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Passivity and $\mathcal{L}_2$ Stability of Networked Dissipative Systems

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Abstract— **It is well known that a negative feedback interconnection of passive systems is passive. However, the extension of this fundamental property to the case of negative feedback interconnection of dissipative systems, which is a more general case, remains largely unaddressed. In this paper, we demonstrate that a negative feedback interconnection of dissipative systems, under appropriate assumptions, could be passive or just $\mathcal{L}_2$ stable; we further propose a way to relax these assumptions. The case when there are time delays in communication is then addressed, and it is shown that under appropriate assumptions, the closed-loop system is passive or $\mathcal{L}_2$ stable for non-increasing time delays; moreover, passivity and $\mathcal{L}_2$ stability can be retained by inserting time-varying gains in the communication path provided a bound on the maximum rate of change of delay is known.**

I. Introduction

In this paper, we study passivity and $\mathcal{L}_2$ stability of a feedback interconnection of two dissipative systems when there are time-varying delays in the loop. It is well known that a feedback interconnection of two passive systems is still passive [1]-[2],[14]. Based on this fundamental result, many constructive control designs have appeared in the literature, see [3]-[4]. Some preliminary results which extend these basic passivity/dissipativity results to network control have been reported in [5]-[12].

In the recent work of Chopra [10]-[11], passivity results for interconnected passive and output strictly passive systems when there are time-varying communication delays are presented. These preliminary results provide a good way to retain passivity properties of the feedback interconnection in the presence of time-varying delays, by inserting properly designed gains in the communication path. Furthermore, they have shown that if the passive systems are transformed by using the the scattering representation [15], and if the scattering variables are transmitted as the new outputs, then the feedback interconnection is passive independently of the constant time delays. In the latest paper of Hirche, Matiakis and Bussa [12], the feedback interconnection of IF-OFP (input feed-forward and output feedback passive) systems is further studied, and a "rotation"transformation to retain the passivity results for the closed-loop in the presence of constant time delays is proposed; they have also shown that the scattering transformation is actually a special case of the "rotation"transformation, and the key point behind this kind of transformation is to rotate the conic representation of system's input and output space and use the "small-gain"theorem as a fundamental design guide.

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In the present paper, we study the feedback interconnection of dissipative systems, which is a more general case compared with the work in [10],[11],[12]. We use a characterization of the supply rate of dissipative systems in terms of system's passivity indices. With this characterization of supply rate, we are able to treat a more general case of dissipative systems because it includes both passive and non-passive systems, both $\mathcal{L}_2$ stable and not $\mathcal{L}_2$ stable systems. The "IF-OFP"systems studied in [12] is also a special case of our dissipative systems studied here. The reason for using passivity indices to characterize dissipative systems is because passivity indices not only measure the excess/shortage of passivity of the system[16]-[17], they also contain the stability information of the dissipative systems. Based on these, we demonstrate that the feedback interconnection of two dissipative systems, with non-increasing time delays is either passive or $\mathcal{L}_2$ stable if the systems' passivity indices satisfy certain conditions. In the general case, when the time delay may be increasing or decreasing, the passivity and $\mathcal{L}_2$ stability of the feedback interconnection can no longer be guaranteed. However, if the maximum rate of the change of delay is known, by inserting properly designed gains into the communication path, passivity and $\mathcal{L}_2$ stability of the closed-loop can still be achieved.

The outline of this paper is as follows. We briefly introduce some background material on passivity and passivity indices in Section II, which is followed by our main lemmas in Section III; we present our main theorems which are related to network control in section IV and finally, the conclusion is provided in Section V.

II. Background

Passivity provides us with a useful tool for the analysis of linear/nonlinear systems, which relates nicely to Lyapunov and $\mathcal{L}_2$ stability [2]. Passivity indices, which are defined in terms of excess or shortage of passivity[16]-[17], have been introduced in order to extend the passivity-based stability conditions to the more general cases for both passive and non-passive systems[4]. Most of the discussion presented in this section is related to passivity and passivity indices which lay the foundation of the results developed in this paper. To set the background and notation for what follows, we need to introduce some basic concepts of passivity and passivity indices. Consider the following nonlinear system:

$$H : \begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases} \tag{1}$$

where $x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ and $y \in Y \subset \mathbb{R}^m$ are the state, input and output variables respectively; X, U and Y are state, input and output spaces, respectively. The representation $x(t) = \phi(t, t_0, x_0, u)$ is used to denote the state at time t reached from the initial state x_0 at t_0.

Definition 1 (Supply Rate [13]). *The supply rate $\omega(t) = \omega(u(t), y(t))$ is a real valued function defined on $U \times Y$, such that for any $u(t) \in U$ and $x_0 \in X$ and $y(t) = h(\phi(t, t_0, x_0, u))$, $\omega(t)$ satisfies*

$$\int_{t_0}^{t_1} |\omega(\tau)| d\tau < \infty \tag{2}$$

Definition 2 (Dissipative System [13]). *System H with supply rate $\omega(t)$ is said to be dissipative if there exists a nonnegative real function $V(x) : X \to \mathbb{R}^+$, called the storage function, such that, for all $t_1 \geq t_0 \geq 0$, $x_0 \in X$ and $u \in U$,*

$$V(x_1) - V(x_0) \leq \int_{t_0}^{t_1} \omega(\tau) d\tau \tag{3}$$

where $x_1 = \phi(t_1, t_0, x_0, u)$ and $\mathbb{R}^+$ is a set of nonnegative real numbers.

