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Asymptotic disturbance attenuation properties for uncertain switched linear systems☆

Hai Lin [a,*], Panos J. Antsaklis [b]

[a] ECE Department, National University of Singapore, Singapore 117576, Singapore

[b] EE Department, University of Notre Dame, Notre Dame, IN 46556, USA

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ABSTRACT

In this paper, the disturbance attenuation properties in the sense of uniformly ultimate boundedness are investigated for a class of switched linear systems with parametric uncertainties and exterior disturbances. The aim is to characterize the conditions under which the switched system can achieve a finite disturbance attenuation level. First, arbitrary switching signals are considered, and a necessary and sufficient condition is given. Secondly, conditions on how to restrict the switching signals to achieve finite disturbance attenuation levels are investigated. Two cases are considered here that depend on whether all the subsystems are uniformly ultimately bounded or not. Both discrete-time and continuous-time switched systems are considered, and the techniques are based on multiple polyhedral Lyapunov functions and their extensions.

1. Introduction

The last decade has seen increasing research activities in the field of switched systems, and the main efforts typically focus on the analysis of dynamic behaviors, such as stability, controllability, reachability, and observability, and aim to design controllers with guaranteed stability and optimized performance; see e.g., [1–7]. However, the literature on robust performance of switched systems is still relatively sparse, and most existing results assume that the disturbances are constrained to have finite energy, i.e., bounded $\mathcal{L}_2$ norm; see e.g. [3,8–10]. In practice, there are disturbances that do not satisfy this condition and act more or less continuously over time. Such disturbances are called persistent, and they cannot be treated in the above framework. In this paper, the disturbance attenuation property is in the signal's magnitude sense, i.e., time domain specifications. Moreover, we explicitly consider dynamic uncertainty in the switched system model. Dynamics uncertainty in the plant model is one of the main challenges in control theory, and it is of practical importance to deal with dynamical uncertainties explicitly.

In this paper, we aim to investigate the disturbance attenuation properties for classes of switched linear systems which are perturbed by both parameter variations and exterior disturbances. Both discrete-time and continuous-time cases are considered here. In particular, we are interested in switched linear systems whose subsystems are described by the perturbed difference equations with parametric uncertainties

$$x[k+1] = A_q(w)x[k] + E_q d[k], \quad k \in \mathbb{Z}^+, \tag{1}$$

or a collection of perturbed differential equations with parametric uncertainties

$$\dot{x}(t) = A_q(w)x(t) + E_q d(t), \quad t \in \mathbb{R}^+, \tag{2}$$

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* Corresponding author.
E-mail addresses: elelh@nus.edu.sg (H. Lin), antsaklis.1@nd.edu (P.J. Antsaklis).

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where $q \in Q = \{q_1, q_2, \ldots, q_N\}$, the state variable $x \in \mathbb{R}^n$, and the disturbance input $d \in \mathcal{D} \subset \mathbb{R}^r$. Assume that $\mathcal{D}$ is a C-set. The term C-set stands for a convex and compact set containing the origin in its interior. Assume polytopic uncertainty in (1) and (2), i.e., $A_q(w) = \sum_{j=1}^{v_q} w_j A_q^j$, $w_j \geq 0$ and $\sum_{j=1}^{v_q} w_j = 1$. Notice that A_q^j are known constant $n \times n$ matrices and the coefficients w_j are unknown and possibly time varying. Without loss of generality, we assume that $E_q \in \mathbb{R}^{n\times r}$ is a constant matrix. The logical rule that orchestrates switching between these subsystems generates switching signals, which are usually described as classes of maps, $\sigma : \mathbb{R}^+ \to Q$ (or sequences $\sigma : \mathbb{Z}^+ \to Q$).

It is known that the dynamic properties of a switched system depend not only on its subsystems but also on the switching signals. The aim of this paper is to characterize the dynamics of each subsystem and the properties of switching signals such that the switched systems generate convergent behaviors. Because of parameter variations and exterior disturbances, it is only reasonable to expect that the trajectories of the switched system converge into a neighborhood region of the equilibrium (the origin here), which is the so-called practical stability or uniformly ultimate boundedness in the literature.

Definition 1. The uncertain switched system under the switching signal $\sigma(t)$ (or $\sigma(k)$) is *Uniformly Ultimately Bounded (UUB)* if there exists a C-set $\mathcal{S}$ such that, for every initial condition $x(0) = x_0$, there exists a finite $T(x_0)$, and $x(t) \in \mathcal{S}$ for $t \geq T(x_0)$ (or $x[k] \in \mathcal{S}$ for $k \geq T(x_0)$).

The disturbance attenuation properties considered here are in the sense of the uniformly ultimate boundedness. Given a collection of switching signals, if the switched system is UUB for all these switching signals, then the switched system is said to have finite disturbance attenuation level under this class of switching signals. If the switched system is UUB for all possible switching signals, the switched system is said to have finite disturbance attenuation level under arbitrary switching. We are also interested in characterizing a useful subclass of switching signals such that the switched system achieves finite disturbance attenuation level even when not all its subsystems have finite disturbance attenuation levels. The characterization is mainly in the time domain, and sufficient conditions on the average dwell time and activation ratio are derived for switching signals based on the multiple Lyapunov function method. Here, we propose to use non-quadratic Lyapunov-like functions, namely polyhedral Lyapunov-like functions, to reduce the conservativeness as it is shown that there always exist polyhedral Lyapunov-like functions for each subsystem under certain conditions.

The paper is organized as follows. In Section 2, the disturbance attenuation properties under arbitrary switching are considered, where necessary and sufficient conditions are described. Section 3 studies the case when all the subsystems are UUB while not stable under arbitrary switching, and requirements on switching signals to guarantee UUB are identified based on multiple polyhedral Lyapunov functions and an average dwell time scheme. In Section 4, a more general case, namely when there are unstable subsystems, is investigated. Based on extensions of classical polyhedral Lyapunov functions for these unstable subsystems, multiple polyhedral Lyapunov-like functions are employed to identify conditions to achieve a finite disturbance attenuation level. Both discrete-time and continuous-time switched systems are considered here.

Notation: The letters $\mathcal{E}, \mathcal{P}, \mathcal{S}, \ldots$ denote sets, and $\partial\mathcal{P}$ the boundary of set $\mathcal{P}$. For any real $\lambda \geq 0$, the set $\lambda\mathcal{S}$ is defined as $\{x = \lambda y,\ y \in \mathcal{S}\}$. A polytope (bounded polyhedral set) $\mathcal{P}$ will be presented either by a set of linear inequalities $\mathcal{P} = \{x : F_i x \leq g_i,\ i = 1, \ldots, s\}$, or by the dual representation in terms of the convex hull of its vertex set $\mathrm{vert}(P) = \{x_j\}$, denoted by $\mathrm{Conv}\{x_j\}$.

2. Performance under arbitrary switching

When there is no restriction or *a priori* knowledge on the switching signals, arbitrary switchings are usually assumed. In addition, in the framework of multiple-controller design [11], it is often desirable to retain stability or boundedness under all possible switchings among these multiple controllers. If this can be guaranteed, then one may just focus on switching for better performance in design, and gain more flexibility in operation. Hence, we consider arbitrary switching in this section and derive necessary and sufficient conditions for the switched systems to achieve finite disturbance attenuation properties. For this, it is necessary to require that every subsystem has a finite disturbance attenuation level. However, even when all the subsystems of a switched system are UUB, it is still possible to construct a divergent trajectory from any initial state for such a switched system. Therefore, in general, the above subsystems' UUB assumption is not sufficient to assure a finite disturbance attenuation property for the switched system under arbitrary switching.

