Continuous-time consensus with discrete-time communication

João Almeida, Carlos Silvestre, António M. Pascoal, and Panos J. Antsaklis

Abstract— **Consider a group of agents, each one having its one node variable, that seek to agree on a common value. The node variables are required to be continuous functions of time. The agents are allowed to exchange information among each other but only at specified update times. The topology of the underlying communication graph can change over time and the information received from other agents might be outdated. Our proposed solution includes an extra variable for each agent that is updated instantaneously at update times. Between update times, both variables, node and extra, evolve in a smooth manner. The stability analysis reduces to the study of a discrete-time equivalent of the continuous-time system and the use of previously established discrete-time consensus results to prove convergence.**

I. INTRODUCTION

Analyzing how groups of agents exhibit complex global behaviors by applying simple local rules, is a topic that has attracted a lot of attention in recent years in many scientific fields that range from biology to space exploration. A very simple problem of this kind is the consensus or agreement problem. Consider a group of agents trying to perform a certain task that requires some sort of coordination among them. To successfully complete the task, the agents have to agree on some common knowledge which in its simplest form can be a scalar value. Initially each agent has its own value but through the exchange of information among each other, they are able to arrive at same value, that is, they reach consensus. However, agent's sensing limitations and/or physical limitations imposed by the surrounding environment, may severely hinder the consensus process. Also, agents need to be able to send and receive information through some type of communication channel. This channel naturally has limited capacity and may be unreliable.

The purpose of some of the work around the consensus problem has been to analyze under what conditions can consensus be achieved, while casting it in many different forms [1]–[11]. Some authors consider agents variables that evolve in continuous time (with the result being a smooth curve) [2]–[5], [10], [11], others in discrete-time (resulting in a sequence of values) [1]–[4], [6]–[8]. Another variant is whether the communication topology is fixed over time [2] or if it is allowed to change, that is, if the information available to a certain agent always comes from the same set of agents or if this set changes over time [1], [3], [4], [6]–[11]. Also, different kinds of connectivity assumptions on the network topology are considered, and in some cases time delays in the communication links are also taken into account [1], [2], [6]–[8], [10], [11]. Consensus as also been posed on manifolds [12].

While some applications may require continuous consensus variables, continuous communication links among agents are hard to achieve in practise. Transmission of data naturally occurs in small bursts at some spaced time instants. Taking into account this communication constraint, [9] introduces a setup where agent variables are continuous but information about neighboring agents is only available at discrete-time update instants. Between update times, the evolution of each variable is determined by a waypoint and a pre-specified continuous function. This implies the knowledge of the next update time, or at least a bound on the duration of the time interval. By describing the evolution of the agent variables, during this period, in terms of a differential equation, we do not require such information. Moreover, we have also considered the situation where, due to some a delay of some sort, the information received from other agents might be outdated.

Our solution includes an extra state variable for each agent that is allowed to be discontinuous. Between update times both variables, the consensus variable and the extra state variable, evolve continuously as determined by a differential equation. At update times, the consensus variable is kept at the same value while the extra variable is updated with information received from other agents that might be outdated. Our convergence analysis starts by constructing a discrete-time equivalent of the continuous-time system described, and then proving that it reaches consensus asymptotically in the presence of time delays by resorting to well established discrete-time consensus results.

The paper is organized as follows. In Section II some key concepts from graph theory are introduced that help to understand the modeling of the communication graph and its properties. A brief summary of relevant results in discrete-time consensus is presented in Section III. Our proposed solution and the main result related to its convergence properties are presented in Section IV. Finally, numerical simulations for a simple example are presented in Section V.

II. DIRECTED GRAPHS

This section contains some key concepts and results in graph theory that play an important role in what follows. See

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[13] for an in-depth presentation of this subject. A directed graph or digraph $\mathcal{G} = \mathcal{G}(V, E)$ consists of a finite set $V = \{1, 2, \ldots, n\}$ of n vertices and a finite set $E \subseteq V \times V$ of m ordered pairs of vertices (i, j) named arcs. Given an arc $(i, j) \in E$, its first and second elements are called the tail and head of the arc, respectively. If (i, j) belongs to E then we say that i is adjacent to j. A path in $\mathcal{G}$ from i to j is a sequence of distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. A vertex i is a root if there is a path in $\mathcal{G}$ from vertex i to every other vertex in $\mathcal{G}$. If a graph has at least one root, we say that it is a rooted graph. Given a sequence of graphs $\{\mathcal{G}_k = (V, E_k)\}_{k=1}^{B}$ (with the same vertex set), the union of these graphs (union graph) is defined as

$$\bigcup_{k=1}^{B} \mathcal{G}_k = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_B = (V, E_1 \cup E_2 \cup \cdots \cup E_B).$$

III. DISCRETE-TIME CONSENSUS

In this section we introduce a well-known discrete-time consensus problem and a related convergence result, adapted from [7] and [8], that will be used later in Section IV.