The above definition states that a system is dissipative if the increase in its energy (storage function) during the interval (t_0, t_1) is no greater than the energy supplied (via the supply rate $\omega(u(t), y(t))$) to it. If the storage function is C^1, then we can write (3) as

$$\frac{dV(x(t))}{dt} \leq \omega(u(t), y(t)) \tag{4}$$

Definition 3 (Passive System [3]). *A system is said to be passive if it is dissipative with respect to the following supply rate:*

$$\omega(u(t), y(t)) = u^T(t) y(t) \tag{5}$$

and the storage function $V(x)$ satisfies $V(0) = 0$.

We can see from the definition that a passive system is a special case of dissipative systems.

Definition 4 (Excess/Shortage of Passivity [4]). *Let $H : u(t) \mapsto y(t)$. System H is said to be:*

- *Input Feed-forward Passive (IFP) if it is dissipative with respect to supply rate $\omega(u(t), y(t)) = u^T(t) y(t) - \nu u^T(t) u(t)$ for some $\nu \in \mathbb{R}$, denoted as IFP(ν).*
- *Output Feedback Passive (OFP) if it is dissipative with respect to the supply rate $\omega(u(t), y(t)) = u^T(t) y(t) - \rho y^T(t) y(t)$ for some $\rho \in \mathbb{R}$, denoted as OFP(ρ).*

ν and ρ are defined as *Input Feed-forward Passivity (IFP) index* and *Output Feedback Passivity (OFP) index*, respectively. A positive ν or ρ means that the system has an excess of passivity. In this case, the system is said to be **strictly input passive** ($\nu > 0$) or **strictly output passive**($\rho > 0$). Clearly, if a system is IFP(ν) or OFP(ρ), then it is also IFP($\nu - \varepsilon$), or OFP($\rho - \varepsilon$) $\forall \varepsilon > 0$. Moreover, if the system has simultaneous IFP and OFP indices, then the system is dissipative with respect to the supply rate given by:

$$\omega(u(t), y(t)) = (1 + \rho\nu) u^T(t) y(t) - \nu u^T(t) u(t) - \rho y^T(t) y(t) \tag{6}$$

which is in a more general form. We can see that if $\rho \geq 0$ and $\nu \geq 0$, then the dissipative system is passive; it is strictly output passive if $\rho > 0$ and $\nu \geq 0$ and it is strictly input passive if $\nu > 0$ and $\rho \geq 0$; for the other cases, the system is only dissipative but not passive. In this paper, we assume that the supply rates for all the dissipative systems considered have the form in (6), with static passivity indices ρ and ν.

III. Main Lemmas

Before we present our main theorems, we first introduce the following lemmas.

Lemma 1 . *Let a dynamical system satisfy the dissipative inequality given by*

$$\dot{V}(x) \leq (1 + \rho\nu) u^T(t) y(t) - \nu u^T(t) u(t) - \rho y^T(t) y(t) \tag{7}$$

where $u(t), y(t) \in \mathbb{R}^m$, ρ and ν are the IFP and OFP passivity indices, and $V(x)$ is the storage function of the system. Then if $\rho > 0$ and $|\nu| < \infty$, the system is $\mathcal{L}_2$ stable.

Proof.

$$\begin{aligned} \dot{V}(x) &\leq (1 + \rho\nu) u^T(t) y(t) - \nu u^T(t) u(t) - \rho y^T(t) y(t) \\ &\leq |1 + \rho\nu| \|u(t)\|_2 \|y(t)\|_2 + |\nu| \|u(t)\|_2^2 - \rho \|y(t)\|_2^2 \end{aligned} \tag{8}$$

then we have

$$\begin{aligned} \dot{V}(x) \leq & -\frac{1}{2\rho}\Big(|1 + \rho\nu| \|u(t)\|_2 - \rho \|y(t)\|_2\Big)^2 \\ & + \Big(\frac{(1 + \rho\nu)^2}{2\rho} + |\nu|\Big) \|u(t)\|_2^2 - \frac{\rho}{2} \|y(t)\|_2^2 \\ \leq & \frac{k^2}{2\rho} \|u(t)\|_2^2 - \frac{\rho}{2} \|y(t)\|_2^2 \end{aligned} \tag{9}$$

where $k^2 = (1 + \rho\nu)^2 + 2\rho|\nu|$. Integrating $\dot{V}(x)$ over $[0, \tau]$ and using $V(x) \geq 0$, and taking the square roots, we arrive at

$$\|y_\tau\|_{\mathcal{L}_2} \leq \frac{k}{\rho} \|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{2V(x(0))}{\rho}} \tag{10}$$

which shows that if $\rho > 0$ and $|\nu| < \infty$ the dissipative system is $\mathcal{L}_2$ stable. Here y_τ and u_τ denote the truncated signal of $y(t)$ and $u(t)$. ■

In view of Lemma 1, the passivity index ρ indicates whether the dissipative system is $\mathcal{L}_2$ stable; but in general, a $\mathcal{L}_2$ stable dissipative system may not be passive because for passive system, we need both $\rho \geq 0$ and $\nu \geq 0$; for output strictly passive system where we have $\rho > 0$, the system is also $\mathcal{L}_2$ stable.

Lemma 2 . *If a dynamical system satisfies the dissipative inequality given by*

$$\dot{V}(x) \leq u^T(t) A y(t) - y^T(t) B y(t) - u^T(t) C u(t) \tag{11}$$

where $u(t), y(t) \in \mathbb{R}^m$, A, B and C are $m \times m$ matrixes and B is positive definite, then the system is $\mathcal{L}_2$ stable.