It is known that a linear system is UUB if and only if the corresponding autonomous system is asymptotically stable [12]. Therefore, this problem is transformed into a stability analysis problem for an autonomous switched system under arbitrary switching,[1] which has been studied in the literature extensively; see, e.g., [1,2] and the references there. However, most existing results are either based on or imply the existence of a common quadratic Lyapunov function, which is sufficient only. Here, necessary and sufficient conditions will be given below.

For such a purpose, we first consider the discrete-time case and introduce the following polytopic uncertain linear time-variant (LTV) system

$$x[k+1] = A(k)x[k], \tag{3}$$

[1] Since switched systems under restricted switching signals are basically nonlinear, it remains unclear whether the UUB property of a switched system under restricted switchings is equivalent to its stability or not. Hence, we will treat UUB directly in the next sections when we consider the performance under restricted switchings.

where $A(k) \in \mathcal{A} \hat{=} \mathrm{Conv}\{A_{q_1}^1, A_{q_1}^2, \ldots, A_{q_1}^{v_{q1}}, A_{q_2}^1, \ldots, A_{q_2}^{v_{q2}}, \ldots, A_{q_N}^1, \ldots, A_{q_N}^{v_{q_N}}\}$. In other words, the state matrix $A(t)$ of the above LTV system (3) is constructed by convex combinations (with time-variant coefficients) of all the subsystems' vertex matrices of the switched linear system (1). Let us denote the index set for these vertices as $i \in \mathfrak{I}$. A necessary and sufficient condition for the asymptotic stability of the above linear time-variant system (3) is stated as follows [13].

Lemma 1 *([13]). The LTV system (3) is globally asymptotically stable if and only if there exists a finite integer* $\mathbf{n}$ *such that*

$$\|A_{i_1}A_{i_2}\cdots A_{i_\mathbf{n}}\| < 1,$$

for all $\mathbf{n}$*-tuple* $A_{i_j} \in \mathrm{vert}(\mathcal{A})$*, where* $j = 1, \ldots, \mathbf{n}$*. Here the norm* $\|\cdot\|$ *stands for either the 1 norm or* ∞ *norm of a matrix.*

As an immediate consequence, necessary and sufficient conditions for the discrete-time switched systems (1) to achieve a finite asymptotic disturbance attenuation level under arbitrary switching can be expressed as the following theorem.

Proposition 1. *The following statements are equivalent:*

(1) *The discrete-time switched linear system (1) has a finite disturbance attenuation level;*
(2) *the autonomous switched linear system* $x[k+1] = A_{\sigma(k)}x[k]$*, where* $A_{\sigma(k)}(w) \in \{A_{q_1}(w), A_{q_2}(w), \ldots, A_{q_N}(w)\}$*, is globally asymptotically stable under arbitrary switching;*
(3) *the LTV system (3)* $x[k+1] = A(k)x[k]$*, where* $A(k) \in \mathcal{A}$*, is asymptotically stable;*
(4) *there exists a finite integer* $\mathbf{n}$ *such that*

$$\|A_{i_1}A_{i_2}\cdots A_{i_\mathbf{n}}\| < 1,$$

for all $\mathbf{n}$*-tuple* $A_{i_j} \in \mathrm{vert}(\mathcal{A})$*, where* $j = 1, \ldots, \mathbf{n}$. □

It is quite interesting that the robust stability of a polytopic uncertain LTV system, which has infinite number of possible dynamics (modes), is equivalent to the stability of a switched system under arbitrary switching between its finite number of vertex dynamics. Note that this is not a surprising result since this fact has already been implied by the finite vertex stability criteria for robust stability in the literature, e.g., [14,15]. By explicitly exploring this equivalence relationship, we may obtain some "new" stability criteria for switched linear systems using the existing robust stability results [14,15]. For example,

Proposition 2. *The discrete-time switched linear system (1) has a finite disturbance attenuation level under arbitrary switching if and only if there exists an integer* $m \geq n$ *and* $L \in \mathbb{R}^{n\times m}$*,* $\mathrm{rank}(L) = n$*, such that, for all* A_i*,* $i \in \mathcal{I}$*, there exists* $\bar{A}_i \in \mathbb{R}^{m\times m}$ *with the following properties:*

(1) $A_i^T L = L\bar{A}_i^T$,
(2) *each column of* $\bar{A}_i$ *has no more than n nonzero elements and*

$$\|\bar{A}_i\|_\infty = \max_{1\leq k\leq m} \sum_{l=1}^{m} |\hat{a}_{kl}| < 1. \quad \square$$

Following similar arguments, the above equivalence also holds for the continuous-time case, namely,

Proposition 3. *The following statements are equivalent:*

(1) *The continuous-time switched linear system (2) achieves a finite disturbance attenuation level under arbitrary switching;*
(2) *the undisturbed continuous-time switched linear system* $\dot{x}(t) = A_{\sigma(t)}(w)x(t)$*, where* $A_{\sigma(t)}(w) \in \{A_{q_1}(w), A_{q_2}(w), \ldots, A_{q_N}(w)\}$*, is globally asymptotically stable under arbitrary switching;*
(3) *the LTV system* $\dot{x}(t) = A(t)x(t)$*, where* $A(t) \in \mathcal{A}$*, is asymptotically stable;*
(4) *there exist a full column rank matrix* $M \in \mathbb{R}^{m\times n}$*,* $m \geq n$*, and a family of matrices* $\{\bar{A}_i \in \mathbb{R}^{m\times n} : i \in \mathfrak{I}\}$ *with strictly negative row dominating diagonal, such that the matrix relations* $MA_i = \bar{A}_i M$ *are satisfied.* □

The last necessary and sufficient algebraic condition originates from [14], which studied the uniform asymptotic stability for differential and difference inclusions. Based on the equivalence between the asymptotic stability of arbitrary switching linear systems and the robust stability of polytopic uncertain LTV systems, some well established converse Lyapunov theorems for LTV systems can be introduced for arbitrary switching linear systems. For example, the following result was adopted from [14].

Proposition 4. *If a (discrete-time or continuous-time) switched linear system is asymptotically stable under arbitrary switching signals, then there exists a polyhedral Lyapunov function, which is monotonically decreasing along the switched linear system's trajectories.* □

Comparing with existing converse Lyapunov theorems, e.g. [16–19], the above result has the following advantages. First, it shows that one may focus on polyhedral Lyapunov function without loss of generality. Second, there exist automated computational methods to calculate polyhedral Lyapunov functions [20–22]. In the following sections, we will employ multiple polyhedral Lyapunov functions and their extensions to study the stability issues for switched systems under constrained, instead of arbitrary, switching signals.

3. Performance under slow switching: UUB subsystems

If the finite disturbance attenuation level is not preserved under arbitrary switching, it is still possible to restrict the switching signals so as to achieve a finite disturbance attenuation level. It is shown in [8,23] that the stability and performance could be preserved under certain constrained switching signals, such as slow switching with bounded average dwell time. Therefore, it is interesting to classify the classes of switching signals under which the switched system remains UUB. The stability analysis with constrained switching has been usually pursued in the framework of multiple Lyapunov functions (MLFs) [1,2,24,25].

