov stability of this system.

Theorem 6.4: *The system described in (48) and (49) is Lyapunov asymptotically stable for* $h \in [h_{\min}, h_{\max}]$ *and* $\tau \in [\tau_{\min}, \tau_{\max}]$ *(with* $\tau_{\max} < h_{\min}$*) if there exists a symmetric positive definite matrix* X *such that* $Q = X - MXM^T$ *is positive definite for all* $h \in [h_{\min}, h_{\max}]$ *and* $\tau \in [\tau_{\min}, \tau_{\max}]$*, where* $M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

6.3.2 Mean square stability of discrete MB-NCS with IF with i.i.d update times

Now, let us consider the case where τ is constant, but $h(k)$ are independent identically distributed with probability distribution $F(h)$. This corresponds to the situation where we might not know how frequently we can access the network, but when we do obtain access to it, we continue to have access to it for a fixed amount of time, so as to, for example, complete a given task or transmit a certain set of packets. We present a stability condition for this case:

Theorem 6.5: *The system described in (48) and (49) with update times* $h(j)$ *independent identically distributed random variable with probability distribution* $F(h)$ *is globally mean square asymptotically stable around the solution* $z = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *if* $K = E\left[\left(\Lambda_{Do}^{(h-\tau)} \right)^2 \right] < \infty$ *and the maximum singular value of the expected value* $M^T M$, $\left\| E\left[M^T M\right] \right\| = \bar{\sigma}\left(E\left[M^T M\right] \right)$ *is strictly less than one, where* $M = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

The proof is similar to that for the continuous case.

6.3.3 Mean square stability of discrete MB-NCS with IF with Markov chain-driven update times

We now consider the situation where the parameter h is driven by a Markov chain and provide a stability condition.

Theorem 6.6: *The system described in (48) and (49) with update times* $h(k) = h_{\omega_k} \neq \infty$ *driven by a finite state Markov chain* $\{\omega_k\}$ *with state space* $\{1, 2, ..., N\}$ *and transition probability matrix* Γ *with elements* $p_{i,j}$ *is globally mean square asymptotically stable around the solution* $z = \left[x^T \ e^T\right]^T = \mathbf{0}$ *if there exist positive definite matrices* $P(1), P(2), \ldots, P(N)$ *such that* $\left(\sum_{j=1}^{N} p_{i,j} \left(H(i)^T P(j) H(i) \right) - P(i) \right) < 0 \ \forall i, j \in 1, ..., N$ *with* $H(i) = \Lambda_{Do}^{(h_i-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$

Once again, the proof follows that of the continuous case.

7 Conclusions

We have introduced the concept of model-based control with intermittent feedback. We proposed a basic architecture, focusing first on the continuous time case, and derived a complete description of the output of the system, as well as necessary and sufficient conditions for stability. We have then extended our results to cases with state observers and nonlinear plants. Extensions of our results to cases with delays can also be found in our previous work. Additionally, we considered the situation where the update times τ and h are time-varying, first addressing the case where they have upper and lower bounds, then moving on to the case where their distributions are i.i.d or driven by a Markov chain, providing stability conditions in each case. We also obtained an analogous set of results for the discrete-time case.

The focus of the present paper was on stability, but the area of performance of networked control systems, both under the model-based architecture and otherwise, remains a relatively unexplored ground for research. In future work, we expect to provide results on performance of model-based networked control systems with intermittent feedback, and will consider other issues, such as robustness, tracking, filtering, and improving control as time elapses (that is, to use intermittent feedback to improve performance, by updating the model during the times when the system is running closed loop, with the aim of enabling the user to run the system closed loop for progressively shorter intervals), as well.

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Appendix A: Proofs

A.1 *Proof of Theorem 2.3*

Proof Sufficiency. Taking the norm of the solution described as in Proposition 2.1:

$$\|z(t)\| = \left\| e^{\Lambda_c(t-t_k)} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 \right\| \leq \tag{A1}$$
$$\left\| \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k \right\| \|z_0\|$$

Notice we are only doing this part for the case when $t \in [t_k, t_k + \tau)$, but the process is exactly the same for the intervals where $t \in (t_k + \tau, t_k + 1)$. Analysing the first term on the right hand side:

$$\left\| e^{\Lambda_c(t-t_k)} \right\| \leq 1 + (t - t_k)\,\bar{\sigma}(\Lambda_c) + \frac{(t-t_k)^2}{2!} \cdots = e^{\bar{\sigma}(\Lambda_c)(t-t_k)} \leq e^{\bar{\sigma}(\Lambda_c)(\tau)} = K_1 \tag{A2}$$

where $\bar{\sigma}(\Lambda_c)$ is the largest singular value of Λ_c. In general this term can always be bounded as the time difference $t - t_k$ is always smaller than τ. That is, even when Λ_c has eigenvalues with positive real part, $\left\| e^{\Lambda_c(t-t_k)} \right\|$ can only grow a certain amount. This growth is completely independent of k.

We now study the term $\left\| \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k \right\|$. It is clear that this term will be bounded if and only if the eigenvalues of $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ lie inside the unit circle:

$$\left\| \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k \right\| \le K_2 e^{-\alpha_1 k} \tag{A3}$$

with $K_2, \alpha_1 > 0$.