Proof. Let $a = \|A\|_2 \geq 0$, $c = \|C\|_2 \geq 0$ and $b = \underline{\lambda}(B) > 0$.

Then we have

$$\begin{aligned}\dot{V}(x) &\le a\|u(t)\|_2\|y(t)\|_2 + c\|u(t)\|_2^2 - b\|y(t)\|_2^2 \\ &\le -\frac{1}{2b}\Big(a\|u(t)\|_2 - b\|y(t)\|_2\Big)^2 + \Big(\frac{a^2}{2b}+c\Big)\|u(t)\|_2^2 - \frac{b}{2}\|y(t)\|_2^2 \\ &\le \frac{k^2}{2b}\|u\|_2^2 - \frac{b}{2}\|y(t)\|_2^2\end{aligned} \tag{12}$$

where $k^2 = a^2 + 2bc$. Integrating $\dot{V}(x)$ over $[0,\tau]$, using $V(x) \ge 0$ and taking the square roots, we arrive at

$$\|y_\tau\|_{\mathcal{L}_2} \le \frac{k}{b}\|u_\tau\|_{\mathcal{L}_2} + \sqrt{\frac{2V(x(0))}{b}} \tag{13}$$

which shows the $\mathcal{L}_2$ stability of this dissipative system. Here y_τ and u_τ denote the truncated signal of $y(t)$ and $u(t)$. ■

In view of Lemma 2, if $B > 0$, then the dissipative system is $\mathcal{L}_2$ stable; moreover, if $A > 0$, $B \ge 0$ and $C \ge 0$, the dissipative system is passive because $\dot{V}(x) \le u^T(t)Ay(t)$.

Lemma 3 . *Assume that H is dissipative with the supply rate given in (6), then $\tilde{H} = R^T HR$ with R a nonsingular matrix as shown in Figure 1, is still a dissipative system. Moreover, the new system $\tilde{H}$ is $\mathcal{L}_2$ stable if and only if the system H is $\mathcal{L}_2$ stable.*

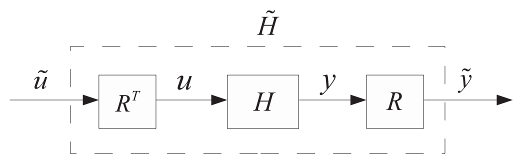

Fig. 1: Transformation of Dissipative System

Proof. Since H is dissipative with the supply rate as given in (6), we have

$$\dot{V}(x) \le (1+\rho\nu)u^T(t)y(t) - \nu u^T(t)u(t) - \rho y^T(t)y(t) \tag{14}$$

with $u(t) = R^T\tilde{u}(t)$, and $y(t) = R^{-1}\tilde{y}(t)$, so

$$\begin{aligned}\dot{\tilde{V}}(x) \le\ & (1+\rho\nu)(R^T\tilde{u}(t))^T R^{-1}\tilde{y}(t) - \nu(R^T\tilde{u}(t))^T(R^T\tilde{u}(t)) \\ & -\rho(R^{-1}\tilde{y}(t))^T(R^{-1}\tilde{y}(t))\end{aligned} \tag{15}$$

and we get

$$\dot{\tilde{V}}(x) \le (1+\rho\nu)\tilde{u}^T(t)\tilde{y}(t) - \nu\tilde{u}^T(t)RR^T\tilde{u}(t) - \rho\tilde{y}^T(t)(R^{-1})^T R^{-1}\tilde{y}(t) \tag{16}$$

this shows that the new system $\tilde{H}$ is still a dissipative system with the input $\tilde{u}$ and the output $\tilde{y}$. Moreover, if $\rho > 0$, then $\rho(R^{-1})^T R^{-1}$ is positive definite. Then from Lemma 2 and Lemma 3, it can been seen that $\tilde{H}$ is $\mathcal{L}_2$ stable if and only if the original system H is $\mathcal{L}_2$ stable. ■

IV. Main Theorems

Consider the feedback interconnection as shown in Figure 2, where system H_1 and H_2 are both dissipative systems with the supply rate given in (6). We first present our passivity and $\mathcal{L}_2$ stability results of the closed-loop system for the no time delays case.

Theorem 4.1 . *Consider the feedback interconnection as shown in Figure 2, where system H_1 is dissipative with respect to the supply rate $\omega_1(t) = (1+\rho_1\nu_1)u_1^T(t)y_1(t) - \nu_1 u_1^T(t)u_1(t) - \rho_1 y_1^T(t)y_1(t)$ and system H_2 is dissipative with respect to the supply rate $\omega_2(t) = (1+\rho_2\nu_2)u_2^T(t)y_2(t) - \nu_2 u_2^T(t)u_2(t) - \rho_2 y_2^T(t)y_2(t)$. The feedback interconnection is*

- *Passive from $r_i(t)$ to $y_i(t)$, for $i = 1,2$ and passive from $r = [r_1^T(t)\ r_2^T(t)]^T$ to $y = [y_1^T(t)\ y_2^T(t)]^T$ if*

$$\begin{aligned}\rho_1 \ge 0, \quad \nu_1 \ge 0, \\ \rho_2 \ge 0, \quad \nu_2 \ge 0;\end{aligned} \tag{17}$$