Following [26], we call a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ a *gauge function* if $\Psi(x) \geq 0$, $\Psi(x) = 0 \Leftrightarrow x = 0$; for $\mu > 0$, $\Psi(\mu x) = \mu\Psi(x)$; and $\Psi(x+y) \leq \Psi(x) + \Psi(y)$, $\forall x, y \in \mathbb{R}^n$. A gauge function is convex and it defines a distance of x from the origin which is linear in any direction. If Ψ is a gauge function, we define the closed set (possibly empty) $\bar{N}[\Psi, \xi] = \{x \in \mathbb{R}^n : \Psi(x) \leq \xi\}$. It is easy to show that the set $\bar{N}[\Psi, \xi]$ is a C-set for all $\xi > 0$. On the other hand, any C-set $\mathcal{S}$ induces a gauge function $\Psi_{\mathcal{S}}(x)$ (known as a Minkowski function of $\mathcal{S}$), which is defined as $\Psi(x) = \inf\{\mu > 0 : x \in \mu\mathcal{S}\}$. Therefore a C-set $\mathcal{S}$ can be thought of as the unit ball $\mathcal{S} = \bar{N}[\Psi, 1]$ of a gauge function Ψ and $x \in \mathcal{S} \Leftrightarrow \Psi(x) \leq 1$.

3.1. Discrete-time case

3.1.1. Polyhedral Lyapunov function

First, consider the discrete-time case and assume that each subsystem

$$x[k+1] = A(w)x[k] + Ed[k] \tag{4}$$

is UUB along with a Lyapunov function in the following sense.

Definition 2. Given a C-set $\mathcal{S}$, a Lyapunov function outside $\mathcal{S}$ for the subsystem (4) is defined as a continuous function $\Psi_q : \mathbb{R}^n \to \mathbb{R}^+$ such that $\bar{N}[\Psi_q, \kappa] \subset \mathcal{S}$, for some positive scalar κ, for which the following condition holds: if $x \notin \bar{N}[\Psi_q, \kappa]$ then $\exists 0 < \lambda_q < 1$ such that

$$\Psi_q(A_q(w)x + E_q d) \leq \lambda_q \Psi_q(x),$$

for all $w \in \mathcal{W}$ and $d \in \mathcal{D}$. □

It can be derived from the above Lyapunov function definition that

$$\Psi_q(x[k]) \leq \max\{\lambda_q^{k-k_0}\Psi_q(x[k_0]), \kappa\} \quad \forall k > k_0$$

for a trajectory $x[k]$ of (4) starting from $x[k_0]$ at time k_0. This further implies the following result.

Lemma 2 ([20]). *If there exists a Lyapunov function outside $\mathcal{S}$ for the subsystem (4), then it is uniformly ultimately bounded (UUB) in $\mathcal{S}$.* □

It can be shown that there always exists a polyhedral C-set and a polyhedral Lyapunov function outside this set for UUB LTV systems. Therefore, without loss of generality, we assume that (4) has a polyhedral Lyapunov function Ψ_q outside $\mathcal{S}$.

The next question is how to compute the polyhedral Lyapunov function $\Psi_q(\cdot)$. For this, we need the concept of a contractive set, which is defined as follows.

Definition 3. Given a scalar $0 < \lambda < 1$, a set $\mathcal{S}$ is said λ-*contractive* with respect to the system (4) if, for any $x \in \mathcal{S}$, $\mathrm{post}_q(x, \mathcal{W}, \mathcal{D}) \subseteq \lambda\mathcal{S}$. Here $\mathrm{post}_q(\cdot)$ is defined as

$$\mathrm{post}_q(x, \mathcal{W}, \mathcal{D}) = \{x' : x' = A_q(w)x + E_q d; \quad \forall w \in \mathcal{W}, d \in \mathcal{D}\},$$

which represents all the possible next step states of (4) from the current state x.

The determination of $\Psi_q(\cdot)$ outside $\mathcal{S}$ is through the calculation of a λ-contractive set inside $\mathcal{S}$, which can be achieved through the following iterative procedure [26].

Consider the following sequence of sets:

$$\{\mathcal{X}_k\} : \mathcal{X}_0 = \mathcal{S}, \qquad \mathcal{X}_k = \mathrm{pre}_q(\lambda\mathcal{X}_{k-1}) \cap \mathcal{S}; \quad k = 1, 2, \ldots \tag{5}$$

where $\mathrm{pre}_q(\mathcal{S})$ is defined as

$$\mathrm{pre}_q(\mathcal{S}) = \{x \in \mathbb{R}^n : \mathrm{post}_q(x, \mathcal{W}, \mathcal{D}) \subseteq \mathcal{S}\}. \tag{6}$$

Once the above procedure terminates, in the sense of $\mathcal{X}_{k+1} = \mathcal{X}_k$ (assume non-empty and contains the origin), then it returns a λ-contractive set contained in $\mathcal{S}$. The following result is adopted from [20,26].

Theorem 1. *If the above procedure (5) terminates and returns a nonempty C-set $\mathcal{P}_\lambda$ for some $0 < \lambda < 1$, then the system (4) is uniformly ultimately bounded (UUB) in $\mathcal{P}_\lambda \subseteq \mathcal{S}$. In addition, $\mathcal{P}_\lambda$ is a C-set, whose induced Minkowski functional $\Psi_{\mathcal{P}_\lambda}(x)$ serves as a Lyapunov function for (4) outside $\bar{N}[\Psi_{\mathcal{P}_\lambda}, 1] \subset \mathcal{S}$.* □

Such a Lyapunov function is uniquely generated from the target set $\mathcal{S}$ for any fixed λ, so it is named a Set-induced Lyapunov Function (SILF) in the literature; see [20,26] and its references. For systems with linearly constrained uncertainties, it can be shown that such a function may be derived by numerically efficient algorithms involving polyhedral sets; see e.g. [21,22].

3.1.2. UUB analysis

In this subsection, it is assumed that each subsystem is UUB with decay rate λ_q along with a polyhedral Lyapunov function, $\Psi_q(\cdot)$. Now, define the multiple Lyapunov function candidate as

$$V(x[k]) = \Psi_{\sigma(k)}(x[k]).$$

Let $k_1, k_2, \ldots$ stand for the time points at which switching occurs, and write q_j for the value of $\sigma(k)$ on $[k_{j-1}, k_j)$. Then, for any k satisfying $k_0 = 0 < \cdots < k_i \leq k < k_{i+1}$, we obtain

$$V(x[k]) \leq \max\{\lambda_{q_i}^{k-k_i}\Psi_{q_i}(x[k_i]), 1\}.$$

Also, there exists a constant scalar μ such that $\Psi_i(x) \leq \mu\Psi_j(x)$ and $\Psi_j(x) \leq \mu\Psi_i(x)$, for all $x \in \mathbb{R}^n$. A possible choice for μ is the largest value among $\Psi_j(v_j)$, $\forall v_j \in \mathrm{vert}\{\bar{N}[\Psi_i, 1]\}$, and $\Psi_i(v_j)$, $\forall v_j \in \mathrm{vert}\{\bar{N}[\Psi_j, 1]\}$. This can be verified by exploring the geometric property of the level sets of $\Psi_i(x)$ and $\Psi_j(x)$, which is shown in the Appendix.