Since k is a function of time we can bounded the right term of the previous inequality in terms of t :

$$K_2 e^{-\alpha_1 k} < K_2 e^{-\alpha_1 \frac{t-1}{h}} = K_2 e^{\frac{\alpha_1}{h}} e^{-\frac{\alpha_1}{h} t} = K_3 e^{-\alpha t} \tag{A4}$$

with $K_{3,}\alpha > 0$.

So from (A1), using (A2) and (A4) we conclude that:

$$\|z(t)\| = \left\| e^{\Lambda_c(t-t_k)} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 \right\| \le K_1 K_3 e^{-\alpha t} \|z_0\| \, . \tag{A5}$$

Necessity. We will now provide the necessity part of the theorem. We will do this by contradiction. Assume the system is stable and that $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ has at least one eigenvalue outside the unit circle. Let us define $\Sigma(h) = e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)}$. Since the system is stable, a periodic sample of the response should converge to zero with time. We will take the samples at times t_{k+1}^-, that is, just before the loop is closed again. We will concentrate on a specific term: the state of the plant $x\left(t_{k+1}^-\right)$, which is the first element of $z\left(t_{k+1}^-\right)$. We will call $x\left(t_{k+1}^-\right)$, $\xi(k)$.

Now assume $\Sigma(\eta)$ has the following form:

$$\Sigma(\eta) = \begin{bmatrix} W(\eta) & X(\eta) \\ Y(\eta) & Z(\eta) \end{bmatrix} .$$

Then we can express the solution $z(t)$ as:

$$\begin{aligned} & e^{\Lambda_c(t-t_k)} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma(h) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 \\ &= \begin{bmatrix} W(t-t_k) & X(t-t_k) \\ Y(t-t_k) & Z(t-t_k) \end{bmatrix} \begin{bmatrix} (W(h))^k & 0 \\ 0 & 0 \end{bmatrix} z_0 \\ &= \begin{bmatrix} W(t-t_k)(W(h))^k & 0 \\ Y(t-t_k)(W(h))^k & 0 \end{bmatrix} z_0 . \end{aligned} \tag{A6}$$

Now, the values of the solution at times t_{k+1}^-, that is, just before the loop is closed again, are

$$z\left(t_{k+1}^-\right) = \begin{bmatrix} W(h)(W(h))^k & 0 \\ Y(h)(W(h))^k & 0 \end{bmatrix} z_0 = \begin{bmatrix} (W(h))^{k+1} & 0 \\ Y(h)(W(h))^k & 0 \end{bmatrix} z_0 \tag{A7}$$

We also know that $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ has at least eigenvalue outside the unit circle, which means that those unstable eigenvalues must be in $W(\tau)$. This means that the first element of $z\left(t_{k+1}^{-}\right)$, which we call $\xi(k)$, will in general grow with k. In other words we cannot ensure $\xi(k)$ will converge to zero for general initial condition x_0.

$$\left\|x\left(t_{k+1}^{-}\right)\right\| = \|\xi(k)\| = \left\|\left(W(h)\right)^{k+1} x_0\right\| \to \infty \qquad \text{as} \;\; k \to \infty\,, \tag{A8}$$

which clearly means the system cannot be stable. Thus, we have a contradiction. □

A.2 *Proof of Theorem 3.2*

Proof Sufficiency. We will perform the proof for $[t_k, t_k + \tau)$, but this holds true for the other interval as well.

$$\begin{aligned} \|z(t)\| &= \left\|e^{\Lambda_c(t-t_k)}\Sigma^k z_0\right\| \\ &\le \left\|e^{\Lambda_c(t-t_k)}\right\| \left\|\Sigma^k\right\| \|z_0\| \end{aligned} \tag{A9}$$

$$\begin{aligned} \left\|e^{\Lambda_c(t-t_k)}\right\| &\le 1 + (t-t_k)\,\bar{\sigma}(\Lambda_c) + \frac{(t-t_k)^2}{2!} + \cdots \\ &= e^{\bar{\sigma}(\Lambda_c)(t-t_k)} \le e^{\bar{\sigma}(\Lambda_c)(\tau)} = K_1 \end{aligned} \tag{A10}$$

And $\left\|\left(\begin{bmatrix} I\,0\,0 \\ 0\,I\,0 \\ 0\,0\,0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I\,0\,0 \\ 0\,I\,0 \\ 0\,0\,0 \end{bmatrix}\right)^k\right\|$ is clearly bounded if and only if the eigenvalues of Σ are within the unit circle.

$$\left\|\begin{bmatrix} I\,0\,0 \\ 0\,I\,0 \\ 0\,0\,0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I\,0\,0 \\ 0\,I\,0 \\ 0\,0\,0 \end{bmatrix}\right\|^k \le K_2 e^{-\alpha_1 k}\,, \quad K_2, \alpha_1 > 0 \tag{A11}$$

Since k is a function of time, we can bound the right term in terms of t.