Consider a set of N agents, labeled 1 through N, each one with its own scalar node variable x_i. Each agent updates its node variable according to the equation

$$x_i(k+1) = \sum_{j=1}^{N} a_{ij}(k) x_j(k - \tau_{ij}(k)), \tag{1}$$

where $a_{ij}(k)$ are nonnegative coefficients, and $\tau_{ij}(k)$ are nonnegative integer time delays. The coefficients are assumed to satisfy the following:

Assumption 1 (Nontrivial convex interaction): There exists a positive constant α such that for all $1 \leq i, j \leq N$ and all $k \geq 0$:

1) $a_{ij}(k) \geq \alpha$;
2) $a_{ij}(k) \in \{0\} \cup [\alpha, 1)$;
3) $\sum_{j=1}^{N} a_{ij}(k) = 1$;
4) the set $\{a_{ij}(k) : k \geq 0\}$ is finite.

Part 1 of the assumption states that $x_i(k)$ is used in every iteration, while other agents variables might not (since they might not be available at every time instant, part 2). Part 3 and the fact that all coefficients are nonnegative implies that the combination of node variables is convex. Part 4 means that each a_{ij} can only take a finite number of different values over time.

As for the time delays affecting the information received from other agents, we assume that the following holds:

Assumption 2 (Bounded discrete-time delays): There exists a positive integer $\overline{\tau}$ such that for all $1 \leq i, j \leq N$ and $k \geq 0$:

1) $0 \leq k - \tau_{ij}(k) \leq k$;
2) $0 \leq \tau_{ij}(k) \leq \overline{\tau}$;
3) and, $\tau_{ii}(k) = 0$.

This assumption states that the system is causal (part 1), that the time delays are upper bounded (part 2), and that each agent has access to its current state (part 3). For convenience, if $a_{ij}(k) = 0$, then $\tau_{ij}(k) = k$. Introducing the state vector

$$x(k) = \begin{bmatrix} x_1(k) & x_2(k) & \cdots & x_N(k) \end{bmatrix}^\top,$$

the consensus iterations (1) can be written compactly in vector form as

$$x(k+1) = A(k) x(k - \tau(k)), \tag{2}$$

using the notation

$$x(k - \tau(k)) = \begin{bmatrix} x_1(k - \tau_{11}(k)) & \cdots & x_1(k - \tau_{N1}(k)) \\ \vdots & \ddots & \vdots \\ x_N(k - \tau_{1N}(k)) & \cdots & x_N(k - \tau_{NN}(k)) \end{bmatrix},$$

and, where $A(k)$ and $\tau(k)$ denote matrices whose entries are $a_{ij}(k)$ and $\tau_{ji}(k)$, respectively.

The communication topology at each iteration can be described in terms of a directed graph $\mathcal{G}(k) = (\{1, \ldots, N\}, E(k))$, where $(j, i) \in E(k)$ if and only if $a_{ij}(k) > 0$. That is, the structure of the directed graph $\mathcal{G}(k)$ and of matrix $A(k)$ are linked. The following assumption is made on the sequence of communication patterns:

Assumption 3 (Periodically rooted digraph): For any sequence of directed graphs $\{\mathcal{G}(k)\}_{k=0}^{+\infty}$, there exists a positive constant B such that the union graph

$$\bigcup_{k=k_0}^{k_0+B-1} \mathcal{G}(k)$$

is rooted for all $k_0 \geq 0$.

We say that consensus is achieved asymptotically if the following holds: for every $x(0) \in \mathbb{R}^N$, and for every sequence $\{A(k)\}_{k=0}^{+\infty}$ allowed by our assumptions, there exists some $d \in \mathbb{R}$ such that $x_i(k) \to d$ as $k \to +\infty$ for all i. Given the previous conditions, the following result can be established.

Theorem 1 ([7], [8]): Under Assumptions 1, 2, and 3, the discrete-time iterations described by (2) reach consensus asymptotically.

IV. CONTINUOUS-TIME CONSENSUS WITH DISCRETE-TIME UPDATES

In this section, we start by introducing the problem of continuous-time consensus with discrete-time updates, and then present our proposed solution to this problem and analyze its convergence properties using the results of the previous section.

Consider again N agents each one with its own node variable x_i expect that now, instead of being discrete in time, the node variables must be continuous in time. Nonetheless, information about other agent's node variables is still only available at update times. These update times are a given sequence of time instants $\{t_k\}_{k=0}^{+\infty}$ that satisfy the following assumption:

Assumption 4 (Communication intervals): For all $k \geq 0$, $t_{k+1} - t_k > 0$ and the set $\{t_{k+1} - t_k : k \geq 0\}$ is finite.

This assumption implies that there exist positive constants τ_l and τ_u such that $0 < \tau_l \leq t_{k+1} - t_k \leq \tau_u$ for all $k \geq 0$, that is, that the communication intervals are bounded.