Denote $\lambda_0 = \max_{q\in Q}\{\lambda_q\}$. Then

$$\begin{aligned} V(x[k]) &\leq \max\{\lambda_0^{k-k_i}\Psi_{q_i}(x[k_i]), 1\} \\ &\leq \max\{\lambda_0^{k-k_{i-1}}\mu\Psi_{q_{i-1}}(x[k_i]), 1\} \\ &\leq \cdots \\ &\leq \max\{\lambda_0^k\mu^i\Psi_{q_0}(x[0]), 1\} \\ &= \max\{\lambda_0^k\mu^{N_\sigma(k)}V(x[0]), 1\} \end{aligned}$$

where $N_\sigma(k)$ denotes the number of switchings of $\sigma(k)$ over the interval $[0, k)$. Assume that there exists a scalar $0 < \lambda^* < 1$ such that

$$\lambda_0^k\mu^{N_\sigma(k)} \leq (\lambda^*)^k. \tag{7}$$

This inequality is equivalent to

$$N_\sigma(k) \leq \frac{k}{\tau_a^*}, \qquad \tau_a^* = \frac{\ln\mu}{\ln\lambda^* - \ln\lambda_0} \tag{8}$$

which is exactly an average dwell time scheme. The constant τ_a^* is called the *average dwell time*. The idea is that there may exist consecutive switchings separated by less than τ_a^*, but the average time interval between consecutive switchings is not less than τ_a^*. Note that the concept of average dwell time between subsystems was originally proposed for continuous-time switched systems in [23], and was extended to the discrete-time case in [27].

From the average dwell switching scheme, we obtain

$$V(x[k]) \leq \max\{(\lambda^*)^kV(x[0]), 1\}.$$

This implies that the entire system is UUB. In summary, we have

Theorem 2. *If all subsystems of the discrete-time switched system (1) are UUB, then the switched system (1) achieves a finite asymptotic disturbance attenuation property under switching signals with average dwell time no less than $\frac{\ln\mu}{\ln\lambda^*-\ln\lambda_0}$ in the sense of (8).*

3.2. Continuous-time case

3.2.1. Polyhedral Lyapunov function

The technique of multiple polyhedral Lyapunov functions and the procedure (5) for discrete-time subsystem (4) can be extended in parallel to the continuous-time case.

It is also assumed that each continuous-time subsystem

$$\dot{x}(t) = A_q(w)x(t) + E_qd(t) \tag{9}$$

is UUB along with a Lyapunov function in the following sense.

Definition 4. Given a C-set $\mathcal{S}$, a Lyapunov function outside $\mathcal{S}$ for the subsystem (9) is defined as a continuous function $\Psi_q : \mathbb{R}^n \to \mathbb{R}^+$ such that $\bar{N}[\Psi_q, \kappa] \subset \mathcal{S}$, for some positive scalar κ, for which the following condition holds: if $x \notin \bar{N}[\Psi, \kappa]$ then $\exists \beta_q > 0$ such that

$$\mathcal{D}^+\Psi_q(x(t)) \leq -\beta_q \Psi_q(x(t)). \quad \Box$$

Here $\mathcal{D}^+\Psi_q(x(t))$ stands for the upper right Dini derivative of $\Psi_q(x(t))$ along the trajectories of system (9), which is defined as

$$\mathcal{D}^+\Psi_q(x(t)) = \lim_{\tau \to 0^+} \sup \frac{\Psi_q(x(t+\tau)) - \Psi_q(x(t))}{\tau}.$$

Under the assumption that $d(t)$ and $w(t)$ are continuous, then the value of the Dini derivative of the point $x(t) = x$ equals

$$\mathcal{D}^+\Psi_q(x(t)) = \lim_{\tau \to 0^+} \sup \frac{\Psi_q(x + \tau[A_q(w)x + E_q d]) - \Psi_q(x)}{\tau}$$

where $x(t) = x$, $d(t) = d$ and $w(t) = w$ [20].

Based on differential inequality theory, it can be derived from the above Lyapunov function definition that

$$\Psi_q(x(t)) \leq \max\{e^{-\beta_q(t-t_0)}\Psi_q(x(t_0)), \kappa\} \quad \forall t > t_0$$

for a trajectory $x(t)$ of (9) starting from $x(t_0)$ at time t_0.

Similar to the discrete-time case, the existence of a Lyapunov function outside $\mathcal{S}$ for the continuous-time subsystem (9) implies that (9) is uniformly ultimately bounded (UUB) in $\mathcal{S}$. In addition, it is without loss of generality to assume that each subsystem (9) has a polyhedral Lyapunov function Ψ_q outside $\mathcal{S}$. To calculate Ψ_q for the continuous-time system (9), we need the Euler approximating system (EAS) of (9) as the following discrete-time system:

$$x(t+1) = [I + \tau A_q(w)]x(t) + \tau E_q d(t). \tag{10}$$

The connection between the EAS (10) and its original continuous-time system (9) is through the lemma [20].

Lemma 3 (*[20]*). *If there exists a C-set $\mathcal{P}$ that is λ-contractive with respect to the EAS (10) for some $0 < \lambda < 1$ and $\tau > 0$, then the Minkowski function of $\mathcal{P}$, denoted as $\Psi_{\mathcal{P}}(x)$, has negative upper right Dini derivative along the trajectories of the continuous-time system (9).* $\Box$

Therefore, the determination of a polyhedral Lyapunov function Ψ_q (outside $\mathcal{S}$) reduces to the calculation of a λ-contractive set with respect to the EAS (10) (contained in $\mathcal{S}$), which can be solved by applying the procedure (5) to the EAS (10). Then, the following result holds.

Proposition 5. *If the procedure (5) for the EAS (10) terminates and returns a nonempty C-set $\mathcal{P}_\lambda$ for some $0 < \lambda < 1$ and $\tau > 0$, then the system (9) is uniformly ultimately bounded (UUB) in $\mathcal{P}_\lambda \subseteq \mathcal{S}$. In addition, $\mathcal{P}_\lambda$ is a C-set, whose induced Minkowski functional $\Psi_{\mathcal{P}_\lambda}(x)$ serves as a Lyapunov function for (9) outside $\bar{N}[\Psi_{\mathcal{P}_\lambda}, 1] \subset \mathcal{S}$. In particular, $N[\Psi_{\mathcal{P}_\lambda}, 1] \subset \mathcal{S}$, and for $x \notin N[\Psi, 1]$*

$$\mathcal{D}^+\Psi_{\mathcal{P}_\lambda}(x(t)) \leq -\frac{1-\lambda}{\tau}\Psi_{\mathcal{P}_\lambda}(x(t)). \quad \Box$$

The proof of this proposition can be found in [20,26].

3.2.2. UUB analysis

In what follows, it is assumed that each subsystem (9) is UUB with decay rate β_q along with a polyhedral Lyapunov function, $\Psi_q(\cdot)$. Now, define the multiple Lyapunov function candidate as

$$V(x(t)) = \Psi_{\sigma(t)}(x(t)).$$

Let $t_1, t_2, \ldots$ stand for the time points at which switching occurs, and write q_j for the value of $\sigma(t)$ on $[t_{j-1}, t_j)$. Then, for any t satisfying $t_0 = 0 < \cdots < t_i \leq t < t_{i+1}$, we obtain

$$V(x(t)) \leq \max\{e^{-\beta_{q_i}(t-t_i)}\Psi_{q_i}(x(t_i)), 1\}.$$

There exists a constant scalar μ such that $\Psi_{q_i}(x) \leq \mu\Psi_{q_j}(x)$ and $\Psi_{q_j}(x) \leq \mu\Psi_{q_i}(x)$, for all $x \in \mathbb{R}^n$.
Denote $\beta_0 = \min_{q\in Q}\{\beta_q\}$. Then

$$\begin{aligned}
V(x(t)) \leq \max\{e^{-\beta_{q_i}(t-t_i)}\Psi_{q_i}(x(t_i)), 1\} &\leq \max\{e^{-\beta_0(t-t_i)}\Psi_{q_i}(x(t_i)), 1\} \\
&\leq \max\{e^{-\beta_0(t-t_i)}\mu\Psi_{q_{i-1}}(x(t_i)), 1\} \\
&\leq \max\{e^{-\beta_0(t-t_i)}\mu e^{-\beta_0(t_i-t_{i-1})}\Psi_{q_{i-1}}(x(t_{i-1})), 1\} \\
&\leq \cdots \\
&\leq \max\{e^{-\beta_0 t}\mu^i\Psi_{q_0}(x(0)), 1\} \\
&= \max\{e^{-\beta_0 t}\mu^{N_\sigma(t)}V(x(0)), 1\}
\end{aligned}$$

where $N_\sigma(t)$ denotes the number of switchings of $\sigma(t)$ over the interval $[0, t)$. Assume that there exists a scalar $0 < \beta^* < \beta_0$ such that

$$\mathrm{e}^{-\beta_0 t}\mu^{N_\sigma(t)} \le \mathrm{e}^{-\beta^* t}. \tag{11}$$