$$K_2 e^{-\alpha_1 k} \le K_2 e^{-\alpha_1 \frac{t-1}{h}} \le K_2 e^{\alpha_1/h} e^{-\alpha_1 t/h} = K_3 e^{-\alpha t}\,, \quad K_3, \alpha_1 > 0 \tag{A12}$$

Thus,

$$\|z(t)\| = \left\|e^{\Lambda_c(t-t_k)}\Sigma^k z_0\right\| \le K_1 K_3 e^{-\alpha t}\|z_0\| \tag{A13}$$

Necessity. Assume that $\begin{bmatrix} I\,0\,0 \\ 0\,I\,0 \\ 0\,0\,0 \end{bmatrix} e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)} \begin{bmatrix} I\,0\,0 \\ 0\,I\,0 \\ 0\,0\,0 \end{bmatrix}$ has at least one eigenvalue outside the unit circle. We will take samples... as we did in the case without the observer. Let's call $\Sigma(h) = e^{\Lambda_o(h-\tau)} e^{\Lambda_c(\tau)}$. We will concentrate on $\xi(k) = \begin{bmatrix} x\left(t_{k+1}^{-}\right) \\ \bar{x}\left(t_{k+1}^{-}\right) \end{bmatrix}$.

Assume $\Sigma(\eta) = \begin{bmatrix} W_1(\eta) & W_2(\eta) & X_1(\eta) \\ W_3(\eta) & W_4(\eta) & X_2(\eta) \\ Y_1(\eta) & Y_2(\eta) & Z(\eta) \end{bmatrix}$

For simplicity, let's call

$$W(\eta) = \begin{bmatrix} W_1(\eta) \; W_2(\eta) \\ W_3(\eta) \; W_4(\eta) \end{bmatrix}, \; X(\eta) = \begin{bmatrix} X_1(\eta) \\ X_2(\eta) \end{bmatrix}, \; Y(\eta) = \begin{bmatrix} Y_1(\eta) \; Y_2(\eta) \end{bmatrix} \tag{A14}$$

Then we can express $z(t)$ as

$$\begin{aligned} & e^{\Lambda_c(t-t_k)} \left(\begin{bmatrix} I\,0\,0 \\ 0\,I\,0 \\ 0\,0\,0 \end{bmatrix} \Sigma(h) \begin{bmatrix} I\,0\,0 \\ 0\,I\,0 \\ 0\,0\,0 \end{bmatrix} \right)^k z_0 \\ &= \begin{bmatrix} W(t-t_k) \; X(t-t_k) \\ Y(t-t_k) \;\; Z(t-t_k) \end{bmatrix} \begin{bmatrix} (W(h))^k & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0\,0 \end{bmatrix} & 0 \end{bmatrix} z_0 \\ &= \begin{bmatrix} W(t-t_k)(W(h))^k & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ Y(t-t_k)(W(h))^k & 0 \end{bmatrix} z_0 \end{aligned} \tag{A15}$$

We know $\begin{bmatrix} I\,0\,0 \\ 0\,I\,0 \\ 0\,0\,0 \end{bmatrix} \Sigma(h) \begin{bmatrix} I\,0\,0 \\ 0\,I\,0 \\ 0\,0\,0 \end{bmatrix}$ has at least one eigenvalue outside the unit circle, thus those unstable eigenvalues must be in $W(h)$. This means that the first two elements of $z\left(t_{k+1}^-\right)$, which we call $\xi(k)$, will in general grow with k (if one selects initial condition z_0 along the eigenvector of the corresponding eigenvalue).

Thus, we cannot ensure $\xi(k)$ will converge to zero for a general condition.

$$\left\| \begin{bmatrix} x\left(t_{k+1}^-\right) \\ \bar{x}\left(t_{k+1}^-\right) \end{bmatrix} \right\| = \|\xi(k)\| = \left\| (W(h))^k \begin{bmatrix} x_0 \\ \bar{x}_0 \end{bmatrix} \right\| \to \infty \text{ as } k \to \infty \tag{A16}$$

This means the system is unstable; thus we have a contradiction.

□

A.3 *Proof of Theorem 4.1*

Proof We will now analyse the behavior of the plant state norm when the loop is open. The stability of the system can be guaranteed if $\|x(t)\|$ decreases such that $\|x(t_k+\tau)\| > \|x(t_{k+1})\|$, where $t_k+\tau$ is the time the loop is opened and t_{k+1} is the next time the loop is closed, with $t_{k+1} - t_k + \tau = h - \tau$.