At each update time, t_k, the node variables x_i are shared among agents. To each time instant, t_k, we associate a directed graph $\mathcal{G}_k = \mathcal{G}(t_k) = (V, E_k)$ with $V = \{1, 2, \ldots, N\}$ and $E_k \subseteq V \times V$. Each graph represents what information is available to each agent. This sequence of graphs is assumed to satisfy:

Assumption 5 (Periodically rooted digraph): For any sequence of directed graphs $\{\mathcal{G}_k\}_{k=0}^{+\infty}$, there exists a positive constant B such that the union graph

$$\bigcup_{k=k_0}^{k_0+B-1} \mathcal{G}_k$$

is rooted for all $k_0 \geq 0$.

Beside being only available at a discrete set of time instants, the state of the agents when received by another agent might refer to a past value of that state, that is, due to some sort of delay (which could be the sum of measurement, computation, and transmission delays) the information received is outdated. These delays are assumed to satisfy the following.

Assumption 6 (Bounded time delays): For all $1 \leq i, j \leq N$ and $k \geq 0$,

1) $t_0 \leq t_k - \tau_{ij}(t_k) \leq t_k$;
2) $0 \leq \tau_{ij}(t_k)$;
3) $\tau_{ii}(t_k) = 0$;
4) and, the set $\{\tau_{ij}(k) : k \geq 0\}$ is finite.

Part 2 and 4 of the previous assumption imply that there exists a positive constant $\overline{\tau}$ such that $\tau_{ij}(t_k) \leq \overline{\tau}$, for all $1 \leq i, j \leq N$ and $k \geq 0$.

We say that consensus is reached asymptotically if, for every $x_i(t_0) \in \mathbb{R}$, we have

$$\lim_{t \to +\infty} |x_i(t) - x_j(t)| = 0$$

for all $i, j \in V$. We are now ready to state the consensus problem addressed in this paper.

Problem: Given N identical agents described by the model

$$\dot{x}_i = u_i,$$

find a (distributed) control law such that consensus is reached asymptotically for all sequences of: i) directed graphs $\{\mathcal{G}_k\}_{k=0}^{+\infty}$ satisfying Assumption 5; ii) update times $\{t_k\}_{k=0}^{+\infty}$ satisfying Assumption 4; iii) and, time delays $\{\tau(k)\}_{k=0}^{+\infty}$ satisfying Assumption 6.

Remark: Our setup is indeed asynchronous since each agent performs its computations independently of the other agents or of time. There are no synchronized clocks between agents. Suppose each agent has associated to it a sequence update times $\{t_k^i\}$ that satisfies Assumption 4. When analyzing the convergence properties of the consensus protocol, we would inevitably consider a sequence of update times that would be the union of the sequence of update times of each agent, properly sorted of course. With a slight abuse of notation, this could be represented as

$$\{t_p\} = \bigcup_{i=1}^{N} \{t_k^i\}.$$

This sequence would also satisfy Assumption 4, although not necessarily with the same upper and lower bounds. The same reasoning can be applied to the sequence of digraphs $\{G_k\}$ and to the sequence of time delays $\{\tau_{ij}(k)\}$.

A. Proposed solution

We begin by introducing an additional state variable $X_i(t)$, one for each agent, that might have discontinuities at update times, unlike $x_i(t)$ that must be continuous at all times. Let

$$z(t) = \begin{bmatrix} x_1(t) & X_1(t) & \cdots & x_N(t) & X_N(t) \end{bmatrix}^\top \in \mathbb{R}^{2N}$$

represent the state of the whole system. The dynamics of either variable (x_i or X_i) can depend on the values of other state variables. This dependence is represented by a graph, an interaction graph, with vertex set $\overline{V} = \{1, 2, \ldots, 2N\}$ (one vertex for each variable). Odd vertices are associated to x_i variables while even vertices are associated to X_i variables, with $i \in V$.

Between update times, say t_k and t_{k+1}, the only information available to each agent is its own, that is, agent i only has access to the values of $x_i(t)$ and $X_i(t)$. During this time interval, the evolution of $x_i(t)$ and $X_i(t)$ is dictated by

$$\dot{x}_i(t) = -b_i\big(x_i(t) - X_i(t)\big) \tag{3}$$
$$\dot{X}_i(t) = c_i\big(x_i(t) - X_i(t)\big) \tag{4}$$

where b_i and c_i are positive constants. This type of dynamics lead to a decrease of the absolute difference between x_i and X_i. Let

$$L_i = \begin{bmatrix} -b_i & b_i \\ c_i & -c_i \end{bmatrix},$$

and

$$\begin{aligned} \Phi_i(t, t_0) &= \exp\{L_i(t - t_0)\} \\ &= \frac{1}{b_i + c_i} \begin{bmatrix} c_i + b_i f(t, t_0) & b_i(1 - f(t, t_0)) \\ c_i(1 - f(t, t_0)) & b_i + c_i f(t, t_0) \end{bmatrix} \end{aligned}$$

where $f(t, t_0) = e^{-(b_i + c_i)(t - t_0)}$, and $\exp\{\cdot\}$ denotes matrix exponential. In terms of the aggregated stated, (3) and (4) yield

