Time Domain and Frequency Domain Conditions For Passivity

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Abstract

This technical report summarizes various key relationships in regards to continuous time and discrete time *passive* systems. It includes relationships for both the linear time invariant case and the non-linear case. For the linear time invariant case we specifically discuss relationships between minimal causal *positive real* *strictly-positive real* *extended strictly-positive real* systems and *passive* systems.

Index Terms

passivity theory, *positive real*, *strictly-positive real*, *extended strictly-positive real*, digital control systems, *passive* systems, *strictly-output passive* systems, *strictly-input passive* systems

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I. INTRODUCTION

Passive systems can be thought of as systems which only store or release energy which was provided to the system. *Passive* systems have been analyzed by studying their input output relationships. In particular the definitions used to describe *positive* and *strongly positive* systems [1] are essentially equivalent to the definitions used for *passive* and *strictly-input passive* systems in which the available storage $\beta = 0$ [2, Definition 6.4.1].

II. KEY RESULTS FOR PASSIVE, DISSIPATIVE, AND POSITIVE REAL SYSTEMS

A. Passive Systems

Let $\mathcal{T}$ be the set of time of interest in which $\mathcal{T} = \mathbb{R}^+$ for continuous time signals and $\mathcal{T} = \mathbb{Z}^+$ for discrete time signals. Let $\mathcal{V}$ be a linear space $\mathbb{R}^n$ and denote by the space $\mathcal{H}$ of all functions $u : \mathcal{T} \to \mathcal{V}$ which satisfy the following:

$$\|u\|_2^2 = \int_0^\infty u^\mathsf{T}(t)u(t)dt < \infty, \tag{1}$$

for continuous time systems (L_2^m), and

$$\|u\|_2^2 = \sum_0^\infty u^\mathsf{T}(i)u(i) < \infty, \tag{2}$$

for discrete time systems (l_2^m).

Similarly we will denote by $\mathcal{H}_e$ as the extended space of functions as $u : \mathcal{T} \to \mathcal{V}$ by introducing the truncation operator:

$$x_T(t) = \begin{cases} x(t), \ t < T, \\ 0, \ t \geq T \end{cases}$$

for continuous time, and

$$x_T(i) = \begin{cases} x(i), \ i < T, \\ 0, \ i \geq T \end{cases}$$

for discrete time. The extended space $\mathcal{H}_e$ satisfies the following:

$$\|u_T\|_2^2 = \int_0^T u^\mathsf{T}(t)u(t)dt < \infty; \ \forall T \in \mathcal{T} \tag{3}$$

for continuous time systems (L_{2e}^m), and

$$\|u_T\|_2^2 = \sum_0^{T-1} u^\mathsf{T}(i)u(i) < \infty; \ \forall T \in \mathcal{T}$$

for discrete time systems (l_{2e}^m). The inner product over the interval $[0, T]$ for continuous time is denoted as follows:

$$\langle y, u\rangle_T = \int_0^T y^\mathsf{T}(t)u(t)dt$$

similarly the inner product over the discrete time interval $\{0, 1, \ldots, T-1\}$ is denoted as follows:

$$\langle y, u\rangle_T = \sum_0^{T-1} y^\mathsf{T}(i)u(i).$$

For simplicity of discussion we note the following equivalence for our inner-product space:

$$\langle (Hu)_T, u_T\rangle = \langle (Hu)_T, u\rangle = \langle Hu, u_T\rangle = \langle Hu, u\rangle_T.$$

The symbol, H denotes a relation on $\mathcal{H}_e$, and if u is a given element of $\mathcal{H}_e$, then the symbol Hu denotes an image of u under H [1]. Furthermore $Hu(t)$ denotes the value of Hu at continuous time t and $Hu(i)$ denotes the value of Hu at discrete time i.

Definition 1: A dynamic system $H : \mathcal{H}_e \to \mathcal{H}_e$ is L_2^m stable if

$$u \in L_2^m \implies Hu \in L_2^m.$$

Remark 1: A proper *LTI* system described by the square transfer function matrix $H(s) \in \mathbb{R}^{m\times m}(s)$ is L_2^m stable if and only if all poles have negative real parts [3, Theorem 9.5 p.488] (uniform BIBO stability) combined with [4, Theorem 6.4.45 p.301]. Therefore a system $H(s)$ with a corresponding minimal realization $\Sigma \triangleq \{A, \ B, \ C, \ D\}$ described by (4) and (5) will also be asymptotically stable.

$$\dot{x} = Ax(t) + Bu(t), \ x \in \mathbb{R}^n \tag{4}$$
$$y(t) = Cx(t) + Du(t) \tag{5}$$
$$H(s) = C(sI - A)^{-1} + D \tag{6}$$

Definition 2: A dynamic system $H : \mathcal{H}_e \to \mathcal{H}_e$ is l_2^m stable if

$$u \in l_2^m \implies Hu \in l_2^m.$$

Remark 2: An *LTI* system described by the square transfer function matrix $H(z) \in \mathbb{R}^{m\times m}(z)$ is l_2^m stable if and only if all poles are inside the unit circle of the complex plain [3, Theorem 10.17 p.508] (uniform BIBO stability) combined with [4, Theorem 6.7.12 p.366]. Therefore a system $H(z)$ with a corresponding minimal realization $\Sigma_z \triangleq \{A, \ B, \ C, \ D\}$ described by (7) and (8) will also be asymptotically stable.

$$x(k+1) = Ax(k) + Bu(k), \ x \in \mathbb{R}^n \tag{7}$$
$$y(k) = Cx(k) + Du(k) \tag{8}$$
$$H(z) = C(zI - A)^{-1} + D \tag{9}$$

Definition 3: Let $H : \mathcal{H}_e \to \mathcal{H}_e$. We say that H is

i) *passive* if $\exists \beta > 0$ s.t.

$$\langle Hu, u\rangle_T \geq -\beta, \ \forall u \in \mathcal{H}_e, \ \forall T \in \mathcal{T}$$

ii) *strictly-input passive* if $\exists\delta > 0$ and $\exists\beta > 0$ s.t.

$$\langle Hu, u\rangle_T \geq \delta\|u_T\|_2^2 - \beta,\ \forall u \in \mathcal{H}_e,\ \forall T \in \mathcal{T}$$

iii) *strictly-output passive* if $\exists\epsilon > 0$ and $\exists\beta > 0$ s.t.