In general, we see that in any interval $[t_k+\tau, t_{k+1})$ the following holds true:

$$\begin{aligned} \|x\| &= \|\hat{x} + e\| < \|\hat{x}\| + \|e\| \\ \|e(t_k+\tau)\| &= 0 \\ \|x(t_k+\tau)\| &= \|\hat{x}(t_k+\tau)\| \end{aligned} \tag{A17}$$

So, we can guarantee that $\|x\|$ will decrease over the interval $[t_k+\tau, t_{k+1})$ if $\|\hat{x}\| + \|e\|$ decrease. We know that:

$$\dot{e} = \dot{x} - \dot{\hat{x}} = f(x) - f(\hat{x}) - \delta(\hat{x}) \tag{A18}$$

Thus:

$$
\begin{aligned}
e(t) &= e(t_k+\tau)\int_{t_k+\tau}^{t}\left(f(x(s)) - f(\hat{x}(s)) - \delta(\hat{x}(s))\right)ds \\
&= \int_{t_k+\tau}^{t}\left(f(x(s)) - f(\hat{x}(s)) - \delta(\hat{x}(s))\right)ds\,, \;\; \forall t \in [t_k+\tau, t_k)
\end{aligned} \tag{A19}
$$

The last equality holds since at $t_k+\tau$ the plant model state is updated and the error is equal to zero. We will now use the Lipschitz condition to bound the norm of the error.

$$
\begin{aligned}
\|e(t)\| &\le \int_{t_k+\tau}^{t}\left(\|f(x(s)) - f(\hat{x}(s))\| + \|\delta(\hat{x}(s))\|\right)ds \\
&\le \int_{t_k+\tau}^{t}\left(K_f\|(x(s)) - (\hat{x}(s))\| + K_\delta\|\hat{x}(s)\|\right)ds \\
&= K_f\int_{t_k+\tau}^{t}\|(x(s)) - (\hat{x}(s))\|\,ds + K_\delta\int_{t_k+\tau}^{t}\|\hat{x}(s)\|\,ds \\
&= K_f\int_{t_k+\tau}^{t}\|e(s)\|\,ds + K_\delta\int_{t_k+\tau}^{t}\|\hat{x}(s)\|\,ds\,, \;\forall t \in [t_k+\tau, t_k)
\end{aligned} \tag{A20}
$$

Then:

$$
\begin{aligned}
\|e(t)\| &\le K_f\int_{t_k+\tau}^{t}\|e(s)\|\,ds + K_\delta\int_{t_k+\tau}^{t}\|\hat{x}(s)\|\,ds \\
&= K_f\int_{t_k+\tau}^{t}\|e(s)\|\,ds + K_\delta\int_{t_k+\tau}^{t}\alpha\,\|\hat{x}(t_k+\tau)\|\,e^{-\beta(t-t_k+\tau)}ds \\
&= K_f\int_{t_k+\tau}^{t}\|e(s)\|\,ds + K_\delta\frac{\alpha}{\beta}\|\hat{x}(t_k+\tau)\|\left(1-e^{-\beta(t-t_k+\tau)}\right), \;\forall t \in [t_k+\tau, t_k)
\end{aligned} \tag{A21}
$$

We now use the Gronwall-Bellman Inequality [] for the following step. This inequality states that if a continuous real-valued function $y(t)$ satisfies $y(t) < \lambda(t) + \int_a^t \mu(s)\,y(s)\,ds$ with $\lambda(t)$ and $\mu(t)$ continuous real-valued functions and $\mu(t)$ non-negative for $t \in [a,b)$, then $y(t) < \lambda(t) + \int_a^t \lambda(s)\,\mu(s)\,e^{\int_s^t u(\psi)d\psi}ds$ over the same interval. So, we assign $y(t) = \|e(t)\|$, $\lambda(t) =$

$K_\delta \frac{\alpha}{\beta} \left\| \hat{x}\left(t_k+\tau\right)\right\| \left(1-e^{-\beta(t-t_k+\tau)}\right)$, and $\mu\left(t\right)=K_f$, and thus obtain:

$$\left\|e\left(t\right)\right\| \leq K_\delta \frac{\alpha}{\beta} \left\| \hat{x}\left(t_k+\tau\right)\right\| \left(1-e^{-\beta(t-t_k+\tau)}\right) \tag{A22}$$

$$+\int_{t_k+\tau}^{t} K_\delta \frac{\alpha}{\beta} \left\| \hat{x}\left(t_k+\tau\right)\right\| \left(1-e^{-\beta(s-t_k+\tau)}\right) K_f e^{K_f(t-s)} ds \tag{A23}$$

$$= K_\delta \frac{\alpha}{\beta} \left\| \hat{x}\left(t_k+\tau\right)\right\| \left(1-e^{-\beta(t-t_k+\tau)} + \int_{t_k+\tau}^{t} \left(1-e^{-\beta(s-t_k+\tau)}\right) K_f e^{K_f(t-s)} ds\right)$$

$$= K_\delta \frac{\alpha}{\beta} \left\| \hat{x}\left(t_k+\tau\right)\right\| \left(1-e^{-\beta(t-t_k+\tau)} + K_f \int_{t_k+\tau}^{t} e^{K_f(t-s)} - e^{-\beta(s-t_k+\tau)} e^{K_f(t-s)} ds\right)$$

$$= K_\delta \frac{\alpha}{\beta} \left\| \hat{x}\left(t_k+\tau\right)\right\| \left(1-e^{-\beta(t-t_k+\tau)} + K_f \int_{t_k+\tau}^{t} e^{K_f(t-s)} - e^{K_f t - K_f s - \beta s + \beta(t_k+\tau)} ds\right)$$