$$\dot{z}(t) = \begin{bmatrix} L_1 & & \\ & \ddots & \\ & & L_N \end{bmatrix} z(t) = Lz(t) \tag{5}$$

for all $t \in [t_k, \, t_{k+1})$. Therefore

$$\begin{aligned} z(t_{k+1}^-) &= \exp\{L(t_{k+1} - t_k)\} z(t_k) \\ &= \begin{bmatrix} \Phi_1(t_{k+1}, t_k) & & \\ & \ddots & \\ & & \Phi_N(t_{k+1}, t_k) \end{bmatrix} z(t_k) \\ &= \Phi(t_{k+1}, t_k) z(t_k), \end{aligned}$$

where $z(t_{k+1}^-) = \lim_{t \nearrow t_{k+1}} z(t)$. Note that the entries of $\Phi_i(t_{k+1}, t_k)$ satisfy Assumption 1 for all $k \geq 0$. The graph associated to $\Phi(t_{k+1}, t_k)$ is denoted by $\mathcal{H} = (\overline{V}, \overline{E}_{\mathcal{H}})$ where

$$\begin{aligned}\overline{E}_{\mathcal{H}} = \{(2i, 2i-1) : i \in V\} \cup \{(2i-1, 2i) : i \in V\} \\ \cup \{(i,i) : i \in \overline{V}\},\end{aligned}$$

and is represented in Fig. 1.

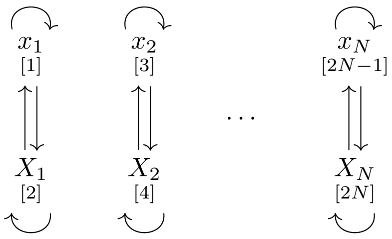


Fig. 1. Interaction graph between state variables for $t \in [t_k, t_{k+1})$, denoted by $\mathcal{H}$ and associated to $\Phi(t_{k+1}, t_k)$. Numbers in brackets represent the vertex in $\overline{V}$ associated to each state variable.

At each update time, t_k, the node variables x_i are shared among agents according to the topology of the directed graph $\mathcal{G}_k$. The value of X_i is updated using the information received from neighboring agents at that time, which may be outdated, while x_i remains unchanged. Formally, we have the update equations

$$x_i(t_k) = x_i(t_k^-) \tag{6}$$

$$X_i(t_k) = a_{ii}(t_k) X_i(t_k^-) + \sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ij}(t_k) x_j(t_k^- - \tau_{ij}(t_k)) \tag{7}$$

where:

- the coefficients $\{a_{ij}(t_k)\}_{i,j=1}^{N}$ satisfy Assumption 1 and are compatible with $\mathcal{G}_k$;
- $\{\tau_{ij}(t_k)\}_{i,j=1}^{N} \in \mathbb{R}$ are time delays affecting the information received from other agents by agent i satisfying Assumption 6;
- and,

$$x_j(t_k^- - \tau_{ij}(t_k)) = \begin{cases} x_j(t_k^-), & \text{if } \tau_{ij}(t_k) = 0 \\ x_j(t_k - \tau_{ij}(t_k)), & \text{if } \tau_{ij}(t_k) > 0 \end{cases}.$$

Using the previously defined aggregate state, we can write the update equations (6)-(7) as

$$z_i(t_k) = \sum_{j=1}^{2N} r_{ij}(t_k) z_j(t_k^- - \sigma_{ij}(t_k)),$$

where

$$r_{ij}(t_k) = \begin{cases} 1, & \text{if } i = j = 2p-1, \text{for some } p \in V \\ a_{pp}(t_k), & \text{if } i = j = 2p, \text{for some } p \in V \\ a_{pq}(t_k), & \text{if } i = 2p \text{ and } j = 2q-1, \\ & \quad \text{for some } p, q \in V \\ 0, & \text{otherwise} \end{cases}$$

and

$$\sigma_{ij}(t_k) = \begin{cases} \tau_{pq}(t_k), & \text{if } i = 2p \text{ and } j = 2q-1, \\ & \quad \text{for some } p, q \in V \\ 0, & \text{otherwise} \end{cases}$$

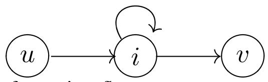
or, in compact notation,

$$z(t_k) = R(t_k) z(t_k^- - \sigma(t_k)). \tag{8}$$

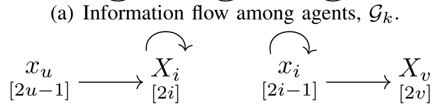
Notice that $\sigma(t_k)$ satisfies Assumption 6 because $\tau(t_k)$ also satisfies that assumption. The structure of $R_k = R(t_k)$ is induced by the topology of the communication graph $\mathcal{G}_k = (V, E_k)$, which in turn implies that the graph associated to R_k is $\overline{\mathcal{G}}_k = (\overline{V}, \overline{E}_k)$ where

$$\overline{E}_k = \{(2i-1, 2j) : (i,j) \in E_k \wedge i \neq j\} \cup \{(i,i) : i \in \overline{V}\}.$$

See Fig. 2 for a graphical interpretation.