$$\langle Hu, u\rangle_T \geq \epsilon\|(Hu)_T\|_2^2 - \beta,\ \forall u \in \mathcal{H}_e,\ \forall T \in \mathcal{T} \quad (10)$$

iv) *non-expansive* if $\exists\hat{\gamma} > 0$ and $\exists\hat{\beta}$ s.t.

$$\|(Hu)_T\|_2^2 \leq \hat{\beta} + \hat{\gamma}^2\|u_T\|_2^2,\ \forall u \in \mathcal{H}_e,\ \forall T \in \mathcal{T}$$

Remark 3: In [2] *strictly-input passive* was referred to as *strictly passive*. Furthermore the definition for (*strictly*) *positive* given in [1] is equivalent to the definition for (*strictly-input*) *passive* with $\beta = 0$ for the continuous time case. We also note that [5] chose to define passive systems for the case when $\beta = 0$. However, we will follow the definition given in [2] and only consider a system as (*strictly*) *positive* using (Definition 3-ii) Definition 3-i in which $\beta = 0$ and $T = \infty$. NB, strictly-positive or strictly-input-passive systems are *not* equivalent to the strictly-positive-real systems whose definitions will be recalled later in the text. Strictly-positive-real systems, as they shall be defined, implicitly require all poles to be strictly in the left-half-plane. For example, $H(s) = \frac{1}{s} + a$, in which $0 < a < \infty$ is obviously strictly-positive and obviously not strictly-positive-real in which their exists no $0 < \epsilon < \infty$ such that $H(s - \epsilon)$ is analytic for all $\mathtt{Re}[s] > 0$ (the first condition required in order for $H(s - \epsilon)$ to be positive-real).

Remark 4: If H is linear then β can be set equal to zero without loss of generality in regards to *passivity*. If H is causal then (*strictly*) *positive* and (*strictly-input*) *passive* are equivalent (assuming $Hu(0) = 0$) [2, p.174, p.200].

Remark 5: A *non-expansive* system H is equivalent to any system which has finite L_2^m (l_2^m) gain in which there exists constants γ and β s.t. $0 < \gamma < \hat{\gamma}$ and satisfy

$$\|(Hu)_T\|_2 \leq \gamma\|u_T\|_2 + \beta,\ \forall u \in \mathcal{H}_e,\ \forall T \in \mathcal{T}. \quad (11)$$

Furthermore a *non-expansive* system implies L_2^m (l_2^m) stability [6, p.4] ([7, Remark 1]).

Theorem 1: [2, p.174-p.175] Assume that H is a linear time invariant system which has a minimal realization Σ (Σ_z) that is asymptotically stable:

(i) then for the continuous time case:
   (a) H is *passive* iff $H(j\omega) + H^*(j\omega) \geq 0,\ \forall\omega \in \mathbb{R}$.
   (b) H is *strictly-input passive* iff

$$H(j\omega) + H^*(j\omega) \geq \delta I,\ \forall\omega \in \mathbb{R}. \quad (12)$$

(ii) and for the discrete time case:
   (a) H is *passive* iff $H(e^{j\theta}) + H^*(e^{j\theta}) \geq 0,\ \forall\theta \in [0, 2\pi]$.
   (b) H is *strictly-input passive* iff

$$H(e^{j\theta}) + H^*(e^{j\theta}) \geq \delta I,\ \forall\theta \in [0, 2\pi]. \quad (13)$$

Remark 6: The theorem stated was left as exercises for the reader to solve in [2, p.174-p.175]. The assumption that the system is a minimal realization and is asymptotically stable is based on their assumption that the impulse response of H is in L_1^m for continuous time or l_1^m for discrete time [2, p.83] and [4, p.353,p.297,p.301].

Theorem 2: Given a single-input single-output *LTI strictly-output passive* system with transfer function $H(s)$ ($H(z)$), real impulse response $h(t)$ ($h(k)$), and corresponding frequency response:

$$H(j\omega) = \mathtt{Re}\{H(j\omega)\} + j\mathtt{Im}\{H(j\omega)\} \quad (14)$$

in which $\mathtt{Re}\{H(j\omega)\} = \mathtt{Re}\{H(-j\omega)\}$ for the real part of the frequency response and $\mathtt{Im}\{H(j\omega)\} = -\mathtt{Im}\{H(-j\omega)\}$ for the imaginary part of the frequency response. The constant ϵ for (10) satisfies:

$$0 < \epsilon \leq \inf_{\omega\in[0,\infty)} \frac{\mathtt{Re}\{H(j\omega)\}}{\mathtt{Re}\{H(j\omega)\}^2 + \mathtt{Im}\{H(j\omega)\}^2} \quad (15)$$

for the continuous time case. Similarly

$$H(e^{j\omega}) = \mathtt{Re}\{H(e^{j\omega})\} + j\mathtt{Im}\{H(e^{j\omega})\} \quad (16)$$

in which $\mathtt{Re}\{H(e^{j\omega})\} = \mathtt{Re}\{H(e^{-j\omega})\}$ in which $0 \leq \omega \leq \pi$ for the real part of the frequency response and $\mathtt{Im}\{H(e^{j\omega})\} = -\mathtt{Im}\{H(e^{-j\omega})\}$ for the imaginary part of the frequency response. The constant ϵ for (10) satisfies:

$$0 < \epsilon \leq \min_{\omega\in[0,\pi]} \frac{\mathtt{Re}\{H(e^{j\omega})\}}{\mathtt{Re}\{H(e^{j\omega})\}^2 + \mathtt{Im}\{H(e^{j\omega})\}^2} \quad (17)$$

for the discrete time case.