This inequality is equivalent to

$$N_\sigma(t) \le \frac{t}{\tau_a^*}, \quad \tau_a^* = \frac{\ln\mu}{\beta_0 - \beta^*} \tag{12}$$

which is exactly an average dwell time scheme.

Then we obtain

$$V(x(t)) \le \max\{\mathrm{e}^{-\beta^* t}V(x(0)), 1\}.$$

This implies that the entire system is UUB. In summary, we have

Theorem 3. *If all the subsystems of switched system (2) are UUB, then the switched system (2) achieves a finite asymptotic disturbance attenuation level under switching signals with average dwell time no less than $\tau_a^* = \frac{\ln\mu}{\beta_0-\beta^*}$ in the sense of (12).* □

4. Performance under slow switching: With non-UUB subsystems

In the previous section, we specified a class of slow switching signals that guarantee the uniformly ultimate boundedness for uncertain switched linear systems with stable subsystems. However, there are some cases that it is unavoidable to switch to unstable subsystems, such as controller failure in fault tolerant systems, packet dropouts in networked control systems, etc.

In this section, we will study the case when not all the subsystems are uniformly ultimately bounded. Without lost of generality, it is assumed that the first r subsystem are UUB along with a Lyapunov function, while the remaining subsystems are not UUB. To make the problem tractable, the expansion rates of these unstable subsystems are limited. In particular, we assume that the expansion of the unstable subsystems are bounded in the sense of polyhedral Lyapunov-like functions, which are introduced below and represent extensions of classical polyhedral Lyapunov functions.

4.1. Discrete-time case

4.1.1. Polyhedral Lyapunov-like function

For a subsystem that is not UUB, there does not exist a polyhedral Lyapunov function as derived in the previous section. Therefore, we generalize the concept of the λ-contractive set and derive a polyhedral Lyapunov-like function. For such a purpose, we first introduce the following definition for a Lyapunov-like function.

Definition 5. Given a C-set $\mathcal{S}$, a Lyapunov-like function outside $\mathcal{S}$ for the subsystem (4) is defined as a continuous function $\Psi_q : \mathbb{R}^n \to \mathbb{R}^+$ such that $\bar{N}[\Psi_q, \kappa] \subset \mathcal{S}$, for some positive scalar κ, for which the following condition holds: if $x \notin \bar{N}[\Psi_q, \kappa]$ then $\exists\lambda_q > 1$ such that

$$\Psi_q(A_q(w)x + E_q d) \le \lambda_q\Psi_q(x),$$

for all $w \in \mathcal{W}$ and $d \in \mathcal{D}$. □

This Lyapunov-like function outside $\mathcal{S}$ definition is quite similar to the definition of Lyapunov function outside $\mathcal{S}$ in the previous section. The difference here is that the value of a Lyapunov-like function increases at every step instead of decreasing. To capture this trend of expansion in the state space, we introduce the following expansive set definition, which is the counterpart to contractive set in the previous section.

Definition 6. Given a scalar $\lambda > 1$, a set $\mathcal{S}$ is said to be λ-*expansive* with respect to the discrete-time subsystem (4) if, for any $x \in \mathcal{S}$ such that $\mathrm{post}_q(x, \mathcal{W}, \mathcal{D}) \subseteq \lambda\mathcal{S}$.

Definition 7. The subsystem (4) is said to have expansive index $\lambda > 1$ to the C-set $\mathcal{S}$ iff there exists a gauge function $\Psi(x)$ and a constant $\xi > 0$ such that the ball $\bar{N}[\Psi, \xi] \subseteq \mathcal{S}$ and, if $x \notin int\{\bar{N}[\Psi, \xi]\}$, then $\Psi(\mathrm{post}_q(x, w, d)) \le \lambda\Psi(x)$ for all $w \in \mathcal{W}$ and $d \in \mathcal{D}$ (or, equivalently, $\bar{N}[\Psi, \mu]$ is λ-expansive for all $\mu \ge \xi$).

Intuitively, the concepts of λ-expansive set and expansive index λ reflect how explosive the unstable subsystems are. For LTI subsystems, this is related to the magnitude of their unstable eigenvalues. It is straightforward to show that a λ-expansive set has the following property, just like a λ-contractive set.

Lemma 4. *If $\mathcal{P}$ is λ-expansive set for the system (9), then $\mu\mathcal{P}$ is so for all $\mu \ge 1$. (If $\mathcal{D} = \{0\}$, for all $\mu \ge 0$.)*

Proof. Let $x \in \mu\mathcal{P}$, hence $\mu^{-1}x \in \mathcal{P}$, so $\text{post}_q(\mu^{-1}x, \mathcal{W}, \mathcal{D}) \subset \lambda\mathcal{P}$. Note that $\mu^{-1}\mathcal{D} \subset \mathcal{D}$, so $\text{post}_q(x, \mathcal{W}, \mathcal{D}) = \mu \times \text{post}_q(\mu^{-1}x, \mathcal{W}, \mu^{-1}\mathcal{D}) \subset \mu \times \text{post}_q(\mu^{-1}x, \mathcal{W}, \mathcal{D}) \subset \lambda\mu\mathcal{P}$. ■

The next question is how to determine such a λ-expansive set for an unstable subsystem. It turns out that the procedure developed for contractive sets can be extended to the case of an expansive set in parallel, which is described in the following.

Let $\mathcal{S}$ be assigned. Consider the following sequence of sets:

$$\{\mathcal{X}_k\} : \mathcal{X}_0 = \mathcal{S}, \qquad \mathcal{X}_k = \text{pre}_q(\lambda\mathcal{X}_{k-1}) \cap \mathcal{S}; \quad k = 1, 2, \dots. \tag{13}$$

We say that a λ-expansive set $\mathcal{P}_\lambda \subseteq \mathcal{S}$ is *maximal* in $\mathcal{S}$ iff every λ-expansive set $\mathcal{P}$ contained in $\mathcal{S}$ is also contained in $\mathcal{P}_\lambda$.

Proposition 6. *The maximal λ-expansive set $\mathcal{P}_\lambda \subseteq \mathcal{S}$ is given by $\mathcal{P}_\lambda = \bigcap_{k=0}^{\infty} \mathcal{X}_k$.*

Proof. First, we show that $\mathcal{X}_{k+1} \subset \mathcal{X}_k$. Indeed, $\mathcal{X}_1 \subset \mathcal{X}_0$. Assume that $\mathcal{X}_k \subset \mathcal{X}_{k-1}$; then $\lambda\mathcal{X}_k \subset \lambda\mathcal{X}_{k-1}$, so $\mathcal{X}_{k+1} = \text{pre}_q(\lambda\mathcal{X}_k) \cap \mathcal{S} \subset \text{pre}_q(\lambda\mathcal{X}_{k-1}) \cap \mathcal{S} = \mathcal{X}_k$.