(a) Information flow among agents, $\mathcal{G}_k$.



(b) Interaction graph between state variables, $\overline{\mathcal{G}}_k$.

Fig. 2. Relation between $\mathcal{G}_k$ and $\overline{\mathcal{G}}_k$. Agent i has one incoming arc from u and one outgoing arc to v in $\mathcal{G}_k$, that give rise to the arcs $(2u-1, 2i)$ and $(2i-1, 2v)$ in $\overline{\mathcal{G}}_k$, respectively. Only self-arcs of i are represented.

Since there is an iteration at $t = t_0$ that requires $z(t_0^-)$, this will be regarded as the initial state of the system.

B. Main result

The following result establishes that the previously described solution leads to consensus.

Theorem 2: Under Assumptions 4, 5, and 6, the dynamical system with state $z(t)$ driven by equations (5) and (8), reaches consensus asymptotically.

The proof of the theorem starts by establishing an equivalent discrete-time description of the system, applying Theorem 1 to this system, and concluding that consensus is reached asymptotically. To see this, consider the case without time delays. The following diagram depicts the sequence of iterations

$$\begin{tikzcd}
y(0) = z(t_0^-) \arrow[d, "R_0"] & y(2) = z(t_1^-) \arrow[d, "R_1"] & y(4) = z(t_2^-) \arrow[d, "R_2"] & \cdots \\
y(1) = z(t_0) \arrow[ur, "\Phi(t_1,t_0)"] & y(3) = z(t_1) \arrow[ur, "\Phi(t_2,t_1)"] & y(5) = z(t_2) \arrow[ur, "\Phi(t_3,t_2)"] &
\end{tikzcd}$$

where $y(p) \in \mathbb{R}^{2N}$ is a new discrete-time state variable, defined as $y(p) = z(t_{p/2}^-)$ for $p = 0, 2, 4, \ldots$, and $y(p) = z(t_{(p-1)/2})$ for $p = 1, 3, 5, \ldots$. We can further write

$$y(p+1) = F(p) y(p)$$

where matrix $F(p)$ is defined as

$$F(p) = \begin{cases} R_{\frac{p}{2}}, & \text{if } p = 0, 2, 4, \ldots \\ \Phi(t_{\frac{p+1}{2}}, t_{\frac{p-1}{2}}), & \text{if } p = 1, 3, 5, \ldots \end{cases}$$

which satisfies Assumption 1 for all $p \geq 0$. The associated graphs, $\mathcal{F}(p)$, are periodically rooted with period $2B$ since, for all $p_0 \geq 0$,

$$\bigcup_{p=p_0}^{p_0+2B-1} \mathcal{F}(p) = \mathcal{H} \cup \bigcup_{k=k_0}^{k_0+B-1} \overline{\mathcal{G}}_k$$

which is rooted due to the following lemma:

Lemma 1: Given $k_0 \geq 0$, if $\bigcup_{k=k_0}^{k_0+B-1} \mathcal{G}_k$ is rooted, then $\mathcal{H} \cup \bigcup_{k=k_0}^{k_0+B-1} \overline{\mathcal{G}}_k$ is also rooted.

Thus, in the absence of time delays, Theorem 1 guarantees that y reaches consensus asymptotically.

To accommodate for time-delays in continuous time, terms of the form $z_j(t_k^- - \sigma_{ij}(t_k))$ must correspond to some $y_j(p - \gamma_{ij}(p))$, where each continuous delay $\sigma_{ij}(t_k)$ needs to be translated into an integer delay $\gamma_{ij}(p)$. In order to accomplish this, each instance of $t_k - \sigma_{ij}(t_k)$ is added to the existing sequence of update times (removing duplicates if necessary), generating a new time instant (a delay event). By reordering the resulting sequence, we get a sequence of increasing time instants

$$\begin{aligned} & t_{[m_0=0]} = t_0 < t_{[1]} < t_{[2]} < \cdots < t_{[m_1]} = t_1 < \cdots \\ & \cdots < t_{[m_k]} = t_k < t_{[m_k+1]} < \cdots \\ & \cdots < t_{[m_{k+1}-1]} < t_{[m_{k+1}]} = t_{k+1} < \cdots \end{aligned}$$

where $\{m_k\}_{k=0}^{+\infty}$ is a subsequence of indices that satisfy $t_{[m_k]} = t_k$. Conceptually, at each delay event $t_{[q]} : q \geq 0 \wedge q \neq m_k$ for all $k \geq 0$, a discrete-time iteration of the form

$$z(t_{[q]}) = z(t_{[q]}^-) = I_{2N} z(t_{[q]}^-)$$

is performed, where I_{2N} is the identity matrix of dimension $2N$. The graph associated to this matrix is denoted by $\mathcal{G}^0 = (\overline{V}, \{(i,i) : i \in \overline{V}\})$ which is a graph with all (and only) self-arcs. Since $\{t_k\}_{k=0}^{+\infty}$ satisfies Assumption 4 and $\{\sigma(k)\}_{k=0}^{+\infty}$ satisfies Assumption 6, the new sequence of time instants $\{t_{[p]}\}_{p=0}^{+\infty}$ satisfies Assumption 4.