Proof: Since a *strictly-output passive* system has a finite integrable (summable) impulse response (ie. $\int_0^\infty h^2(t)dt < \infty$ ($\sum_{i=0}^\infty h^2[i] < \infty$)) then (10) can be written as:

$$\int_{-\infty}^{\infty} H(j\omega)|U(j\omega)|^2 d\omega \geq \epsilon \int_{-\infty}^{\infty} |H(j\omega)|^2|U(j\omega)|^2 d\omega \quad (18)$$

for the continuous time case or

$$\int_{-\pi}^{\pi} H(e^{j\omega})|U(e^{j\omega})|^2 d\omega \geq \epsilon \int_{-\pi}^{\pi} |H(e^{j\omega})|^2|U(e^{j\omega})|^2 d\omega \quad (19)$$

for the discrete time case. (18) can be written in the following simplified form:

$$\int_{-\infty}^{\infty} \mathtt{Re}\{H(j\omega)\}|U(j\omega)|^2 d\omega \geq \epsilon \int_{-\infty}^{\infty} (\mathtt{Re}\{H(j\omega)\}^2 + \mathtt{Im}\{H(j\omega)\}^2)|U(j\omega)|^2 d\omega \quad (20)$$

in which (15) clearly satisfies (20). Similarly (19) can be written in the following simplified form:

$$\int_{-\pi}^{\pi} \mathtt{Re}\{H(e^{j\omega})\}|U(e^{j\omega})|^2 d\omega \geq \epsilon \int_{-\pi}^{\pi} (\mathtt{Re}\{H(e^{j\omega})\}^2 + \mathtt{Im}\{H(e^{j\omega})\}^2)|U(e^{j\omega})|^2 d\omega \quad (21)$$

in which (17) clearly satisfies (21). ■

B. Dissipative Systems

Dissipative systems are concerned with relating β to an appropriate *storage function* $s(u(t), y(t))$ based on the internal states $x \in \mathbb{R}^n$ of the systems ((4),(5)) or ((7),(8)) such that $\beta(x) : \mathbb{R}^n \to \mathbb{R}^+$. The discussion can be generalized for non-linear systems, however for simplicity we will focus on the linear time invariant cases.

Definition 4: A state space system Σ is dissipative with respect to the supply rate $s(u, y)$ if there exists a matrix $P = P^{\mathsf{T}} > 0$, such that for all $x \in \mathbb{R}^n$, all $t_2 \geq t_1$, and all input functions u

$$x^{\mathsf{T}}(t_2)Px(t_2) \leq x^{\mathsf{T}}(t_1)Px(t_1) + \int_{t_1}^{t_2} s(u(t), y(t))dt, \text{ holds.} \tag{22}$$

By dividing both sides of (22) by $t_2 - t_1$ and letting $t_2 \to t_1$ it follows that $\forall t \geq 0$

$$\begin{aligned} x^{\mathsf{T}}(t)\dot{P}x(t) &\leq s(u(t), y(t)) \\ \dot{x}^{\mathsf{T}}(t)Px(t) + x^{\mathsf{T}}(t)P\dot{x}(t) &\leq s(u(t), y(t)) \\ x^{\mathsf{T}}(t)[A^{\mathsf{T}}P + PA]x(t) + 2x^{\mathsf{T}}(t)PBu(t) &\leq s(u(t), y(t)) \end{aligned} \tag{23}$$

therefore (23) can be used as an alternative definition for a dissipative system.

Remark 7: We chose an appropriate storage function $\beta(x) = x^{\mathsf{T}}Px$ and substitute it into [6, (3.3)] which results in (22). Since $\beta(x) \in C^1$ we can derive (23) as was shown for the nonlinear case [8, (5.83)].

We note that since Σ is a minimal realization of $H(s)$ then from [6, Corollary 3.1.8] we can state the *equivalent* definitions for *passivity* based on P and Σ.

Definition 5: Assume that Σ is a dissipative system with a storage function $s(u(t), y(t))$ of the following form:

$$s(u(t), y(t)) = y^{\mathsf{T}}(t)Qy(t) + 2y^{\mathsf{T}}(t)Su(t) + u^{\mathsf{T}}(t)Ru(t) \tag{24}$$

then Σ:

i) is *passive* iff

$$Q = R = 0, \text{ and } S = \frac{1}{2}I \tag{25}$$

ii) is *strictly-input passive* iff $\exists \delta > 0$ and

$$Q = 0, \ R = -\delta I, \text{ and } S = \frac{1}{2}I \tag{26}$$

iii) is *strictly-output passive* iff $\exists \epsilon > 0$ and

$$Q = -\epsilon I, \ R = 0, \text{and } S = \frac{1}{2}I \tag{27}$$

iv) is *non-expansive* iff $\exists \hat{\gamma} > 0$ and

$$Q = -I, \ R = \hat{\gamma}^2 I, \text{and } S = 0 \tag{28}$$

Remark 8: The reason that these conditions are necessary and sufficient is that the system Σ is a minimal realization of $H(s)$ which is controllable and observable and satisfies either [9, Theorem 1] or [5, Theorem 16] for the *LTI* case.

From the above discussion the following corollary can be stated.

Corollary 1: A necessary and sufficient test for Definition 5 to hold is that there $\exists P = P^{\mathsf{T}} > 0$ such that the following *LMI* is satisfied:

$$\begin{bmatrix} A^{\mathsf{T}}P + PA - \hat{Q} & PB - \hat{S} \\ (PB - \hat{S})^{\mathsf{T}} & -\hat{R} \end{bmatrix} \leq 0 \ , \tag{29}$$

in which

$$\hat{Q} = C^{\mathsf{T}}QC \tag{30}$$

$$\hat{S} = C^{\mathsf{T}}S + C^{\mathsf{T}}QD \tag{31}$$

$$\hat{R} = D^{\mathsf{T}}QD + (D^{\mathsf{T}}S + S^{\mathsf{T}}D) + R. \tag{32}$$

An analogous discussion can be made for the discrete time case similar to that given in [10, Appendix C].