Next, we prove that, if $\mathcal{P}_\lambda$ is nonempty, then it is λ-expansive. If $x \in \mathcal{P}_\lambda$, then $x \in \mathcal{X}_k$ for all k. For $h \geq 0$ and $k \geq h$, $post(x, \mathcal{W}, \mathcal{D}) \subset \lambda\mathcal{X}_{k-1} \subset \lambda\mathcal{X}_{h-1}$. For all $w \in \mathcal{W}$ and $d \in \mathcal{D}$, $\text{post}_q(x, w, d) \in \lambda\mathcal{X}_{h-1}$. Since h is arbitrary, so $\text{post}_q(x, w, d) \in \lambda\mathcal{P}_\lambda$. Therefore, $\mathcal{P}_\lambda$ is λ-expansive.

Finally, we prove that $\mathcal{P}_\lambda$ is maximal. Let $\mathcal{P} \subset \mathcal{X}_0$ be λ-expansive. Assume $\mathcal{P} \subset \mathcal{X}_k$. For any $x \in \mathcal{P}$, $\text{post}_q(x, \mathcal{W}, \mathcal{D}) \subset \lambda\mathcal{P} \subset \lambda\mathcal{X}_k$. Hence, $x \in \mathcal{X}_{k+1}$, then $\mathcal{P} \subset \mathcal{X}_{k+1}$. Therefore, $\mathcal{P} \subset \mathcal{X}_k$ for all k. Thus, $\mathcal{P} \subset \mathcal{P}_\lambda$. ■

The above iterative procedure for determining the maximal λ-expansive set may fail to terminate in finite steps. However, under certain conditions, the maximal λ-expansive set $\mathcal{P}_\lambda$ could be determined by finite iterations as shown below.

Proposition 7. *Assume that $\mathcal{P}_\lambda$ is a C-set, and $\lambda > 1$. Then for every λ^* such that $1 < \lambda < \lambda^*$, there exists $\mathbf{k}$ such that $\mathcal{X}_k$ is λ^*-expansive for all $k \geq \mathbf{k}$.*

Proof. Let $\xi = \frac{\lambda^*}{\lambda} > 1$. There exists $\mathbf{k}$ such that $\mathcal{P}_\lambda \subset \mathcal{X}_k \subset \xi\mathcal{P}_\lambda$, for $k \geq \mathbf{k}$. Since $\xi\mathcal{P}_\lambda$ is λ-expansive (from Lemma 4), for any $x \in \mathcal{X}_k$, so $x \in \xi\mathcal{P}_\lambda$, and $\text{post}_q(x, \mathcal{W}, \mathcal{D}) \subset \lambda\xi\mathcal{P}_\lambda \subset \lambda\xi\mathcal{X}_k = \lambda^*\mathcal{X}_k$. ■

With the existence and determination of a non-empty maximal λ-expansive set $\mathcal{P}_\lambda \subseteq \mathcal{S}$, we may induce a Lyapunov-like function from $\mathcal{P}_\lambda$.

Proposition 8. *If $\mathcal{P}_\lambda = \bigcap_{k=0}^{\infty} \mathcal{X}_k$ is a nonempty C-set, then its Minkowski function $\psi(x) = \Psi_{\mathcal{P}_\lambda}(x)$ is a Lyapunov-like function for the subsystem (9) outside $\mathcal{P}_\lambda \subseteq \mathcal{S}$.*

This is straightforward to verify based on the definitions of a λ-expansive set and Lyapunov-like functions.

4.1.2. UUB analysis

Based on similar techniques in [20], it can be argued that it does not cost generality to focus only on the polyhedral Lyapunov-like functions under the existence assumption of a Lyapunov-like function for (4). Therefore, it is assumed that each unstable subsystem has an expansive index $\lambda_q > 1$ to $\mathcal{S}$ along with a polyhedral Lyapunov-like function, $\Psi_q(\cdot)$. For the UUB subsystems, the existence of polyhedral Lyapunov-like functions $\Psi_q(\cdot)$ outside $\mathcal{S}$ is still assumed. Similarly, define the multiple Lyapunov function candidate as

$$V(x[k]) = \Psi_{\sigma(k)}(x[k]). \tag{14}$$

For any switching signal $\sigma(k)$ and any $k > 0$, let $K_i(k)$ denote the total period that the q_i-th subsystem is activated during $[0, k)$. Define $K^-(k) = \sum_{i \leq r,\ q_i \in Q} K_i(k)$, which stands for the total activation period of the UUB subsystems. On the other hand, $K^+(k) = \sum_{i > r,\ q_i \in Q} K_i(k)$ denotes the total activation period of the non-UUB subsystems. We have $K^-(k) + K^+(k) = k$.

For any k satisfying $k_0 = 0 < \dots < k_i \leq k < k_{i+1}$, we obtain

$$V(x[k]) \leq \max\{\lambda_{q_i}^{k-k_i}\Psi_{q_i}(x[k_i]), 1\}. \tag{15}$$

Let us define $\lambda_s = \max_{1 \leq i \leq r}\{\lambda_{q_i}\} < 1$, and $\lambda_u = \max_{r < i \leq N}\{\lambda_{q_i}\} \geq 1$. Also, there exists a constant scalar μ such that $\Psi_i(x) \leq \mu\Psi_j(x)$ and $\Psi_j(x) \leq \mu\Psi_i(x)$, for all $x \in \mathbb{R}^n$. The scalar μ can be selected as in the previous section.

Therefore, by induction, we have

$$V(x[k]) \leq \max\{\lambda_s^{K^-(k)}\lambda_u^{K^+(k)}\mu^{N_\sigma(k)}V(x[0]), 1\}.$$

If there exists a positive scalar $0 < \lambda < 1$ such that

$$\frac{K^+(k)}{k} \leq \frac{\ln\lambda - \ln\lambda_s}{\ln\lambda_u - \ln\lambda_s} \tag{16}$$

which is a condition on the percentage of time interval that the unstable subsystems are activated, then we obtain

$$\left(\frac{\lambda_u}{\lambda_s}\right)^{K^+(k)} \leq \left(\frac{\lambda}{\lambda_s}\right)^k \Leftrightarrow \lambda_s^{K^-(k)}\lambda_u^{K^+(k)} \leq \lambda^k. \tag{17}$$

And thus

$$V(x[k]) \leq \max\{\lambda^k \mu^{N_\sigma(k)} V(x[0]), 1\}.$$

Assume that

$$\lambda^k \mu^{N_\sigma(k)} \leq (\lambda^*)^k \tag{18}$$

for some $0 < \lambda^* < 1$. This inequality is equivalent to

$$N_\sigma(k) \leq \frac{k}{\tau_a^*}, \quad \tau_a^* = \frac{\ln \mu}{\ln \lambda^* - \ln \lambda} \tag{19}$$

which is an average dwell time scheme as well. Therefore

$$V(x[k]) \leq \max\{(\lambda^*)^k V(x[0]), 1\}.$$

This implies that the entire system is UUB.