- *$\mathcal{L}_2$ stable with input $r = [r_1^T(t)\ r_2^T(t)]^T$ and output $y = [y_1^T(t)\ y_2^T(t)]^T$ if*

$$\begin{aligned}\rho_1 + \nu_2 > |\frac{1}{2}(1+\rho_1\nu_1) - \frac{1}{2}(1+\rho_2\nu_2)| \\ \rho_2 + \nu_1 > |\frac{1}{2}(1+\rho_1\nu_1) - \frac{1}{2}(1+\rho_2\nu_2)|\ .\end{aligned} \tag{18}$$

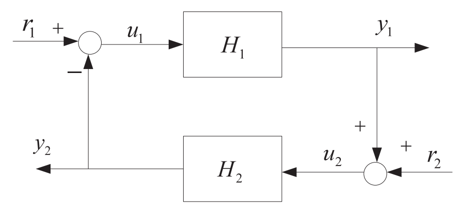

Fig. 2: Feedback Interconnection of Dissipative Systems

Proof. Since H_1 and H_2 are dissipative with storage functions $V_1(x)$ and $V_2(x)$ satisfying

$$\begin{aligned}\dot{V}_1(x) \le (1+\rho_1\nu_1)u_1^T(t)y_1(t) - \nu_1 u_1^T(t)u_1(t) - \rho_1 y_1^T(t)y_1(t) \\ \dot{V}_2(x) \le (1+\rho_2\nu_2)u_2^T(t)y_2(t) - \nu_2 u_2^T(t)u_2(t) - \rho_2 y_2^T(t)y_2(t)\end{aligned} \tag{19}$$

then, if $\rho_1, \nu_1 \ge 0$ and $\rho_2, \nu_2 \ge 0$

$$\dot{V}_1(x) \le (1+\rho_1\nu_1)u_1^T(t)y_1(t), \quad \dot{V}_2(x) \le (1+\rho_2\nu_2)u_2^T(t)y_2(t) \tag{20}$$

or

$$\begin{aligned}\dot{\hat{V}}_1(x) = \frac{\dot{V}_1}{1+\rho_1\nu_1} \le u_1^T(t)y_1(t) \\ \dot{\hat{V}}_2(x) = \frac{\dot{V}_2}{1+\rho_2\nu_2} \le u_2^T(t)y_2(t)\end{aligned} \tag{21}$$

which shows that the feed-forward path from $r_1(t)$ to $y_1(t)$ and the feedback path from $r_2(t)$ to $y_2(t)$ are both passive. Since $u_1(t) = r_1(t) - y_2(t)$, $u_2(t) = r_2(t) + y_1(t)$, we can obtain

$$\dot{\hat{V}} = \dot{\hat{V}}_1 + \dot{\hat{V}}_2 \le r_1^T(t)y_1(t) + r_2^T(t)y_2(t) \tag{22}$$

which shows that the closed-loop is passive from input $r = [r_1^T(t)\ r_2^T(t)]^T$ to output $y = [y_1^T(t)\ y_2^T(t)]^T$. Moreover, in view of (19), we obtain

$$\dot{V} = \dot{V}_1 + \dot{V}_2 \le r^T(t)Ay(t) - y^T(t)By(t) - r^T(t)Cr(t) \tag{23}$$

where $V(x)$ is the storage function for the closed-loop and

$$A = \begin{pmatrix}1+\rho_1\nu_1 & 2\nu_1 \\ -2\nu_2 & 1+\rho_2\nu_2\end{pmatrix} \quad B = \begin{pmatrix}b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix} \tag{24}$$

$$C = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} \tag{25}$$

with

$$b_{11} = \rho_1 + \nu_2,\; b_{12} = \frac{1}{2}(1+\rho_1\nu_1) - \frac{1}{2}(1+\rho_2\nu_2),$$

$$b_{22} = \rho_2 + \nu_1,\; b_{21} = \frac{1}{2}(1+\rho_1\nu_1) - \frac{1}{2}(1+\rho_2\nu_2).$$

According to Lemma 3, if the symmetric matrix $B > 0$, the closed-loop is $\mathcal{L}_2$ stable, thus the sufficient conditions for the feedback interconnection to be $\mathcal{L}_2$ stable with input r and output y are given by (18). ■

As presented by Theorem 4.1, for the no time delays case, we have shown conditions under which the feedback interconnection of two dissipative systems will be passive or $\mathcal{L}_2$ stable. Our next theorem shows a way to relax the above conditions by pre-multiplying and post-multiplying both the feed-forward and feedback systems with a diagonal matrix.

Theorem 4.2 . *Consider the feedback interconnection as shown in Figure 3, where the system H_1 is dissipative with respect to the supply rate $\omega_1(t) = (1+\rho_1\nu_1)u_1^T(t)y_1(t) - \nu_1 u_1^T(t)u_1(t) - \rho_1 y_1^T(t)y_1(t)$ and the system H_2 is dissipative with respect to the supply rate $\omega_2(t) = (1+\rho_2\nu_2)u_2^T(t)y_2(t) - \nu_2 u_2^T(t)u_2(t) - \rho_2 y_2^T(t)y_2(t)$, $d_1,\ d_2 \in \mathbb{R}$. Then the closed-loop is*