Definition 6: A state space system Σ_z is dissipative with respect to the supply rate $s(u, y)$ iff there exists a matrix $P = P^{\mathsf{T}} > 0$, such that for all $x \in \mathbb{R}^n$, all $l > k \geq 0$, and all input functions u

$$x^{\mathsf{T}}(l)Px(l) \leq x^{\mathsf{T}}(k)Px(k) + \sum_{i=k}^{l-1} s(u[i], y[i]), \text{ holds.} \tag{33}$$

Lemma 1: A state space system Σ_z is dissipative with respect to the supply rate $s(u, y)$ iff there exists a matrix $P = P^{\mathsf{T}} > 0$, such that for all $x \in \mathbb{R}^n$, all $k \geq 0$, and all input functions $u(k)$ such that

$$x^{\mathsf{T}}[k+1]Px[k+1] - x^{\mathsf{T}}[k]Px[k] \leq \quad s(u[k], y[k]) \tag{34}$$

$$\begin{aligned} \{Ax[k] + Bu[k]\}^{\mathsf{T}}P\{Ax[k] + Bu[k]\} - & \\ x^{\mathsf{T}}[k]Px[k] \leq & \quad s(u[k], y[k]) \\ x^{\mathsf{T}}[k]\{A^{\mathsf{T}}PA - P\}x[k] + 2x^{\mathsf{T}}[k]A^{\mathsf{T}}PBu[k] + & \\ u^{\mathsf{T}}[k]B^{\mathsf{T}}PBu[k] \leq & \quad s(u(k), y(k)) \end{aligned} \tag{35}$$

holds.

Proof: (33) $\implies$ (34) can be shown be setting $l = k+1$. (34) $\implies$ (33):

Taking (34) we can write

$$\sum_{i=k}^{l-1}(x^{\mathsf{T}}[i+1]Px[i+1] - x^{\mathsf{T}}[i]Px[i]) \leq \sum_{i=k}^{l-1} s(u[i], y[i])$$

which can then be expressed as

$$\begin{aligned} \sum_{i=k+1}^{l} x^{\mathsf{T}}[i]Px[i] - \sum_{i=k}^{l-1} x^{\mathsf{T}}[i]Px[i] &\leq \textstyle\sum_{i=k}^{l-1} s(u[i], y[i]) \\ x^{\mathsf{T}}[l]Px[l] - x^{\mathsf{T}}[k]Px[k] &\leq \textstyle\sum_{i=k}^{l-1} s(u[i], y[i]). \end{aligned}$$

∎

We note that since Σ_z is a minimal realization of $H(z)$ then a similar argument can be made as was done in [6, Corollary 3.1.8] for the discrete time which allows us to state the *equivalent* definitions for *passivity* based on P and Σ_z.

Definition 7: Assume that Σ_z is a dissipative system with a storage function $s(u[k], y[k])$ of the following form:

$$s(u[k], y[k]) = y^{\mathsf{T}}[k]Qy[k] + 2y^{\mathsf{T}}[k]Su[k] + u^{\mathsf{T}}[k]Ru[k] \tag{36}$$

then Σ_z:

i) is *passive* iff

$$Q = R = 0, \text{ and } S = \frac{1}{2}I \tag{37}$$

ii) is *strictly-input passive* iff $\exists \delta > 0$ and

$$Q = 0, \ R = -\delta I, \text{ and } S = \frac{1}{2}I \tag{38}$$

iii) is *strictly-output passive* iff $\exists \epsilon > 0$ and

$$Q = -\epsilon I, \ R = 0, \text{and } S = \frac{1}{2}I \tag{39}$$

iv) is *non-expansive* iff $\exists \hat{\gamma} > 0$ and

$$Q = -I, \ R = \hat{\gamma}^2 I, \text{and } S = 0 \tag{40}$$

Therefore the following corollary can be stated.

Corollary 2: [10, Lemma C.4.2] A necessary and sufficient test for Definition 7 to hold is that there $\exists P = P^{\mathsf{T}} > 0$ such that the following *LMI* is satisfied:

$$\begin{bmatrix} A^{\mathsf{T}}PA - P - \hat{Q} & A^{\mathsf{T}}PB - \hat{S} \\ (A^{\mathsf{T}}PB - \hat{S})^{\mathsf{T}} & -\hat{R} + B^{\mathsf{T}}PB \end{bmatrix} \leq 0 \ , \quad (41)$$

in which

$$\hat{Q} = C^{\mathsf{T}}QC \quad (42)$$
$$\hat{S} = C^{\mathsf{T}}S + C^{\mathsf{T}}QD \quad (43)$$
$$\hat{R} = D^{\mathsf{T}}QD + (D^{\mathsf{T}}S + S^{\mathsf{T}}D) + R. \quad (44)$$

C. Positive Real Systems

Positive real systems $H(s)$ have the following properties:

Definition 8: [11, p.51] [8, Definition 5.18] An $n \times n$ rational and proper matrix $H(s)$ is termed *positive real* (*PR*) if the following conditions are satisfied:

i) All elements of $H(s)$ are analytic in $\text{Re}[s] > 0$.
ii) $H(s)$ is real for real positive s.
iii) $H^*(s) + H(s) \geq 0$ for $\text{Re}[s] > 0$.

furthermore $H(s)$ is strictly positive real (*SPR*) if there $\exists \epsilon > 0$ s.t. $H(s-\epsilon)$ is positive real. Finally, $H(s)$ is strongly positive real if $H(s)$ is strictly positive real and $D + D^{\mathsf{T}} > 0$ where $D \triangleq H(\infty)$.

The test for positive realness can be simplified to a frequency test as follows:

Theorem 3: [11, p.216] [8, Theorem 5.11] Let $H(s)$ be a square, real rational transfer function. $H(s)$ is positive real iff the following conditions hold:

i) All elements of $H(s)$ are analytic in $\text{Re}[s] > 0$.
ii) $H^*(j\omega) + H(j\omega) \geq 0$ for $\forall \omega \in \mathbb{R}$ for which $j\omega$ is not a pole for any element of $H(s)$.
iii) Any pure imaginary pole $j\omega_o$ of any element of $H(s)$ is a simple pole, and the associated residue matrix $H_o \triangleq \lim_{s \to j\omega_o}(s - j\omega_o)H(s)$ is nonnegative definite Hermitian (i.e. $H_o = H_o^* \geq 0$).

A similar test is given for strict positive realness.

Theorem 4: [8, Theorem 5.17] Let $H(s)$ be a $n \times n$, real rational transfer function and suppose $H(s)$ is not singular. Then $H(s)$ is strictly positive real iff the following conditions hold:

i) All elements of $H(s)$ are analytic in $\text{Re}[s] \geq 0$.
ii) $H^*(j\omega) + H(j\omega) > 0$ for $\forall \omega \in \mathbb{R}$.
iii) Either $D + D^{\mathsf{T}} > 0$ or both $D + D^{\mathsf{T}} \geq 0$ and $\lim_{\omega \to \infty} \omega^2 Q^{\mathsf{T}}[(H^*(j\omega) + H(j\omega)]Q > 0$ for every $Q \in \mathbb{R}^{n \times (n-q)}$, where $q = \text{rank}(D + D^{\mathsf{T}})$, such that $Q^{\mathsf{T}}(D + D^{\mathsf{T}})Q = 0$.