Theorem 4. *The system is UUB under a switching signal $\sigma(k)$ if there exist two scalars λ^* and λ such that the $K^+(k)$ and $N_\sigma(k)$ satisfy the following two conditions:*

(1) $\frac{K^+(k)}{k} \leq \frac{\ln \lambda_s + \ln \lambda}{\ln \lambda_s + \ln \lambda_u}$ *holds for some scalar* $\lambda_s < \lambda < 1$*;*
(2) *the average dwell time is not smaller than τ_a^*, i.e., $N_\sigma(k) \leq \frac{k}{\tau_a^*}$, where $\tau_a^* = \frac{\ln \mu}{\ln \lambda^* - \ln \lambda}$.*

Compared with the result for all subsystems being UUB, the switching signals in this section have one condition on the percentage of the activation periods of unstable subsystems in addition to a similar average dwell time condition. In other words, a switched system that stays too long in the unstable mode may lead to an unbounded disturbance attenuation performance.

4.2. Continuous-time case

4.2.1. Polyhedral Lyapunov-like functions

First, the definition of Lyapunov-like functions for the continuous-time unstable subsystems can be given as follows.

Definition 8. A Lyapunov-like function outside $\mathcal{S}$ for the continuous-time subsystem (1) can be defined as a continuous function $\Psi_q : \mathbb{R}^n \to \mathbb{R}^+$ such that $\bar{N}[\Psi_q, \kappa] \subset \mathcal{S}$, for some positive scalar κ, for which the following condition holds: if $x \notin \bar{N}[\Psi_q, \kappa]$ then there exists $\beta_q > 0$ such that $\mathcal{D}^+\Psi_q(x(t)) \leq \beta_q \Psi_q(x(t))$. □

Applying the procedure (13) to a continuous-time unstable subsystem's EAS (10), one may obtain a λ-expansive set with respect to the EAS (10). Similar to the case of contractive set, we may induce a Lyapunov-like function from $\mathcal{P}_\lambda$ for the continuous-time unstable subsystem (1) as the following result implies.

Proposition 9. *Given a C-set $\mathcal{S}$, if the procedure (13) terminates and returns a nonempty C-set $\mathcal{P}_\lambda$ for some $\lambda > 1$ and $\tau > 0$, then the Minkowski function of $\mathcal{P}_\lambda$, $\Psi_{\mathcal{P}_\lambda}(x)$, is a Lyapunov-like function for the system (1) outside $\mathcal{P}_\lambda \subseteq \mathcal{S}$. In particular, $\bar{N}[\Psi_{\mathcal{P}_\lambda}, 1] \subset \mathcal{S}$, and for $x \notin \bar{N}[\Psi_{\mathcal{P}_\lambda}, 1]$, $\mathcal{D}^+\Psi_{\mathcal{P}_\lambda}(x(t)) \leq \frac{\lambda-1}{\tau}\Psi_{\mathcal{P}_\lambda}(x(t))$.* □

Proof. Since $\mathcal{P}_\lambda$ is a nonempty C-set, for $x \notin \bar{N}[\Psi_{\mathcal{P}_\lambda}, 1]$, there exists a $\mu \geq 1$ such that x lies on the boundary of set $\bar{N}[\Psi_{\mathcal{P}_\lambda}, \mu]$, i.e., $x \in \partial\bar{N}[\Psi_{\mathcal{P}_\lambda}, \mu]$. In addition, $\mathcal{P}_\lambda$ is λ-expansive with respect to system (10), so is $\mu\mathcal{P}_\lambda$ for $\mu \geq 1$. Therefore, for $x \in \partial\bar{N}[\Psi_{\mathcal{P}_\lambda}, \mu]$, $\mathrm{post}_q(x, \mathcal{W}, \mathcal{D}) \subseteq \lambda\mu\mathcal{P}_\lambda$. Hence, for $w \in \mathcal{W}$, $d \in \mathcal{D}$, $\Psi_{\mathcal{P}_\lambda}([I + \tau A(w)]x + \tau Ed) \leq \lambda\Psi_{\mathcal{P}_\lambda} = \lambda\mu$ holds for some $\tau > 0$. So,

$$\frac{\Psi_{\mathcal{P}_\lambda}([I + \tau A_q(w)]x + \tau E_q d) - \Psi_{\mathcal{P}_\lambda}(x)}{\tau} \leq \frac{\lambda - 1}{\tau}\Psi_{\mathcal{P}_\lambda}(x)$$

holds for some $\tau > 0$. Since $\Psi_{\mathcal{P}_\lambda}(x)$ is a convex function, and $\Psi_{\mathcal{P}_\lambda}(x) = \mu$ is finite, then the difference quotient is a nondecreasing function for τ [28, Section 23]. In addition, $\mathcal{D}^+\Psi_{\mathcal{P}_\lambda}(x(t))$ exists and

$$\begin{aligned}\mathcal{D}^+\Psi_{\mathcal{P}_\lambda}(x(t)) &= \inf_{\tau > 0} \frac{\Psi_{\mathcal{P}_\lambda}(x + \tau[A_q(w)x + E_q d]) - \Psi_{\mathcal{P}_\lambda}(x)}{\tau} \\ &\leq \frac{\lambda - 1}{\tau}\Psi_{\mathcal{P}_\lambda}(x).\end{aligned}$$

Since this inequality holds for any $w \in \mathcal{W}$ and $d \in \mathcal{D}$, the Dini derivative

$$\mathcal{D}^+\Psi_{\mathcal{P}_\lambda}(x(t)) \le \frac{\lambda - 1}{\tau}\Psi_{\mathcal{P}_\lambda}(x(t)),$$

at point $x(t) = x$. Since x is an arbitrary point outside $\mathcal{P}_\lambda$, we conclude that the function $\Psi_{\mathcal{P}_\lambda}(x)$ is a Lyapunov-like function for system (1) outside $\mathcal{P}_\lambda \subseteq \mathcal{S}$. ■

4.2.2. UUB analysis

For any switching signal $\sigma(t)$ and any $t > 0$, let $K_i(t)$ denote the total period in which the q_i-th subsystem is activated during $[0, t)$. Define $K^-(t) = \sum_{i \le r,\ q_i \in Q} K_i(t)$, which stands for the total activation period of the UUB subsystems. On the other hand, $K^+(t) = \sum_{i > r,\ q_i \in Q} K_i(t)$ denotes the total activation period of the non-UUB subsystems. We have $K^-(t) + K^+(t) = t$.

Similarly, define the multiple Lyapunov function candidate as

$$V(x(t)) = \Psi_{\sigma(t)}(x(t)).$$

For any t satisfying $t_0 = 0 < \cdots < t_i \le t < t_{i+1}$, we obtain

$$V(x(t)) \le \begin{cases} \max\{e^{-\beta_{q_i}(t-t_i)}\Psi_{q_i}(x(t_i)), 1\} & q_i \le r \\ \max\{e^{\beta_{q_i}(t-t_i)}\Psi_{q_i}(x(t_i)), 1\} & r < q_i \le N. \end{cases}$$

Let us define $\beta_s = \min_{1 \le i \le r}\{\beta_{q_i}\}$, and $\beta_u = \max_{r < i \le N}\{\beta_{q_i}\}$. Also, there exists a constant scalar μ such that $\Psi_{q_i}(x) \le \mu\Psi_{q_j}(x)$ and $\Psi_{q_j}(x) \le \mu\Psi_{q_i}(x)$, for all $x \in \mathbb{R}^n$. The scalar μ can be selected as in the previous section.