Let n_k denote for the number of delay generated time instants in the time interval (t_k, t_{k+1}). Since the time delays are bounded, in any time interval (t_k, t_{k+1}), only a finite number of these "dummy" iterations take place, as shown in the following lemma:

Lemma 2 (Finite number of delay events): The number of delay generated time instants in any interval (t_k, t_{k+1}), n_k, is upper bounded by

$$\overline{n} = \left\lceil \frac{\overline{\tau}}{\tau_l} \right\rceil N(N-1),$$

where $\lceil x \rceil$ stands for the smallest integer greater than or equal to x.

We are now ready to prove our main result.

Proof: [of Theorem 2] In order to apply the discrete-time consensus result, we introduce a new state variable $y(p) \in \mathbb{R}^{2N}$, defined as

$$y(p) = \begin{cases} z(t_{[p/2]}^-), & \text{if } p = 0, 2, 4, \ldots \\ z(t_{[(p-1)/2]}), & \text{if } p = 1, 3, 5, \ldots \end{cases}$$

This state variable evolves in discrete-time, according to

$$y_i(p+1) = \sum_{j=1}^{2N} f_{ij}(p) y(p - \gamma_{ij}(p)),$$

or in vector form as

$$y(p+1) = F(p) y(p - \gamma(p)), \tag{9}$$

where

- $F(p)$ can be R_k, $\Phi(t,s)$ for some $t_0 \leq s < t$, or I_{2N}, with the associated graphs, $\mathcal{F}(p)$, being $\overline{\mathcal{G}}_k$, $\mathcal{H}$, or $\mathcal{G}^0$, respectively;
- $\gamma(p)$ is a time-varying integer delay matrix to be defined.

The idea of the proof is to show that $y(p)$, driven by (9), satisfies all conditions necessary to convergence to consensus, and that this implies that $z(t)$ also reaches consensus asymptotically.

Formally, $F(p)$ is defined as

$$F(p) = \begin{cases} R_{\frac{p}{2}}, & \text{if } p = 0, 2, 4, \ldots \wedge p = 2m_k \\ I_{2N}, & \text{if } p = 0, 2, 4, \ldots \wedge p \neq 2m_k \\ \Phi(t_{[\frac{p+1}{2}]}, t_{[\frac{p-1}{2}]}), & \text{if } p = 1, 3, 5, \ldots \end{cases}$$

For all $p \geq 0$, $F(p)$ satisfies Assumption 1.

At each update time t_k, in order to compute $z_i(t_k) = z_i(t_{[m_k]})$ we need, among others, the value of $z_j(t_k - \sigma_{ij}(t_k))$ with $\sigma_{ij}(t_k) > 0$, or equivalently, the value of $z_j(t_{[q_{ij}]})$ with $t_{[q_{ij}]} = t_k - \sigma_{ij}(t_k)$. In terms of the discrete-time variable y, we are trying to compute $y_i(2m_k + 1)$ and want to access $y_j(2q_{ij} + 1)$. The discrete delay is then $\gamma_{ij}(2m_k + 1) = 2m_k + 1 - (2q_{ij} + 1) = 2(m_k - q_{ij}) - 1$. Formally, the iteration delays $\gamma_{ij}(p)$ are defined, for all $i, j \in \overline{V}$, as:

- $\gamma_{ij}(p) = 0$, if $p \neq 2m_k$ or if $p = 2m_k$ and $\sigma_{ij}(t_k) = 0$;
- $\gamma_{ij}(p) = p - (2q_{ij}(p) + 1) = 2(m_k - q_{ij}(p)) - 1$, if $p = 2m_k$ and $\sigma_{ij}(t_k) > 0$, where $0 \leq q_{ij}(p) \leq m_k - 1$ is such that $t_k - \sigma_{ij}(t_k) = t_{[q_{ij}(p)]}$.

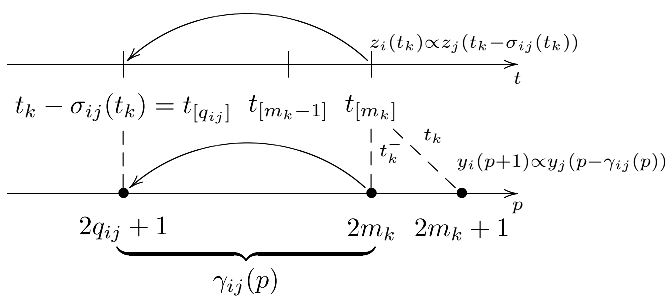


Fig. 3. Definition of $\gamma_{ij}(p)$ when $p = 2m_k$ and $\sigma_{ij}(t_k) > 0$.