Remark 9: The following theorem suggest that (12) and *strong positive realness* are equivalent. Which we will now show.

Lemma 2: Let $H(s)$ (with a corresponding minimal realization Σ) be a $n \times n$, real rational transfer function and suppose $H(s)$ is not singular. Then the following are equivalent:

i) $H(s)$ is *strongly positive real*
ii) Σ is asymptotically stable and

$$H(j\omega) + H^*(j\omega) \geq \delta I, \ \forall \omega \in \mathbb{R} \quad (45)$$

Proof: ii $\implies$ i:
Since Σ is asymptotically stable then all poles are in the open left half plane, therefore Theorem 4-i is satisfied. Next (45) clearly satisfies Theorem 4-ii. Also, (45) implies that $D + D^{\mathsf{T}} > \delta I > 0$ which satisfies 4-iii which satisfies the final condition to be *strictly-positive real* and also *strongly positive real* as noted in Definition 8.
i $\implies$ ii:
First we note that Theorem 4-i implies Σ will be asymptotically stable. Next, from Definition 8 we note that $\exists \delta_1 > 0$ s.t.

$$H^*(j\infty) + H(j\infty) = D^{\mathsf{T}} + D \geq \delta_1 I > 0$$

Lastly, we assume that $\exists \delta_2 \leq 0$ s.t.

$$H^*(j\omega) + H(j\omega) \geq \delta_2 I, \ \forall \omega(-\infty, \infty) \quad (46)$$

however this contradicts Theorem 4-ii therefore $\exists \delta_2 > 0$ s.t. (46) is satisfied which implies (45) is satisfied in which $\delta = \min\{\delta_1, \delta_2\} > 0$. ■

Remark 10: Note that Lemma 2-ii is equivalent to Σ being asymptotically stable and $H(s)$ being *strictly-input passive* as stated in Theorem 1-ib.

Finally, we state the Positive Real Lemma and the Strictly Positive Real Lemmas for the continuous time case.

Lemma 3: [11, p.218] Let $H(s)$ be an $n \times n$ matrix of real rational functions of a complex variable s, with $H(\infty) < \infty$. Let Σ be a minimal realization of $H(s)$. Then $H(s)$ is positive real iff there exists $P = P^{\mathsf{T}} > 0$ s.t.

$$\begin{bmatrix} A^{\mathsf{T}}P + PA & PB - C^{\mathsf{T}} \\ (PB - C^{\mathsf{T}})^{\mathsf{T}} & -(D^{\mathsf{T}} + D) \end{bmatrix} \leq 0 \quad (47)$$

Lemma 4: [12, Lemma 2.3] Let $H(s)$ be an $n \times n$ matrix of real rational functions of a complex variable s, with $H(\infty) < \infty$. Let Σ be a minimal realization of $H(s)$. Then $H(s)$ is strongly positive real iff there exists $P = P^{\mathsf{T}} > 0$ s.t. Σ is asymptotically stable and

$$\begin{bmatrix} A^{\mathsf{T}}P + PA & PB - C^{\mathsf{T}} \\ (PB - C^{\mathsf{T}})^{\mathsf{T}} & -(D^{\mathsf{T}} + D) \end{bmatrix} < 0. \quad (48)$$

Discrete time positive real systems $H(z)$ have the following properties:

Definition 9: [8, Definition 13.16] [13, Definition 2.4] A square matrix $H(z)$ of real rational functions is a *positive real* matrix if:

i) all the entries of $H(z)$ are analytic in $|z| > 1$ and,
ii) $H_o = H(z) + H^*(z) \geq 0, \ \forall |z| > 1$.

Furthermore $H(z)$ is *strictly-positive real* if $\exists 0 < \rho < 1$ s.t. $H(\rho z)$ is *positive real*. Unlike for the continuous time case there is no need to denote that $H(z)$ is strongly positive real when $H(z)$ is strictly positive real and $(D + D^{\mathsf{T}}) > 0$ where $D \triangleq H(\infty)$. For the discrete time case $(D + D^{\mathsf{T}}) > 0$ is implied as is noted in [14, Remark 4].

The test for a *positive real* system can be simplified to a frequency test as follows:

Theorem 5: [8, Theorem 13.26] Let $H(z)$ be a square, real rational $n \times n$ transfer function matrix. $H(z)$ is *positive real* iff the following conditions hold:

i) No entry of $H(z)$ has a pole in $|z| > 1$.
ii) $H(e^{j\theta}) + H^*(e^{j\theta}) \geq 0, \ \forall \theta \in [0, 2\pi]$, in which $e^{j\theta}$ is not a pole for any entry of $H(z)$.
iii) If $e^{j\hat{\theta}}$ is a pole of any entry of $H(z)$ it is at most a simple pole, and the residue matrix $H_o \triangleq \lim_{z \to e^{j\hat{\theta}}} (z - e^{j\hat{\theta}}) G(z)$ is nonnegative definite.

The test for a *strictly-positive real* system can be simplified to a frequency test as follows:

Theorem 6: [13, Theorem 2.2] Let $H(z)$ be a square, real rational $n \times n$ transfer function matrix in which $H(z) + H^*(z)$ has rank n almost everywhere in the complex z-plane. $H(z)$ is *strictly-positive real* iff the following conditions hold:

i) No entry of $H(z)$ has a pole in $|z| \geq 1$.
ii) $H(e^{j\theta}) + H^*(e^{j\theta}) \geq \epsilon I > 0, \ \forall \theta \in [0, 2\pi], \ \exists \epsilon > 0$.

Remark 11: Note that since $\theta \in [0, 2\pi]$ in the definition for *strictly-positive real*, then the stronger inequality with ϵ can be used as well. The following theorem suggest that (13) and *strictly-positive real* are equivalent which we now show.