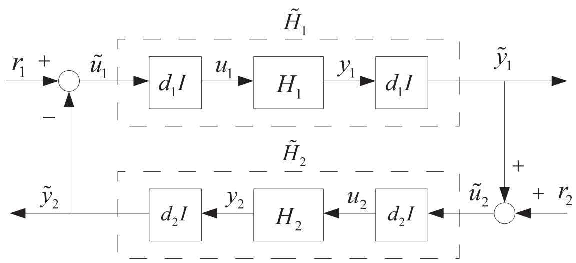

Fig. 3: Feedback Interconnection of Dissipative Systems

- *Passive from input $r = [r_1^T(t)\ \ r_2^T(t)]^T$ to output $\tilde{y} = [\tilde{y}_1^T(t)\ \tilde{y}_2^T(t)]^T$, if*

$$\begin{aligned} \rho_1 \geq 0,\quad \nu_1 \geq 0,\\ \rho_2 \geq 0,\quad \nu_2 \geq 0; \end{aligned} \tag{26}$$

- *$\mathcal{L}_2$ stable with input r and output $\tilde{y}$, if*

$$\begin{aligned} \frac{\rho_1}{d_1^2} + d_2^2\nu_2 > |\frac{1}{2}(1+\rho_1\nu_1) - \frac{1}{2}(1+\rho_2\nu_2)| \\ \frac{\rho_2}{d_2^2} + d_1^2\nu_1 > |\frac{1}{2}(1+\rho_1\nu_1) - \frac{1}{2}(1+\rho_2\nu_2)|\ . \end{aligned} \tag{27}$$

Proof. The proof is similar to the proof in Theorem 4.1 by using Lemma 3. ■

Remarks: Theorem 4.2 shows that in the case when the passivity indices of the feed-forward and feedback systems do not satisfy the conditions shown in Theorem 4.1, a good choice of the weights d_1 and d_2 may enable us to make the closed-loop system $\mathcal{L}_2$ stable.

Next we also consider time-varying delays in the communication network and we derive conditions under which passivity and $\mathcal{L}_2$ stability of the feedback interconnection of two dissipative systems can be preserved.

Theorem 4.3 . *Consider the feedback interconnection as shown in Figure 4, with H_1 and H_2 being the same dissipative systems as discussed in Theorem 4.1. If the delays $T_1(t)$ and $T_2(t)$ are non-increasing, then the closed-loop is*

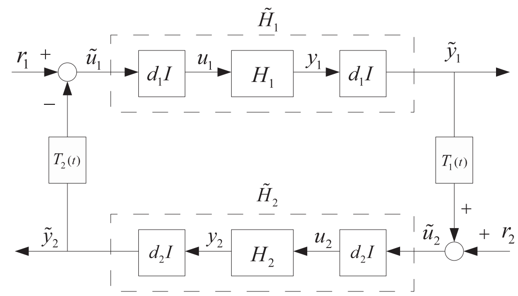

Fig. 4: Feedback Interconnection of Dissipative Systems

- *Passive from $r = [r_1^T(t),\ r_2^T(t)]^T$ to $\tilde{y} = [\tilde{y}_1^T(t),\ \tilde{y}_2^T(t)]^T$ if*

$$\begin{aligned} 1+\rho_1\nu_1 > 0,\quad \nu_1 \geq 0,\quad \frac{\rho_1}{d_1^2} \geq \frac{1}{2}|1+\rho_1\nu_1| + \frac{1}{2}|1+\rho_2\nu_2| \\ 1+\rho_2\nu_2 > 0,\quad \nu_2 \geq 0,\quad \frac{\rho_2}{d_2^2} \geq \frac{1}{2}|1+\rho_1\nu_1| + \frac{1}{2}|1+\rho_2\nu_2|\ ; \end{aligned} \tag{28}$$

- *$\mathcal{L}_2$ stable with input $r = [r_1^T(t)\ r_2^T(t)]^T$ and output $\tilde{y} = [\tilde{y}_1^T(t)\ \tilde{y}_2^T(t)]^T$ if*

$$\begin{aligned} \frac{\rho_1}{d_1^2} + d_2^2\nu_2 > \frac{1}{2}|1+\rho_1\nu_1| + \frac{1}{2}|1+\rho_2\nu_2| + d_2^2|\nu_2| \\ \frac{\rho_2}{d_2^2} + d_1^2\nu_1 > \frac{1}{2}|1+\rho_1\nu_1| + \frac{1}{2}|1+\rho_2\nu_2| + d_1^2|\nu_1|\ . \end{aligned} \tag{29}$$

Proof. According to Lemma 3, the storage function for $\tilde{H}_1$ and $\tilde{H}_2$ satisfy

$$\begin{aligned} \dot{\tilde{V}}_1(x) \leq (1+\rho_1\nu_1)\tilde{u}_1^T(t)\tilde{y}_1(t) - d_1^2\nu_1\tilde{u}_1^T(t)\tilde{u}_1(t) - \frac{\rho_1}{d_1^2}\tilde{y}_1^T(t)\tilde{y}_1(t) \\ \dot{\tilde{V}}_2(x) \leq (1+\rho_2\nu_2)\tilde{u}_2^T(t)\tilde{y}_2(t) - d_2^2\nu_2\tilde{u}_2^T(t)\tilde{u}_2(t) - \frac{\rho_2}{d_2^2}\tilde{y}_2^T(t)\tilde{y}_2(t) \end{aligned} \tag{30}$$

with $\tilde{u}_1(t) = r_1(t) - \tilde{y}_2(t - T_2(t))$ and $\tilde{u}_2(t) = r_2(t) + \tilde{y}_1(t - T_1(t))$, we can get