$$= K_\delta \frac{\alpha}{\beta} \left\| \hat{x}\left(t_k+\tau\right)\right\| \Bigg(1-e^{-\beta(t-t_k+\tau)} + K_f \Bigg(\frac{-1}{K_f}\left(1-e^{K_f(t-(t_k+\tau))}\right) + \frac{1}{K_f+\beta}\Big(e^{-\beta(t-(t_k+\tau))} - e^{K_f(t-(t_k+\tau}$$

$$= K_\delta \frac{\alpha}{\beta} \left\| \hat{x}\left(t_k+\tau\right)\right\| \left(1-e^{-\beta(t-t_k+\tau)} - 1 + e^{K_f(t-(t_k+\tau))} + \frac{K_f}{K_f+\beta}\left(e^{-\beta(t-(t_k+\tau))} - e^{K_f(t-(t_k+\tau))}\right)\right)$$

$$= K_\delta \frac{\alpha}{\beta} \left\| \hat{x}\left(t_k+\tau\right)\right\| \left(e^{K_f(t-(t_k+\tau))} - e^{-\beta(t-(t_k+\tau))}\right)\left(1-\frac{K_f}{K_f+\beta}\right)$$

$$= K_\delta \left\| \hat{x}\left(t_k+\tau\right)\right\| \left(e^{K_f(t-(t_k+\tau))} - e^{-\beta(t-(t_k+\tau))}\right)\left(\frac{\alpha}{K_f+\beta}\right), \ \forall t \in [t_k+\tau, t_k)$$

Note that he error signal will be zero if the update time $h-\tau = t_{k+1}-(t_k+\tau)$ is zero (or if the model is perfect, that is, same dynamics as the plant). With this bound over the error signal we can proceed to calculate the bound over the plant state.

$$\left\|x\left(t\right)\right\| \leq \left\|\hat{x}\left(t\right)\right\| + \left\|e\left(t\right)\right\|$$

$$\leq \alpha \left\| \hat{x}\left(t_k+\tau\right)\right\| e^{-\beta(t-(t_k+\tau))} + K_\delta \left\| \hat{x}\left(t_k+\tau\right)\right\| \left(e^{K_f(t-(t_k+\tau))} - e^{-\beta(t-(t_k+\tau))}\right)\left(\frac{\alpha}{K_f+\beta}\right) \tag{A24}$$

$$= \alpha \left\| \hat{x}\left(t_k+\tau\right)\right\| \left(e^{-\beta(t-(t_k+\tau))} + \left(e^{K_f(t-(t_k+\tau))} - e^{-\beta(t-(t_k+\tau))}\right)\left(\frac{K_\delta}{K_f+\beta}\right)\right) \tag{A25}$$

$$, \ \forall t \in [t_k+\tau, t_k)$$

For stability, we need$\left\| \hat{x}\left(t_k+\tau\right)\right\| > \left\| \hat{x}\left(t_{k+1}\right)\right\|$. Therefore, we require:

$$\left\| \hat{x}\left(t_k+\tau\right)\right\| - \alpha \left\| \hat{x}\left(t_k+\tau\right)\right\| \left(e^{-\beta(h-\tau)} + \left(e^{K_f(h-\tau)} - e^{-\beta(h-\tau)}\right)\left(\frac{K_\delta}{K_f+\beta}\right)\right) > 0 \tag{A26}$$

$$\left\| \hat{x}\left(t_k+\tau\right)\right\| \left(1-\left(e^{-\beta(h-\tau)} + \left(e^{K_f(t-(t_k+\tau))} - e^{-\beta(h-\tau)}\right)\left(\frac{K_\delta}{K_f+\beta}\right)\right)\right) > 0$$

$$\left(1-\left(e^{-\beta(h-\tau)} + \left(e^{K_f(h-\tau)} - e^{-\beta(h-\tau)}\right)\left(\frac{K_\delta}{K_f+\beta}\right)\right)\right) > 0$$

□

A.4 *Proof of Theorem 4.2*

Proof Note that the error can be bounded as follows:

$$\|e(t)\| \leq \int_{t_k+\tau}^{t} \left(\left(K_f + K_{m,\max}\right) \|x(s) - \hat{x}(s)\| + \left(K_{\delta_f} + K_{\delta_m}\right) \|\hat{x}(s)\| \right) ds \ , \quad \forall t \in [t_k + \tau, t_{k+1}) \ . \tag{A27}$$

The rest of the proof is done as in the previous theorem. □

A.5 *Proof of Theorem 5.2*

Proof The proof is similar to the corresponding development for constant h and τ. On the closed loop interval, the system response is:

$$z(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = e^{\Lambda_c(t-t_k)} \begin{bmatrix} x(t_k) \\ 0 \end{bmatrix} = e^{\Lambda_c(t-t_k)} z(t_k) \ , \ \forall t \in [t_k, t_k + \tau). \tag{A28}$$

And on the open loop interval, the response is:

$$z(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = e^{\Lambda_o(t-(t_k+\tau))} e^{\Lambda_c(t-t_k)} \begin{bmatrix} x(t_k) \\ 0 \end{bmatrix} = e^{\Lambda_o(t-(t_k+\tau))} e^{\Lambda_c(t-t_k)} z(t_k) \ , \tag{A29}$$
$$\forall t \in [t_k + \tau, t_{k+1})$$