Lemma 5: Let $H(z)$ (with a corresponding minimal realization Σ_z) be a square, real rational $n \times n$ transfer function matrix in which $H(z) + H^*(z)$ has rank n almost everywhere in the complex z-plane. Then the following are equivalent:

i) $H(z)$ is *strictly positive real*
ii) Σ_z is asymptotically stable and

$$H(e^{j\theta}) + H^*(e^{j\theta}) \geq \delta I, \ \forall \theta \in [0, 2\pi] \tag{49}$$

Proof: ii $\implies$ i:
Since Σ_z is asymptotically stable then all poles are strictly inside the unit circle, therefore Theorem 6-i is satisfied. Next (49) clearly satisfies Theorem 6-ii.
i $\implies$ ii:
First we note that Theorem 6-i implies Σ_z will be asymptotically stable. Finally Theorem 6-ii clearly satisfies (49). ■
Finally, we state the Positive Real Lemma and the Strictly Positive Real Lemmas for the discrete time case.

Lemma 6: [13, Theorem 3.7] Let $H(z)$ be an $n \times n$ matrix of real rational functions and let Σ_z be a stable realization of $H(z)$. Then $H(z)$ is *positive real* iff there exists $P = P^\mathsf{T} > 0$ s.t.

$$\begin{bmatrix} A^\mathsf{T} P A - P & A^\mathsf{T} P B - C^\mathsf{T} \\ (A^\mathsf{T} P B - C^\mathsf{T})^\mathsf{T} & -(D^\mathsf{T} + D) + B^\mathsf{T} P B \end{bmatrix} \leq 0. \tag{50}$$

Lemma 7: [14, Corollary 2] Let $H(z)$ be an $n \times n$ matrix of real rational functions and let Σ_z be an asymptotically stable realization of $H(z)$. Then $H(z)$ is *strictly-positive real* iff there exists $P = P^\mathsf{T} > 0$ s.t.

$$\begin{bmatrix} A^\mathsf{T} P A - P & A^\mathsf{T} P B - C^\mathsf{T} \\ (A^\mathsf{T} P B - C^\mathsf{T})^\mathsf{T} & -(D^\mathsf{T} + D) + B^\mathsf{T} P B \end{bmatrix} < 0. \tag{51}$$

III. Main Results

We now state the main result in regards to *passive* and *positive real* systems.

Lemma 8: Let $H(s)$ be an $n \times n$ matrix of real rational functions of a complex variable s, with $H(\infty) < \infty$. Let Σ be a minimal realization of $H(s)$. Furthermore we denote $H(t)$ as an $n \times n$ impulse response matrix of $H(s)$ in which the output $y(t)$ is computed as follows:

$$y(t) = \int_0^t H(t - \tau) u(\tau) d\tau$$

Then the following statements are equivalent:

i) $H(s)$ is *positive real*.
ii) There $\exists P = P^\mathsf{T} > 0$ s.t. (47) is satisfied.
iii) With $Q = R = 0, \ S = \frac{1}{2} I$ there $\exists P = P^\mathsf{T} > 0$ s.t. (29) is satisfied.
iv)
$$\int_0^\infty y^\mathsf{T}(t) u(t) dt \geq 0, \ \text{when } y(0) = 0$$

Proof: i $\Leftrightarrow$ ii:
Shown in Lemma 3.
iii $\Leftrightarrow$ iv:
iv is an equivalent test for *passivity* (see Remark 4) and Corollary 1 provides the necessary and sufficient test for *passivity*.
iii $\implies$ ii:
A *passive* system $H(s)$ is also *passive* iff $kH(s)$ is *passive* for $\forall k > 0$. Therefore (29) for $kH(s)$ in which $\Sigma = \{A, B, kC, kD\}$ and $Q = R = 0, \ S = \frac{1}{2} I, \ \hat{Q} = 0, \ \hat{S} = \frac{k}{2} C^\mathsf{T}, \ \hat{R} = \frac{k}{2}(D^\mathsf{T} + D)$:

$$\begin{bmatrix} A^\mathsf{T} P + P A & P B - \frac{k}{2} C^\mathsf{T} \\ (P B - \frac{k}{2} C^\mathsf{T})^\mathsf{T} & -\frac{k}{2}(D^\mathsf{T} + D) \end{bmatrix} \leq 0 \ , \tag{52}$$

which for $k = 2$ satisfies (47).
ii $\implies$ iii:
The converse argument can be made in which a *positive real* system $H(s)$ is *positive real* iff $kH(s)$ is *positive real* $\forall k > 0$ in which we choose $k = \frac{1}{2}$. ■

Remark 12: This theorem appears as [8, Theorem 5.], however, a different proof is provided which appears only valid when there are no poles on the imaginary axis in order to invoke Parseval's theorem. This stresses the importance which the dissipative definition for *passivity* allows us to make such a strong connection to a *positive real* system.

Lemma 9: Let $H(s)$ be an $n \times n$ matrix of real rational functions of a complex variable s, with $H(\infty) < \infty$. Let Σ be a minimal realization of $H(s)$. Furthermore we denote $H(t)$ as an $n \times n$ impulse response matrix of $H(s)$ in which the output $y(t)$ is computed as follows:

$$y(t) = \int_0^t H(t - \tau) u(\tau) d\tau$$

Then the following statements are equivalent:

i) $H(s)$ is *strongly positive real*.
ii) There $\exists P = P^\mathsf{T} > 0$ s.t. (48) is satisfied.
iii) Σ is asymptotically stable, and for $Q = 0, \ R = -\delta I$,$S = \frac{1}{2} I$ there $\exists P = P^\mathsf{T} > 0$ s.t. (29) is satisfied (*strictly-input passive* and *non-expansive*).
iv) Σ is asymptotically stable, and if $y(0) = 0$ then

$$\int_0^\infty y^\mathsf{T}(t) u(t) \geq \delta \|u(t)\|_2^2$$

in which $\delta = \inf_{-\infty\leq\omega\leq\infty} \mathtt{Re}\{H(j\omega)\}$ for the single input single output case.