In what follows, we show that $\gamma(p)$ satisfies Assumption 2. For $p = 2m_k$ and $\sigma_{ij}(t_k) > 0$, it is easy to see that

$$\gamma_{ij}(p) \geq 1 \geq 0$$

and that

$$p \geq p - 1 \geq p - \gamma_{ij}(p) \geq 1 \geq 0.$$

To show that the iteration delays are bounded, let $k' \geq 0$ be the greatest index such that

$$t_{[m_{k'}]} \leq t_k - \overline{\tau} \leq t_{[q_{ij}(p)]} \leq t_{[m_k]} = t_k.$$

If $t_k - \overline{\tau} < t_{[0]} = t_0$, then we can take $k' = 0$ since $q_{ij}(p) \geq 0$. The sequence of indices $\{m_k\}_{k=0}^{+\infty}$ can be obtained through the recursion

$$m_{k+1} = m_k + n_k + 1,$$

starting with $m_0 = 0$. From this recursion, we can derive the bound

$$m_k - m_{k'} = \sum_{l=k'+1}^{k} n_l + k - k' \leq (\overline{n} + 1) \left\lceil \frac{\overline{\tau}}{\tau_l} \right\rceil$$

where $\overline{n}$ is as defined in Lemma 2, and where we have used the fact that $k - k' \leq \lceil \overline{\tau}/\tau_l \rceil$ (because k' is chosen to be the greatest). Since $m_k - q_{ij}(p) \leq m_k - m_{k'}$, we get

$$\gamma_{ij}(p) = 2(m_k - q_{ij}(p)) - 1 \leq 2(\overline{n}+1) \left\lceil \frac{\overline{\tau}}{\tau_l} \right\rceil - 1 = \overline{\gamma}.$$

For all other cases, Assumption 2 is trivially satisfied since $\gamma(p) = 0$.

Next, we show that the graphs $\mathcal{F}(p)$ are periodically rooted with period $\overline{B} = 2(\overline{n} + 1)B$. Given any sequence of such graphs, any subsequence of length $2(\overline{n}+1)$ contains at least one $\overline{\mathcal{G}}_k$ graph. Thus, any subsequence of length $2B(\overline{n}+1) = \overline{B}$ contains at least a sequence of B graphs $\overline{\mathcal{G}}_k$. Using the fact that $\mathcal{G} \cup \mathcal{G} = \mathcal{G}$ for any graph $\mathcal{G}$, and that $\mathcal{G}^0 \cup \mathcal{H} = \mathcal{H}$, the union of the graphs across any such subsequence is equal to

$$\begin{aligned} \bigcup_{p=p_0}^{p_0+\overline{B}-1} \mathcal{F}(p) &= \mathcal{G}^0 \cup \mathcal{H} \cup \overline{\mathcal{G}}_{k_0} \cup \overline{\mathcal{G}}_{k_0+1} \cup \cdots \cup \overline{\mathcal{G}}_{k_0+B'-1} \\ &= \mathcal{H} \cup \bigcup_{k=k_0}^{k_0+B'-1} \overline{\mathcal{G}}_k \end{aligned}$$

where $p_0 \geq 0$ and $B' \geq B$. Lemma 1 guarantees that the resulting graph has at least one root. We conclude that the sequence of graphs is periodically rooted with period $\overline{B}$, thus satisfying Assumption 3.

We conclude by Theorem 1 that $y(p)$ reaches consensus asymptotically. Therefore,

$$\lim_{p \to +\infty} y(p) = d\mathbf{1}_{2N} \Rightarrow \lim_{k \to +\infty} z(t_k) = d\mathbf{1}_{2N}$$

where $\mathbf{1}_{2N} \in \mathbb{R}^{2N}$ is a vector with all entries equal to one. We have that $z(t) = \Phi(t, t_k) z(t_k)$ for $t \in [t_k, t_{k+1})$. Since the sequence of update times is lower bounded, $t \to +\infty$ implies $t_k \to +\infty$. For the same reason, the entries of $\Phi(t, t_k)$ are bounded. Since $\Phi(t, t_k) d\mathbf{1}_{2N} = d\mathbf{1}_{2N}$ for all t, t_k, we conclude that

$$\lim_{t \to +\infty} z(t) = d\mathbf{1}_{2N}.$$

■

V. SIMULATIONS

In this section we provide an example involving the proposed solution.

Consider $N = 5$ agents whose initial states are $x_i(0) = X_i(0) = i - 3$ with $i = 1, 2, 3, 4, 5$. The sequence of time instants is generated by taking $t_0 = 0$ and randomly picking (with equal probability) the incremental differences $t_{k+1} - t_k$ from the set $\{1 + 0.4d : d = 0, \ldots, 10\}$. The same is done for the time delays but over the set $\{0.5d : d = 0, \ldots, 20\}$.