Now, note that at times t_k,, the error is reset to zero, which corresponds to pre-multiplying by $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

Using the above, we obtain

$$z(t_k) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h(k)-\tau(k))} e^{\Lambda_c \tau(k)} z(t_{k-1}) \ .$$

Then, with initial conditions $t(0) = t_0, \ z(t_0) = z_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$:

$$\begin{aligned} z(t) &= e^{\Lambda_c(t-t_k)} z(t_k) \\ &= e^{\Lambda_c(t-t_k)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h(k)-\tau(k))} e^{\Lambda_c \tau(k)} z(t_{k-1}) \\ &= e^{\Lambda_c(t-t_k)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h(k)-\tau(k))} e^{\Lambda_c \tau(k)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h(k-1)-\tau(k-1))} e^{\Lambda_c \tau(k-1)} z(t_{k-2}) \\ &= e^{\Lambda_c(t-t_k)} \left(\prod_{j=1}^{k} M(j) \right) z_0 \ , \ t \in [t_k, t_k + \tau) \ , \end{aligned}$$

where

$$M(j) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} e^{\Lambda_o(h(j)-\tau(j))} e^{\Lambda_c \tau(j)} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

And similarly for the interval $t \in [t_k + \tau, t_{k+1})$. □

A.6 *Proof of Theorem 5.3*

Proof Note that the output norm can be bounded by

$$\begin{aligned}
&\left\| e^{\Lambda_o(t-(t_k+\tau))} e^{\Lambda_c(\tau)} \left(\prod_{j=1}^{k} M(j) \right) z_0 \right\| \\
&\leq \left\| e^{\Lambda_o(t-(t_k+\tau))} \right\| \left\| e^{\Lambda_c(\tau)} \right\| \left\| \prod_{j=1}^{k} M(j) \right\| \|z_0\| \\
&\leq e^{\bar{\sigma}(\Lambda_o) h_{\max} - \tau_{\min}} \left\| e^{\Lambda_c(\tau)} \right\| \left\| \prod_{j=1}^{k} M(j) \right\| \|z_0\|
\end{aligned}$$

That is, since $e^{\Lambda_o(t-(t_k+\tau))}$ has finite growth and will grow for at most from $\tau_{\min}$ to $h_{\max}$, then convergence of the product of matrices $M(j)$ to zero ensures the stability of the system. Such convergence to zero is guaranteed by the existence of a symmetric positive definite matrix X in the Lyapunov equation. □

A.7 *Proof of Theorem 5.4*

Proof Let us begin by evaluating the expectation of the squared norm of the system. Note that

we are doing this for the interval $t \in [t_k, t_k + \tau_k)$, but the proof is the same for the interval $t \in [t_k + \tau_k, t_{k+1})$.

$$E \left\| e^{\Lambda_o(t-(t_k+\tau_k))} e^{\Lambda_c(\tau_k)} \left(\prod_{j=1}^{k} M(j) \right) z_0 \right\|^2 \tag{A30}$$

$$\begin{aligned}
&= E \left[z_0^T \left(\prod_{j=1}^{k} M(j) \right)^T \left(e^{\Lambda_o(t-(t_k+\tau_k))} e^{\Lambda_c(\tau_k)} \right)^T e^{\Lambda_o(t-(t_k+\tau_k))} e^{\Lambda_c(\tau_k)} \left(\prod_{j=1}^{k} M(j) \right) z_0 \right] \\
&\leq E \left[\bar{\sigma} \left(\left(e^{\Lambda_o(t-(t_k+\tau_k))} e^{\Lambda_c(\tau_k)} \right)^T e^{\Lambda_o(t-(t_k+\tau_k))} e^{\Lambda_c(\tau_k)} \right) z_0^T \left(\prod_{j=1}^{k} M(j) \right)^T \left(\prod_{j=1}^{k} M(j) \right) z_0 \right] \\
&\leq E \left[\left(e^{\bar{\sigma}(\Lambda_o)(h-\tau)(k+1)} \right)^2 z_0^T \left(\prod_{j=1}^{k} M(j) \right)^T \left(\prod_{j=1}^{k} M(j) \right) z_0 \right]
\end{aligned}$$

Now that the expectation is all in terms of the update times, we can use the i.i.d property of the update times and the assumption that K is bounded:

$$E\left[\left(e^{\bar{\sigma}(\Lambda_o)(h-\tau)(k+1)}\right)^2 z_0^T\left(\prod_{j=1}^{k} M(j)\right)^T\left(\prod_{j=1}^{k} M(j)\right)z_0\right] \tag{A31}$$

$$= K\, z_0^T E\left[\left(\prod_{j=1}^{k} M(j)\right)^T M(k)^T M(k)\left(\prod_{j=1}^{k} M(j)\right)\right] z_0$$

$$= K\, z_0^T E\left[\left(\prod_{j=1}^{k} M(j)\right)^T E\left[M^T M\right]\left(\prod_{j=1}^{k} M(j)\right)\right] z_0$$

$$\leq K\,\bar{\sigma}\left(E\left[M^T M\right]\right) z_0^T E\left[\left(\prod_{j=1}^{k} M(j)\right)^T\left(\prod_{j=1}^{k} M(j)\right)\right] z_0$$