Furthermore, iii implies that for $Q=-\epsilon I$, $R=0$, and $S=\frac{1}{2}I$ there $\exists P=P^{\mathsf{T}}>0$ s.t. (29) is also satisfied (*strictly-output passive*). Thus if $y(0)=0$ then

$$\int_0^\infty y^{\mathsf{T}}(t)u(t)dt \geq \epsilon\|y(t)\|_2^2$$

Remark 13: In order for the equivalence between *strongly positive real* and *strictly-input passive* to be stated, the *strictly-input passive* system must also have finite gain (i.e. Σ is asymptotically stable). For example the realization for $H(s)=1+\frac{1}{s},\ \Sigma=\{A=0,B=1,C=1,D=1\},\ \delta=1$ is *strictly-input passive* but is not asymptotically stable. However $H(s)=\frac{s+b}{s+a},\ \Sigma=\{A=-a,B=(b-a),C=D=1\},\delta=\min\{1,\frac{b}{a}\}$ is both *strictly-input passive* and asymptotically stable for all $a,b>0$.

Proof: i $\Leftrightarrow$ ii:
Is stated in Lemma 4.
ii $\Leftrightarrow$ iv:
The equivalence between asymptotic stability, *strictly-input passive* and *strongly positive real* is noted in Remark 10.
iii $\Leftrightarrow$ iv:
As noted in Definition 5. ■

Remark 14: It is well known that a *non expansive* system which is *strictly-input passive* $\implies$ that H is also *strictly-output passive* [6, Remark 2.3.5] [8, Proposition 5.2], the converse however, is not always true (i.e. $\inf_{\forall\omega}\mathtt{Re}\{H(j\omega)\}$ is zero for strictly proper (*strictly-output passive*) systems).
It has been shown for the continuous time case [6, Theorem 2.2.14] and discrete time case [7, Theorem 1] [10, Lemma C.2.1-(iii)] that a *strictly-output passive* system $\implies$ *non expansive* but it remains to be shown if the converse is true or not true. Indeed, we can show that an infinite number of continuous-time and discrete-time linear-time invariant systems do exists which are both passive and *non expansive* and are neither *strictly-output passive* (nor *strictly-input passive*).

Theorem 7: Let $H:\mathcal{H}_e\to\mathcal{H}_e$ (in which $y=Hu$, $y(0)=0$, and for the case when a state-space-description exists for H that it is zero-state-observable ($y=0$ implies that the state $x=0$) and there exists a positive definite storage function $\beta(x)>0,\ x\neq 0,\ \beta(0)=0$) have the following properties:

a) $\|(y)_T\|_2 \leq \gamma\|(u)_T\|_2$
b) $\langle y,u\rangle_T \geq -\delta\|(u)_T\|_2^2$
c) There exists a non-zero-normed input u such that $\langle y,u\rangle_T = -\delta\|(u)_T\|_2^2$ in which $\|(y)_T\|_2^2 \neq \delta^2\|(u)_T\|_2^2$.

Then the following system H_1 in which the output y_1 is computed as follows:

$$y_1 = y+\delta u \tag{53}$$

has the following properties:

I. H_1 is passive,
II. H_1 is non-expansive,
III. H_1 is neither *strictly-output passive* (nor *strictly-input passive*).

Proof: 7-I
Solving for the inner-product between y_1 and u we have

$$\langle y_1,u\rangle_T = \langle y,u\rangle_T + \delta\|(u)_T\|_2^2$$
$$\langle y_1,u\rangle_T \geq (-\delta+\delta)\|(u)_T\|_2^2 \geq 0.$$

7-II
Solving for the extended-two-norm for y_1 we have

$$\|(y_1)_T\|_2^2 = \|(y+\delta u)_T\|_2^2$$
$$\|(y_1)_T\|_2^2 \leq \|(y)_T\|_2^2 + \delta^2\|(u)_T\|_2^2$$
$$\|(y_1)_T\|_2^2 \leq (\gamma^2+\delta^2)\|(u)_T\|_2^2.$$

7-III
Recalling, from our proof for passivity, and our solution for the inner-product between y_1 and u, and substituting our final Assumption-c we have:

$$\langle y_1,u\rangle_T = (-\delta+\delta)\|(u)_T\|_2^2 = 0.$$

It is obvious that no constant $\delta>0$ exists such that

$$\langle y_1,u\rangle_T = 0 \geq \delta\|(u)_T\|_2^2$$

since it is assumed that $\|(u)_T\|_2^2>0$, hence H_1 is not *strictly-input passive*. In a similar manner, noting that with the added restriction that the following rare-case $\|(y)_T\|_2^2=\delta^2\|(u)_T\|_2^2$ does not occur for the same input function u when $\langle y,u\rangle_T = -\delta\|(u)_T\|_2^2$ holds, it is obvious that no constant $\epsilon>0$ exists such that

$$\langle y_1,u\rangle_T = 0 \geq \epsilon\|(y_1)_T\|_2^2$$
$$0 \geq \epsilon\left(\|(y)_T\|_2^2 + 2\delta\langle y,u\rangle_T + \delta^2\|(u)_T\|_2^2\right)$$
$$0 \geq \epsilon\left(\|(y)_T\|_2^2 - \delta^2\|(u)_T\|_2^2\right)$$

holds. ■

Corollary 3: The following continuous-time-system $H(s)$

$$H(s) = \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2},\ 0<\omega_n<\infty \tag{54}$$

satisfies the assumptions listed in Theorem 7 required of system H in which $\delta=\frac{1}{8}$ and an input-sinusoid $u(t)=\sin(\sqrt{3}\omega_n t)$ is a null-inner-product sinusoid such that:

$$H_1(s) = \frac{1}{8}+H(s) = \frac{1}{8}+\frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2},\ 0<\omega_n<\infty$$

is both passive and non-expansive but neither *strictly-output passive* nor *strictly-input passive*.
We now conclude with main results in regards to discrete time *passive* and *positive real* systems (the proofs follow along similar lines for the continuous time case).