$$\begin{aligned} &\dot{\tilde{V}}_1 + \dot{\tilde{V}}_2 \leq (1+\rho_1\nu_1)r_1^T(t)\tilde{y}_1(t) + (1+\rho_2\nu_2)r_2^T(t)\tilde{y}_2(t) \\ &+ \left(\frac{1}{2}|1+\rho_1\nu_1| + \frac{1}{2}|1+\rho_2\nu_2| - \frac{\rho_1}{d_1^2} + |d_2^2\nu_2| - d_2^2\nu_2\right)\|\tilde{y}_1(t)\|_2^2 \\ &+ \left(\frac{1}{2}|1+\rho_1\nu_1| + \frac{1}{2}|1+\rho_2\nu_2| - \frac{\rho_2}{d_2^2} + |d_1^2\nu_1| - d_1^2\nu_1\right)\|\tilde{y}_2(t)\|_2^2 \\ &+ (|d_1^2\nu_1| - d_1^2\nu_1)\|r_1(t)\|_2^2 + (|d_2^2\nu_2| - d_2^2\nu_2)\|r_2(t)\|_2^2 - \phi(t)\ , \end{aligned} \tag{31}$$

with

$$\phi(t)=(\frac{1}{2}|1+\rho_2\nu_2|+|d_2^2\nu_2|-d_2^2\nu_2)\Big(\|\tilde{y}_1(t)\|_2^2-\|\tilde{y}_1(t-T_1(t))\|_2^2\Big)$$
$$+(\frac{1}{2}|1+\rho_1\nu_1|+|d_1^2\nu_1|-d_1^2\nu_1)\Big(\|\tilde{y}_2(t)\|_2^2-\|\tilde{y}_2(t-T_2(t))\|_2^2\Big)\,. \tag{32}$$

Let $\alpha=(\frac{1}{2}|1+\rho_2\nu_2|+|d_2^2\nu_2|-d_2^2\nu_2)$ and $\beta=(\frac{1}{2}|1+\rho_1\nu_1|+|d_1^2\nu_1|-d_1^2\nu_1)$, we can see that $\alpha\geq 0$ and $\beta\geq 0$; then by integrating $\phi(t)$ we have

$$\int_0^t\phi(\tau)d\tau=\alpha\Big(\int_{t-T_1(t)}^t\|\tilde{y}_1(\tau)\|_2^2d\tau+\int_0^{t-T_1(t)}\|\tilde{y}_1(\tau)\|_2^2d\tau$$
$$-\int_0^t\|\tilde{y}_1(\tau-T_1(\tau))\|_2^2d\tau\Big)+\beta\Big(\int_{t-T_2(t)}^t\|\tilde{y}_2(\tau)\|_2^2d\tau$$
$$+\int_0^{t-T_2(t)}\|\tilde{y}_2(\tau)\|_2^2d\tau-\int_0^t\|\tilde{y}_2(\tau-T_2(\tau))\|_2^2d\tau\Big)\,. \tag{33}$$

Let $\sigma_1=\tau-T_1(\tau)$ and $\sigma_2=\tau-T_2(\tau)$, we can get

$$d\tau=\frac{d\sigma_1}{1-\frac{dT_1(\tau)}{d\tau}}\quad,\ d\tau=\frac{d\sigma_2}{1-\frac{dT_2(\tau)}{d\tau}}\,. \tag{34}$$

Then,

$$\int_0^t\phi(\tau)d\tau=\alpha\int_{t-T_1(t)}^t\|\tilde{y}_1(\tau)\|_2^2d\tau+\beta\int_{t-T_2(t)}^t\|\tilde{y}_2(\tau)\|_2^2d\tau$$
$$-\alpha\int_0^{t-T_1(t)}\frac{\frac{dT_1(\tau)}{d\tau}}{1-\frac{dT_1(\tau)}{d\tau}}\|\tilde{y}_1(\sigma_1)\|_2^2d\sigma_1$$
$$-\beta\int_0^{t-T_2(t)}\frac{\frac{dT_2(\tau)}{d\tau}}{1-\frac{dT_2(\tau)}{d\tau}}\|\tilde{y}_2(\sigma_2)\|_2^2d\sigma_2 \tag{35}$$

Since $\alpha\geq 0$ and $\beta\geq 0$, if $\frac{dT_1(t)}{dt}\leq 0$ and $\frac{dT_2(t)}{dt}\leq 0$, then $\int_0^t\phi(\tau)d\tau\geq 0,\ \forall t$, thus $\phi(t)\geq 0$. Then we can get

$$\dot{\tilde{V}}_1+\dot{\tilde{V}}_2\leq r^T\tilde{A}\tilde{y}-\tilde{y}^T\tilde{B}\tilde{y}-r^T\tilde{C}r \tag{36}$$

where

$$\tilde{A}=\begin{pmatrix}1+\rho_1\nu_1 & 0\\ 0 & 1+\rho_2\nu_2\end{pmatrix}\quad \tilde{B}=\begin{pmatrix}\tilde{b}_{11} & 0\\ 0 & \tilde{b}_{22}\end{pmatrix} \tag{37}$$

$$\tilde{C}=\begin{pmatrix}\hat{\nu}_1-|\hat{\nu}_1| & 0\\ 0 & \hat{\nu}_2-|\hat{\nu}_2|\end{pmatrix} \tag{38}$$

with

$$\tilde{b}_{11}=\hat{\rho}_1+\hat{\nu}_2-|\hat{\nu}_2|-\frac{1}{2}|1+\rho_1\nu_1|-\frac{1}{2}|1+\rho_2\nu_2|$$
$$\tilde{b}_{22}=\hat{\rho}_2+\hat{\nu}_1-|\hat{\nu}_1|-\frac{1}{2}|1+\rho_1\nu_1|-\frac{1}{2}|1+\rho_2\nu_2|$$

and with $\hat{\rho}_1=\rho_1/d_1^2$, $\hat{\rho}_2=\rho_2/d_2^2$, $\hat{\nu}_1=d_1^2\nu_1$, $\hat{\nu}_2=d_2^2\nu_2$. Again, for the closed-loop system to be passive, we need $\tilde{A}>0$, $\tilde{B}\geq 0$ and $\tilde{C}\geq 0$, which yields the conditions shown in (28) and for the closed-loop system to be $\mathcal{L}_2$ stable, we need $\tilde{B}>0$ which yields the conditions shown in (29). ■