We repeat the last three steps recursively to obtain

$$E\left\|e^{\Lambda_c(t-t_k)}\left(\prod_{j=1}^{k} M(j)\right)z_0\right\|^2$$
$$\leq K\,\left(\bar{\sigma}\left(E\left[M^T M\right]\right)\right)^k z_0^T z_0$$

From here, we can see that if $\left\|E\left[M^T M\right]\right\| =,\bar{\sigma}\left(E\left[M^T M\right]\right) < 1$, then the limit of the expectation as time goes to infinity approaches zero. □

A.8 *Proof of Theorem 6.2*

Proof Sufficiency. Taking the norm of the solution described as in Proposition 6.1:

$$\|z(n)\| = \left\|\Lambda_{Dc}^{(n-n_k)}\left(\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}\Lambda_{Do}^{(h-\tau)}\Lambda_{Dc}^{\tau}\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}\right)^k z_0\right\| \tag{A32}$$

$$\leq \left\|\Lambda_{Dc}^{(n-n_k)}\right\|\left\|\left(\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}\Lambda_{Do}^{(h-\tau)}\Lambda_{Dc}^{\tau}\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}\right)^k\right\|$$
$$\|z_0\|$$

Notice we are only doing this part for the case when $n \in [n_k, n_k+\tau)$, but the process is exactly the same for the intervals where $n \in (n_k+\tau, n_k+1)$. Analysing the first term on the right hand side:

$$\left\|\Lambda_{Dc}^{(n-n_k)}\right\| \leq \left(\bar{\sigma}\left(\Lambda_{Dc}\right)\right)^{n-n_k} \leq \left(\bar{\sigma}\left(\Lambda_{Dc}\right)\right)^{\tau} = K_1 \tag{A33}$$

where $\bar{\sigma}\left(\Lambda_{Dc}\right)$ is the largest singular value of Λ_{Dc}. In general this term can always be bounded as the time difference $n - n_k$ is always smaller than τ. That is, even when Λ_{Dc} has eigenvalues with positive real part, $\left\|\Lambda_{Dc}^{(n-n_k)}\right\|$ can only grow a certain amount. This growth is completely independent of k.

We now study the term $\left\| \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k \right\|$. It is clear taht this term will be bounded if and only if the eigenvalues of $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ lie inside the unit circle:

$$\left\| \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k \right\| \leq K_2 e^{-\alpha_1 k} \tag{A34}$$

with $K_2, \alpha_1 > 0$.

Since k is a function of time we can bounded the right term of the previous inequality in terms of t :

$$K_2 e^{-\alpha_1 k} < K_2 e^{-\alpha_1 \frac{n-1}{h}} = K_2 e^{\frac{\alpha_1}{h}} e^{-\frac{\alpha_1}{h} n} = K_3 e^{-\alpha n} \tag{A35}$$

with $K_{3,} \alpha > 0$.

So from the above, we conclude that:

$$\begin{aligned} &\|z(n)\| \\ &= \left\| \Lambda_{Dc}^{(n-n_k)} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 \right\| \\ &\leq K_1 K_3 e^{-\alpha n} \left\| z_0 \right\| . \end{aligned} \tag{A36}$$

Necessity. We will now provide the necessity part of the theorem. We will do this by contradiction. Assume the system is stable and that $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ has at least one eigenvalue outside the unit circle. Let us define $\Sigma(h) = \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau}$. Since the system is stable, a periodic sample of the response should converge to zero with time. We will take the samples at times n_{k+1}, that is, just before the loop is closed again. We will concentrate on a specific term: the state of the plant $x(n_{k+1})$, which is the first element of $z(n_{k+1})$. We will call $x(n_{k+1})$, $\xi(k)$.

Now assume $\Sigma(\eta)$ has the following form:

$$\Sigma(\eta) = \begin{bmatrix} W(\eta) & X(\eta) \\ Y(\eta) & Z(\eta) \end{bmatrix} .$$

Then we can express the solution $z(n)$ as:

$$\begin{aligned} &\Lambda_{Dc}^{(n-n_k)} \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Sigma(h) \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)^k z_0 \\ &= \begin{bmatrix} W(n-n_k) & X(n-n_k) \\ Y(n-n_k) & Z(n-n_k) \end{bmatrix} \begin{bmatrix} (W(h))^k & 0 \\ 0 & 0 \end{bmatrix} z_0 \\ &= \begin{bmatrix} W(n-n_k)\,(W(h))^k & 0 \\ Y(n-n_k)\,(W(h))^k & 0 \end{bmatrix} z_0 . \end{aligned} \tag{A37}$$

Now, the values of the solution at times n_{k+1}^{-}, that is, just before the loop is closed again, are

$$z\left(n_{k+1}\right)=\begin{bmatrix} W(h)\left(W(h)\right)^{k} & 0 \\ Y(h)\left(W(h)\right)^{k} & 0 \end{bmatrix} z_0$$
$$=\begin{bmatrix} \left(W(h)\right)^{k+1} & 0 \\ Y(h)\left(W(h)\right)^{k} & 0 \end{bmatrix} z_0 \tag{A38}$$