Lemma 10: Let $H(z)$ be an $n\times n$ matrix of real rational functions of variable z. Let Σ_z be a minimal realization of $H(z)$ which is Lyapunov stable. Furthermore we denote $H[k]$ as an $n\times n$ impulse response matrix of $H(z)$ in which the output $y[k]$ is computed as follows:

$$y[k] = \sum_{i=0}^{k} H[k-i]u[i]$$

Then the following statements are equivalent:

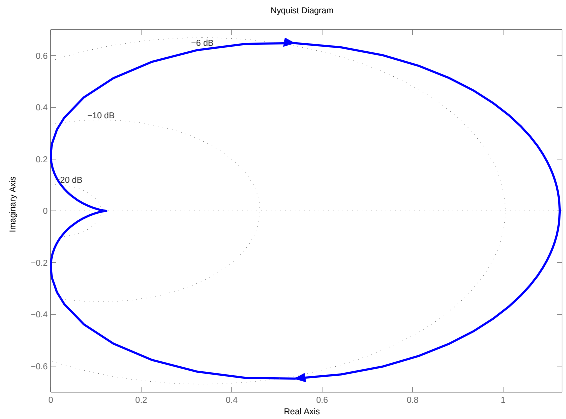

Fig. 1. Nyquist plot for $H_1(s) = \frac{1}{8} + \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2}$, $0 < \omega_n < \infty$.

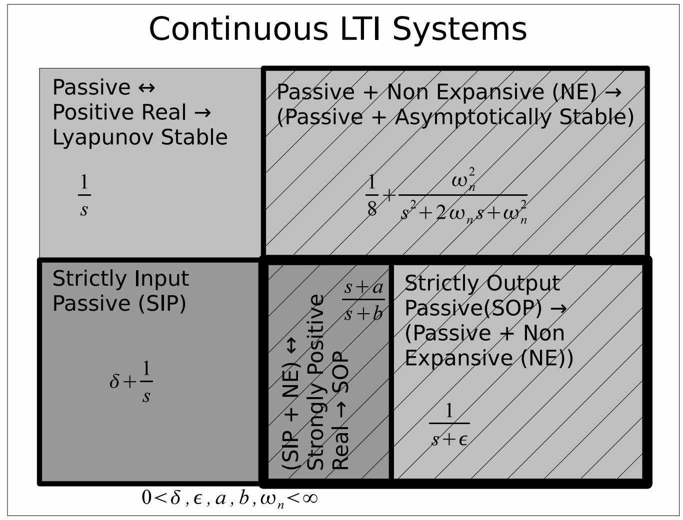

Fig. 2. Venn Diagram relating continuous *LTI* systems to *positive real* systems.

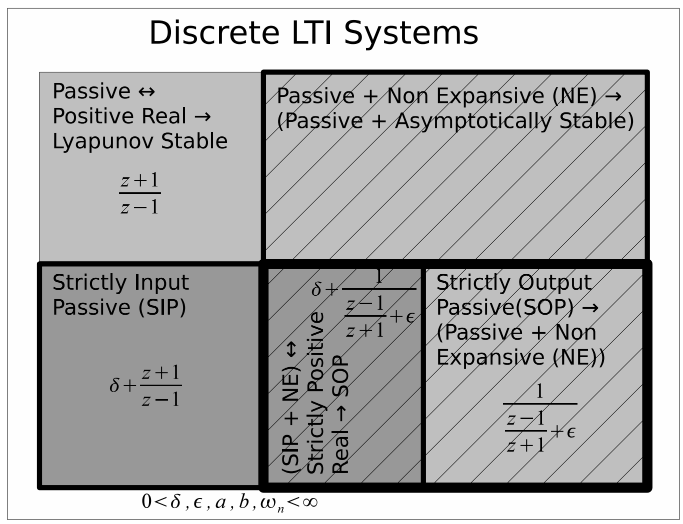

Fig. 3. Venn Diagram relating discrete *LTI* systems to *positive real* systems.

i) $H(z)$ is *positive real*.
ii) There $\exists P = P^\mathsf{T} > 0$ s.t. (50) is satisfied.
iii) With $Q = R = 0$, $S = \frac{1}{2}I$ there $\exists P = P^\mathsf{T} > 0$ s.t. (41) is satisfied.
iv) If $y[0] = 0$ then

$$\sum_{i=0}^{\infty} y^\mathsf{T}(i)u(i) \geq 0$$

Lemma 11: Let $H(z)$ be an $n \times n$ matrix of real rational functions of variable z. Let Σ_z be a minimal realization of $H(z)$ which is Lyapunov stable. Furthermore we denote $H[k]$ as an $n \times n$ impulse response matrix of $H(z)$ in which the output $y[k]$ is computed as follows:

$$y[k] = \sum_{i=0}^{k} H[k-i]u[i]$$

Then the following statements are equivalent:

i) $H(z)$ is *strictly-positive real*.
ii) There $\exists P = P^\mathsf{T} > 0$ s.t. (51) is satisfied.
iii) Σ_z is asymptotically stable, and for $Q = 0$, $R = -\delta I$, $S = \frac{1}{2}I$ there $\exists P = P^\mathsf{T} > 0$, and$\exists \delta > 0$ s.t. (41) is satisfied.
iv) Σ_z is asymptotically stable, and if $y[0] = 0$ then

$$\sum_{i=0}^{\infty} y^\mathsf{T}(i)u(i) \geq \delta \|u(i)\|_2^2$$

IV. Conclusions

Figure 2 (Figure 3) summarize many of the connections between continuous (discrete) time *passive* systems and *positive real* systems as noted in Section III. We believe all the results in Section III are original and unified (clarified many implicit assumptions in various statements) which are distributed around in the literature on this topic. For example the equivalence between a *passive* and bounded real system (Lemma 8) has been conjectured for years in which we note most recently the incomplete proof given in [8, Theorem 5.13] where Parseval's relation is to be used when $H(s)$ has poles only in the open left half complex plane. Many have made reference to [11] for such an equivalent statement however we find that *passivity* implied $H(s)$ to be *positive real* [11, Theorem 2.7.3] (there is also a necessary and sufficient test for a lossless system [11, Theorem 2.7.4]). We believe that [11, p.230, Time-Domain Statement of the Positive Real Property] could be what others are referring to, we offer our proof as a vastly simpler way of showing equivalence between the two systems. We note how much confusion can arise from statements such as those given in [15, Definition 1, Lemma 1, and Lemma 3] which fail to mention the implicit assumption that the *strictly-input passive* system is also *non-expansive* (or its minimal realization is asymptotically stable). Most importantly, Theorem 7 (Corollary 3) demonstrate how to construct an infinite number of (*LTI*) systems which are finite gain stable systems and *passive* but are neither *strictly-output passive* nor *strictly-input passive*.

V. Acknowledgements

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