We have shown that if the time delays in the communication networks are non-increasing, then under the conditions of Theorem 4.3, the feedback interconnection of two dissipative systems could be passive or $\mathcal{L}_2$ stable. Our next result shows a way to deal with time-varying delays by inserting proper gains into the communication network. This is the same method used in [11], but instead of passive or output strictly passive systems, we use this method for the network control of dissipative systems.

Theorem 4.4 . *Consider the feedback interconnection of two dissipative systems H_1 and H_2 as shown in Figure 5, where H_1 and H_2 are the same systems as in Theorem 4.1.*

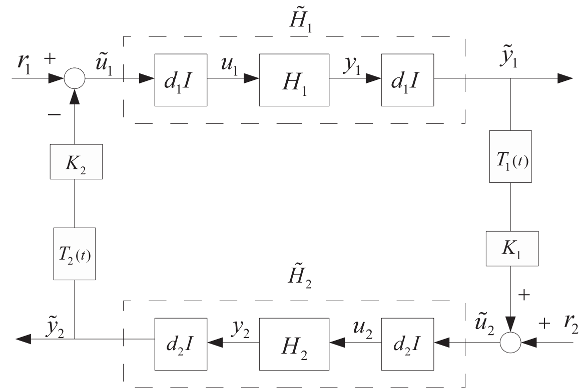

Fig. 5: Feedback Interconnection of Dissipative Systems

Assume that the change rate of the delay is bounded by $T_i'(t)\in[0,1), i=1,2$. If $K_i=1-max\{T_i'(t)\}, i=1,2$, then the closed-loop system is

- *Passive from $r=[r_1^T(t),\ r_2^T(t)]^T$ to $\tilde{y}=[\tilde{y}_1^T(t),\ \tilde{y}_2^T(t)]^T$ if $1+\rho_1\nu_1>0,\ \hat{\nu}_1\geq 0,\ 1+\rho_2\nu_2>0,\ \ \hat{\nu}_2\geq 0$, and*

$$\hat{\rho}_1\geq\frac{1}{2}K_2|1+\rho_1\nu_1|+\frac{1}{2}|1+\rho_2\nu_2|+(1-K_1)\hat{\nu}_2$$
$$\hat{\rho}_2\geq\frac{1}{2}K_1|1+\rho_2\nu_2|+\frac{1}{2}|1+\rho_1\nu_1|+(1-K_2)\hat{\nu}_1\ ; \tag{39}$$

- *$\mathcal{L}_2$ stable with input $r=[r_1^T(t)\ r_2^T(t)]^T$ and output $\tilde{y}=[\tilde{y}_1^T(t)\ \tilde{y}_2^T(t)]^T$ if*

$$\hat{\rho}_1>\frac{1}{2}K_2|1+\rho_1\nu_1|+\frac{1}{2}|1+\rho_2\nu_2|+|\hat{\nu}_2|-K_1\hat{\nu}_2$$
$$\hat{\rho}_2>\frac{1}{2}K_1|1+\rho_2\nu_2|+\frac{1}{2}|1+\rho_1\nu_1|+|\hat{\nu}_1|-K_2\hat{\nu}_1\ ; \tag{40}$$

where $\hat{\rho}_1=\rho_1/d_1^2$, $\hat{\rho}_2=\rho_2/d_2^2$ and $\hat{\nu}_1=d_1^2\nu_1,\ \hat{\nu}_2=d_2^2\nu_2$.

Proof. In the present set-up, with $\tilde{u}_1(t)=r_1(t)-K_2\tilde{y}_2(t-T_2(t))$, $\tilde{u}_2(t)=r_2(t)+K_1\tilde{y}_1(t-T_1(t))$, we can get

$$\dot{\tilde{V}}_1+\dot{\tilde{V}}_2\leq(1+\rho_1\nu_1)r_1^T(t)\tilde{y}_1(t)+(1+\rho_2\nu_2)r_2^T(t)\tilde{y}_2(t)$$
$$+\Big(\frac{1}{2}K_2|1+\rho_1\nu_1|+\frac{1}{2}|1+\rho_2\nu_2|-\frac{\rho_1}{d_1^2}+|d_2^2\nu_2|-K_1d_2^2\nu_2\Big)\|\tilde{y}_1(t)\|_2^2$$
$$+\Big(\frac{1}{2}K_1|1+\rho_2\nu_2|+\frac{1}{2}|1+\rho_1\nu_1|-\frac{\rho_2}{d_2^2}+|d_1^2\nu_1|-K_2d_1^2\nu_1\Big)\|\tilde{y}_2(t)\|_2^2$$
$$+(K_2|d_1^2\nu_1|-d_1^2\nu_1)\|r_1(t)\|_2^2+(K_1|d_2^2\nu_2|-d_2^2\nu_2)\|r_2(t)\|_2^2-\phi(t)\,, \tag{41}$$

with

$$\phi(t) = (\frac{1}{2}|1+\rho_2\nu_2| + |\hat{\nu}_2| - K_1\hat{\nu}_2)\Big(\|\tilde{y}_1(t)\|_2^2 - K_1\|\tilde{y}_1(t-T_1(t))\|_2^2\Big) + (\frac{1}{2}|1+\rho_1\nu_1| + |\hat{\nu}_1| - K_2\hat{\nu}_1)\Big(\|\tilde{y}_2(t)\|_2^2 - K_2\|\tilde{y}_2(t-T_2(t))\|_2^2\Big)\,. \tag{42}$$