We also know that $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)} \Lambda_{Dc}^{\tau} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ has at least one eigenvalue outside the unit circle, which means that those unstable eigenvalues must be in $W(h)$. This means that the first element of $z\left(n_{k+1}\right)$, which we call $\xi\left(k+1\right)$, will in general grow with k. In other words we cannot ensure $\xi\left(k+1\right)$ will converge to zero for general initial condition x_0.

$$\left\|x\left(n_{k+1}\right)\right\| = \left\|\xi\left(k+1\right)\right\| = \left\|\left(W\left(h\right)\right)^{k+1} x_0\right\| \rightarrow \infty$$
$$\text{as } \; k \rightarrow \infty\,, \tag{A39}$$

which clearly means the system cannot be stable. Thus, we have a contradiction. □

A.9 *Proof of Theorem 6.4*

Proof The proof is similar to the corresponding development for constant h and τ. On the closed loop interval, the system response is:

$$z\left(n\right)=\begin{bmatrix} x\left(n\right) \\ e\left(n\right) \end{bmatrix} = \Lambda_{Dc}^{(n-n_k)} \begin{bmatrix} x\left(n_k\right) \\ 0 \end{bmatrix} = \Lambda_{Dc}^{(n-n_k)} z\left(n_k\right)\,,\ \forall n \in \left[n_k, n_k+\tau\right). \tag{A40}$$

And on the open loop interval, the response is:

$$z\left(n\right)=\begin{bmatrix} x\left(n\right) \\ e\left(n\right) \end{bmatrix} = \Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{(n-n_k)} \begin{bmatrix} x\left(n_k\right) \\ 0 \end{bmatrix} = \Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{(n-n_k)} z\left(n_k\right)\,, \tag{A41}$$
$$\forall n \in \left[n_k+\tau, n_{k+1}\right)$$

Now, note that at times $n_{k,}$, the error is reset to zero, which corresponds to pre-multiplying by $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

Using the above, we obtain

$$z\left(t_k\right)=\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)(j)} \Lambda_{Dc}^{\tau(j)} z\left(n_{k-1}\right)\,.$$

Then, with initial conditions $n(0) = t_0,\ z\left(n_0\right) = z_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}$:

$$\begin{aligned} z\left(n\right) &= \Lambda_{Dc}^{(n-n_k)} z\left(n_k\right) \\ &= \begin{bmatrix} I\ 0 \\ 0\ 0 \end{bmatrix} \Lambda_{Do}^{(n-(n_k+\tau_k))} \Lambda_{Dc}^{(n-n_k)} z\left(n_{k-1}\right) \\ &= \begin{bmatrix} I\ 0 \\ 0\ 0 \end{bmatrix} \Lambda_{Do}^{(n-(n_k+\tau_k))} \Lambda_{Dc}^{(n-n_k)} \begin{bmatrix} I\ 0 \\ 0\ 0 \end{bmatrix} \Lambda_{Do}^{(n-(n_{k-1}+\tau_{k-1}))} \Lambda_{Dc}^{(n-n_{k-1})} z\left(n_{k-2}\right) \\ &= \Lambda_{Dc}^{(n-n_k)} \left(\prod_{j=1}^{k} M\left(j\right) \right) z_0\ ,\ n \in \left[n_k, n_k + \tau_k\right), \end{aligned}$$

where

$$M\left(j\right) = \begin{bmatrix} I\ 0 \\ 0\ 0 \end{bmatrix} \Lambda_{Do}^{(h-\tau)(j)} \Lambda_{Dc}^{\tau(j)} \begin{bmatrix} I\ 0 \\ 0\ 0 \end{bmatrix}$$

And similarly for the interval $n \in [n_k + \tau, n_{k+1})$. □

A.10 *Proof of Theorem 6.5*

Proof Note that the output norm can be bounded by

$$\begin{aligned} &\left\| \Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{\tau} \left(\prod_{j=1}^{k} M\left(j\right) \right) z_0 \right\| \\ &\leq \left\| \Lambda_{Do}^{(n-(n_k+\tau))} \Lambda_{Dc}^{\tau} \right\| \left\| \Lambda_{Dc}^{\tau} \right\| \left\| \prod_{j=1}^{k} M\left(j\right) \right\| \left\| z_0 \right\| \\ &\leq \bar{\sigma} \left(\Lambda_{Do}^{h_{\max}-\tau_{\min}} \right) \left\| \Lambda_{Dc}^{\tau} \right\| \left\| \prod_{j=1}^{k} M\left(j\right) \right\| \left\| z_0 \right\| \end{aligned}$$

That is, since $\Lambda_{Do}^{(n-(n_k+\tau))}$ has finite growth and will grow for at most from $\tau_{\min}$ to $h_{\max}$, then convergence of the product of matrices $M\left(j\right)$ to zero ensures the stability of the system. Such convergence to zero is guaranteed by the existence of a symmetric positive definite matrix X in the Lyapunov equation. □