Consider the four kinds of graphs (more precisely of edge sets) shown in Fig. 4(a)-(d), whose union is rooted (Fig. 4(e)). The sequence of graphs is constructed as follows. First, we take the four graphs in sequence $\mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$. Then, we take the same four graphs but change the heads and tails of the arcs according to a circular left shift rule,

$$(1, 2, 3, 4, 5) \to (2, 3, 4, 5, 1) \to (3, 4, 5, 1, 2) \to \cdots.$$

For example, during the first sequence of four, graph $\mathcal{G}_1$ says that agent 1 sends out information to agents 2 and 3. In the second sequence of four, it will be agent 2 sending out information to agents 3 and 4. The repetition of the process leads to a periodically rooted sequence of graphs with period $B = 4$. The values of the coefficients are

$$a_{ij}(t_k) = \frac{1}{1 + |\mathcal{N}_i(t_k)|}, b_i = \frac{4}{5}, \text{ and } c_i = \frac{1}{5},$$

for all $i, j \in \{1, 2, 3, 4, 5\}$, where $|\mathcal{N}_i(t_k)|$ stands for the number of elements in $\mathcal{N}_i(t_k) = \{j \in V \backslash \{i\} : (j, i) \in E_k\}$.

Fig. 5 depicts the time evolution of the difference between the maximum and the minimum of the agents states at each time instant. As can be seen, this value can increase over some intervals of time, but over a large enough interval of time (related to the period over which the sequence of graphs is rooted) the overall difference decreases and tends to zero. Since this difference tends to zero, all states tend to the same value. This is illustrated in Fig. 6.

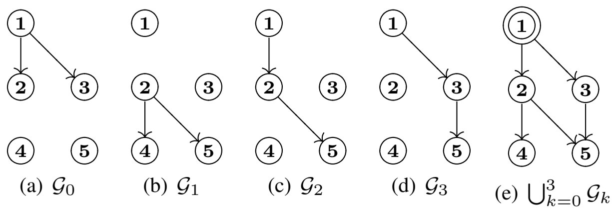


Fig. 4. A sequence of four graphs (a)-(d) whose union (e) has a root (vertex 1).

VI. CONCLUSIONS

The problem of consensus seeking is analyzed in the context of continuous variables with discrete-time updates. Besides the usual node variable for which consensus is sought, each agent has an extra state variable. Between update times, both variables evolve continuously. At update times, the extra state variable is updated using information (possibly outdated) received from other agents, while the node variable is kept at the same value.

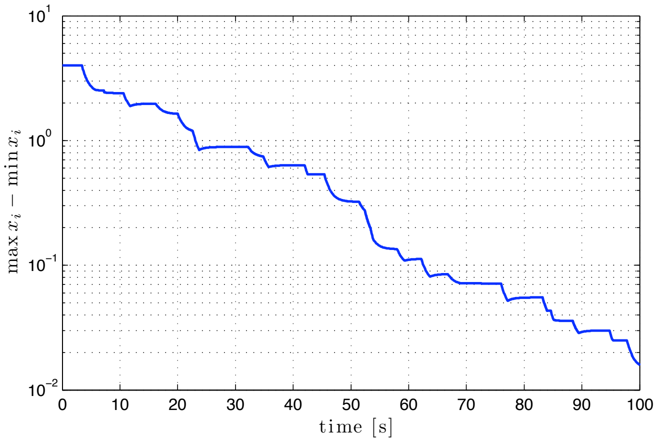


Fig. 5. Time evolution of $\max_{i \in V} x_i - \min_{i \in V} x_i$.

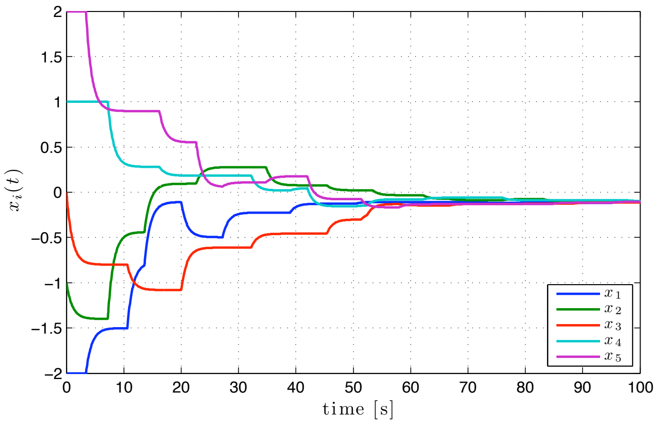


Fig. 6. Time evolution of $x_i(t)$ for $i = 1, 2, 3, 4, 5$.

The evolution of the aggregated state of the system is equivalently described by an appropriately defined discrete-time system. In this setup, both continuous evolution and discrete updates are interpreted as two different types of iterations. Time delays are incorporated by extending the set of update times and performing at each new time instant an identity iteration.

By supporting our proofs on existing discrete-time consensus results, should extensions to these be made available, further developments of the results presented should be possible.

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