Let $\alpha = (\frac{1}{2}|1+\rho_2\nu_2| + |\hat{\nu}_2| - K_1\hat{\nu}_2)$ and $\beta = (\frac{1}{2}|1+\rho_1\nu_1| + |\hat{\nu}_1| - K_2\hat{\nu}_1)$; since we choose $K_i = 1 - max\{T_i^{'}(t)\}$ and $T_i^{'}(t) \in [0,1), i = 1,2$, we have $K_i \in (0,1], i = 1,2$; thus $\alpha \geq 0$ and $\beta \geq 0$. Again, let $\sigma_1 = \tau - T_1(\tau)$ and $\sigma_2 = \tau - T_2(\tau)$, by using the change of variables in the same way as shown in the proof of Theorem 4.3, we can get

$$\begin{aligned}\int_0^t \phi(\tau)d\tau &\geq \alpha\int_{t-T_1(t)}^t \|\tilde{y}_1(\tau)\|_2^2 d\tau + \beta\int_{t-T_2(t)}^t \|\tilde{y}_2(\tau)\|_2^2 d\tau \\ &+ \alpha\int_0^{t-T_1(t)}\Big(\|\tilde{y}_1(\sigma_1)\|_2^2 - \frac{K_1}{1-max\{T_1^{'}(\tau)\}}\|\tilde{y}_1(\sigma_1)\|_2^2\Big)d\sigma_1 \\ &+ \beta\int_0^{t-T_2(t)}\Big(\|\tilde{y}_2(\sigma_2)\|_2^2 - \frac{K_2}{1-max\{T_2^{'}(\tau)\}}\|\tilde{y}_2(\sigma_2)\|_2^2\Big)d\sigma_2\end{aligned} \tag{43}$$

since $K_i = 1 - max\{T_i^{'}(t)\}, i = 1,2$, it follows that

$$\int_0^t \phi(\tau)d\tau \geq \alpha\int_{t-T_1(t)}^t \|\tilde{y}_1(\tau)\|_2^2 d\tau + \beta\int_{t-T_2(t)}^t \|\tilde{y}_2(\tau)\|_2^2 d\tau \geq 0 \tag{44}$$

which implies $\phi(t) \geq 0$. Then we arrive at

$$\dot{\tilde{V}}_1 + \dot{\tilde{V}}_2 \leq r^T\tilde{A}\tilde{y} - \tilde{y}^T\tilde{B}\tilde{y} - r^T\tilde{C}r \tag{45}$$

where

$$\tilde{A} = \begin{pmatrix} 1+\rho_1\nu_1 & 0 \\ 0 & 1+\rho_2\nu_2 \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} \tilde{b}_{11} & 0 \\ 0 & \tilde{b}_{22} \end{pmatrix} \tag{46}$$

and

$$\tilde{C} = \begin{pmatrix} \hat{\nu}_1 - K_2|\hat{\nu}_1| & 0 \\ 0 & \hat{\nu}_2 - K_1|\hat{\nu}_2| \end{pmatrix} \tag{47}$$

with

$$\tilde{b}_{11} = \hat{\rho}_1 + K_1\hat{\nu}_2 - |\hat{\nu}_2| - \frac{1}{2}K_2|1+\rho_1\nu_1| - \frac{1}{2}|1+\rho_2\nu_2|$$

$$\tilde{b}_{22} = \hat{\rho}_2 + K_2\hat{\nu}_1 - |\hat{\nu}_1| - \frac{1}{2}K_1|1+\rho_1\nu_1| - \frac{1}{2}|1+\rho_2\nu_2|.$$

Again, for the closed-loop system to be passive, we need $\tilde{A} > 0$, $\tilde{B} \geq 0$ and $\tilde{C} \geq 0$, which yields the conditions shown in (39); and for the closed-loop system to be $\mathcal{L}_2$ stable, we need $\tilde{B} > 0$ which yields the conditions shown in (40). ■

V. Conclusion

In this paper, we studied passivity and $\mathcal{L}_2$ stability of a negative feedback interconnection of dissipative systems with and without time delays in communication. We characterized the supply rate of dissipative systems in terms of passivity indices, and we have shown that our study of dissipative systems is a more general case since it characterizes both passive and non-passive systems and also both $\mathcal{L}_2$ and not $\mathcal{L}_2$ stable systems. We derived conditions for the closed-loop system to be passive or $\mathcal{L}_2$ stable under three different situations *i*) without considering time delays; *ii*) with non-increasing time delays; *iii*) with increasing time delays where the maximum change rate of the delay is known. We have shown that in the general case, when the time delay may be increasing or decreasing, the passivity and $\mathcal{L}_2$ stability of the feedback interconnection can no longer be guaranteed. However, if the maximum rate of the change of delay is known, by inserting properly designed gains into the communication path, passivity and $\mathcal{L}_2$ stability can be preserved in the closed-loop .

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