Therefore, by induction, we obtain

$$V(x(t)) \le \max\{e^{-\beta_s K^-(t) + \beta_u K^+(t)}\mu^{N_\sigma(t)}V(x(0)), 1\}.$$

If there exists a positive scalar $\beta < \beta_s$ such that

$$\frac{K^+(t)}{t} \le \frac{\beta_s - \beta}{\beta_u + \beta_s} \tag{20}$$

which is a condition on the percentage of time interval that the unstable subsystems are activated, then

$$e^{-\beta_s K^-(t) + \beta_u K^+(t)} \le e^{-\beta t},$$

and thus $V(x(t)) \le \max\{e^{-\beta t}\mu^{N_\sigma(t)}V(x(0)), 1\}$. In addition, if

$$N_\sigma(t) \le \frac{t}{\tau_a^*}, \qquad \tau_a^* = \frac{\ln\mu}{\beta - \beta^*} \tag{21}$$

for some positive scalar $\beta^* < \beta$, which represent a bounded average dwell time requirement. This implies $e^{-\beta t}\mu^{N_\sigma(t)} \le e^{-\beta^* t}$. So, $V(x(t)) \le \max\{e^{-\beta^* t}V(x(0)), 1\}$, and the UUB of (2) follows.

Theorem 5. *The switched system (2) achieves a finite asymptotic disturbance attenuation level under switching signals*

(1) *with percentage of time interval that the unstable subsystems are activated less than $\frac{\beta_s - \beta}{\beta_u + \beta_s}$ in the sense of (20), and*

(2) *with average dwell time no less than $\frac{\ln\mu}{\beta - \beta^*}$ in the sense of (21).*

5. Concluding remarks

In this paper, we have investigated the asymptotic disturbance attenuation properties for a class of switched linear systems with parametric uncertainties and exterior disturbances under various switching signals. Both continuous-time and discrete-time cases were considered. The contributions of the paper are threefold. First, the equivalence between the asymptotic stability of arbitrary switching systems and the robust stability of a corresponding LTV system was emphasized, based on which necessary and sufficient conditions on the subsystems' dynamics were presented for the switched systems to achieve a finite disturbance attenuation level under arbitrary switching. Secondly, restrictions on the switching signals to guarantee the finiteness of the disturbance attenuation level were identified even when there are unstable subsystems. Thirdly, new concepts and a construction procedure for polyhedral Lyapunov-like functions for unstable systems were introduced to characterize the explosiveness of their unbounded behaviors.

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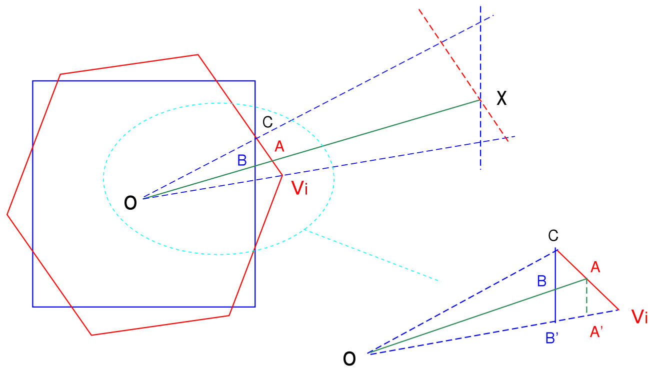

Fig. 1. Illustration of the choice for constant μ.

Appendix

A possible choice for μ is the largest value among $\Psi_{q_j}(v_j), \forall v_j \in \mathrm{vert}\{\bar{N}[\Psi_{q_i}, 1]\}$, and $\Psi_{q_i}(v_j), \forall v_j \in \mathrm{vert}\{\bar{N}[\Psi_{q_j}, 1]\}$, which can be justified as follows.

For simplicity, we consider the planar case, while the argument can be directly extended to the higher dimensional case. In Fig. 1, the rectangle stands for the C-set $\bar{N}[\Psi_{q_i}, 1]$, while the polygon stands for $\bar{N}[\Psi_{q_j}, 1]$. For any point $x \in \mathbb{R}^2$, the value $\Psi_{q_i}(x)$ ($\Psi_{q_j}(x)$) is the Minkowski distance from x to the C-set $\bar{N}[\Psi_{q_i}, 1]$ (or $\bar{N}[\Psi_{q_j}, 1]$ respectively) by the definition of a gauge function. In other words,

$$\Psi_{q_i}(x) = \inf\{\xi > 0 : x \in \xi\bar{N}[\Psi_{q_i}, 1]\},$$
$$\Psi_{q_j}(x) = \inf\{\xi > 0 : x \in \xi\bar{N}[\Psi_{q_j}, 1]\}.$$

Assume that the radius starting from the origin and going through point x intersects the bounds of $\bar{N}[\Psi_{q_j}, 1]$ and $\bar{N}[\Psi_{q_i}, 1]$ at points A and B, respectively. From the definition of Minkowski distance, we may obtain that

$$\Psi_{q_i}(x) = \frac{\|OX\|}{\|OB\|}, \qquad \Psi_{q_j}(x) = \frac{\|OX\|}{\|OA\|},$$

where $\|\cdot\|$ stands for the Euclidian norm. Therefore,

$$\frac{\Psi_{q_i}(x)}{\Psi_{q_j}(x)} = \frac{\frac{\|OX\|}{\|OB\|}}{\frac{\|OX\|}{\|OA\|}} = \frac{\|OA\|}{\|OB\|}.$$

Similarly, $\frac{\Psi_{q_j}(x)}{\Psi_{q_i}(x)} = \frac{\|OB\|}{\|OA\|}$. Therefore, the factor between two gauge functions Ψ_{q_i} and Ψ_{q_j} at any point can be "projected" into the faction of a pair of points at the boundaries of the two C-sets $\bar{N}[\Psi_{q_i}, 1]$ and $\bar{N}[\Psi_{q_j}, 1]$. Note that the pair is not arbitrary: the line connecting these two points should go through the origin and the point x of concern.

Next, we will focus on the boundaries of the C-sets $\bar{N}[\Psi_{q_i}, 1]$ and $\bar{N}[\Psi_{q_j}, 1]$ and prove that the maximum value of the faction constant μ occurs at the vertices. To see this, we zoom out the triangle $\triangle OCV_i$, where point C is the intersection point of two boundaries of $\bar{N}[\Psi_{q_i}, 1]$ and $\bar{N}[\Psi_{q_j}, 1]$, and V_i is a vertex of $\bar{N}[\Psi_{q_j}, 1]$ here. In the zoomed triangle, starting from point A draw a line parallel to CB and intersect line OV_i at A'. Assuming line CB intersects line OV_i at B', it is easy to show that $\frac{\Psi_{q_i}(x)}{\Psi_{q_j}(x)} = \frac{\|OA\|}{\|OB\|} = \frac{\|OA'\|}{\|OB'\|} \le \frac{\|OV_i\|}{\|OB'\|} = \frac{\Psi_{q_i}(v_i)}{\Psi_{q_j}(v_i)} = \Psi_{q_i}(v_i)$, which holds for any pair of A and B within the cone OCV_i. Therefore, the maximum of the faction $\frac{\Psi_{q_i}(x)}{\Psi_{q_j}(x)}$ occurs at the vertex V_i. The arguments can be easily extended to more general cases. This explains why we choose μ as the largest value among $\Psi_{q_j}(v_j), \forall v_j \in \mathrm{vert}\{\bar{N}[\Psi_{q_i}, 1]\}$, and $\Psi_{q_i}(v_j), \forall v_j \in \mathrm{vert}\{\bar{N}[\Psi_{q_j}, 1]\}$.

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