Passivity of Cascaded Systems Based on The Analysis of Passivity Indices

Technical Report of the ISIS Group

at the University of Notre Dame

ISIS-09-008

Nov, 2009

Han Yu and Panos J. Antsaklis

Department of Electrical Engineering

University of Notre Dame

Notre Dame, IN 46556

Interdisciplinary Studies in Intelligent Systems

Passivity of Cascaded Systems Based on The Analysis of Passivity Indices

Han Yu and Panos J. Antsaklis
Department of Electrical Engineering
University of Notre Dame
Notre Dame, IN 46556
e-mail: hyu@nd.edu,antsaklis.1@nd.edu

Abstract

Passivity index is defined in terms of an excess or shortage of passivity, and it has been introduced in order to extend the passivity-based stability conditions to the more general cases for both passive and non-passive systems. In this report, we revisit the secant criterion literature results from the perspective of passivity indices. While most of the passivity-based stability results in literature focus on studying the feedback interconnection of passive or non-passive systems, our results focus on the study of cascaded interconnection. In this report, we show how to use the secant criterion to quantify the excess/shortage of passivity for cascaded system which includes both passive and non-passive systems. We further show that under certain conditions, the cascaded interconnection can be directly stabilized via output feedback.

Index Terms

Passivity Index, Cascaded Systems, Input Feed-forward Passivity(IFP), Output Feedback Passivity(OFP), Secant Criterion, Storage Function.

I. Introduction

In the recent paper of Murat Arcak and Eduardo D.Sontag [1]-[2], the classical secant criterion which is an often-used tool in the analysis of biological feedback loops, see [3] and [4], has been revisited and its advantage in the analysis of a class of output strictly passive (OSP) systems either with a cascade or with a cyclic ineterconnection structure has been shown. In [1], the secant criterion is generalized as the stability analysis tool for a class of cascaded OSP systems. One should notice that the analysis shown in [1] is based on the assumption that each subsystem admits a storage function of the form given by $\|y\|^2$ where $\|\cdot\|$ denotes the $\mathcal{L}_2$ norm, and each subsystem is passive with the supply rate given by $\gamma u^T y$, where γ is a "gain"associated with each OSP system. Under those assumptions, if the product of each subsystem's associated "gain" satisfies the secant criterion, the cascaded system is $\mathcal{L}_2$ stable.

In [2], the authors show that the secant criterion developed earlier in the literature is in fact a necessary and sufficient condition for diagonal stability of a class of matrices. Then they use the secant criterion and the diagonal stability results as a tool to construct a Lyapunov function for the stabilization problem of a class of OSP systems with a cyclic interconnection structure. The reason why "diagonal stability"of the corresponding class of matrices is of special interest is that it enables us to choose the proper weight for each subsystem's storage function and this contributes to the construction of a Lyapunov function, which is a weighted sum of each subsystem's storage function. While [1] restricts the form of each subsystem's storage function and supply rate to yield the $\mathcal{L}_2$ stability results, [2] does not restrict the form of each subsystem's storage function but still employs the systematic use of a "gain"associated to each OSP system's supply rate as in [1]. It is also shown the lack of input feed-forward passivity for this particular class of cascaded OSP systems.

[1] and [2] motivate us to revisit the systematic "gains"associated with each OSP system's supply rate from the perspective of passivity indices. Without restricting the form of each subsystem's storage function and instead of associating each subsystem's supply rate with a "gain", we use passivity indices to quantify the excess or shortage of passivity for each subsystems and explore how this kind of quantification will affect the overall degree of passivity for the entire cascaded interconnection, and furthermore, how this kind of quantification will affect the stability of the entire cyclic interconnection. We believe this kind of quantification is a convenient analysis tool for the design of control systems in the future.

II. Background Material

Much of the discussion presented in this section is related to passivity and passivity index which lay the foundation of the results developed in this paper. In other words, we view systems from an input-output perspective. To set the background and notation for what follows, we need to introduce some basic concepts of passive system and passivity index.

Consider the following nonlinear system [5]:

$$H : \begin{cases} \dot{x} = f(x,u) \\ y = h(x,u) \end{cases} \tag{1}$$

where $x \in X \subset \mathbb{R}^n$, $u \in U \subset \mathbb{R}^m$ and $y \in Y \subset \mathbb{R}^m$ are the state, input and output variables, respectively, and X, U and Y are state, input and output spaces, respectively. The representation $x(t) = \phi(t, t_0, x_0, u)$ is used to denote the state at time t reached from the initial state x_0 at t_0.

Definition 1 (Passive System). *The dynamic system given in (1)is said to be* ***passive*** *if there exists a $\mathcal{C}^1$ storage function $V(x) \geq 0$ such that*

$$\dot{V} = \frac{\partial V(x)}{\partial x} f(x(t), u(t)) \leq -S(x) + u(t)^T y(t) \tag{2}$$

for some positive semi-definite function $S(x)$. We say it is ***strictly passive*** *if $S(x) > 0$. Here $u(t)^T y(t) - S(x)$ is defined as* ***supply rate****, and*

$$\int_{t_0}^{t} |u(t)^T y(t) - S(x)| < \infty \tag{3}$$

Definition 2 (Zero-State Observability and Detectability [6]). *A system as given in (1) is zero-state observable (ZSO) if for any $x \in X$,*

$$y(t) = h(\phi(t, t_0, x, 0)) = 0, \quad \forall t \geq t_0 \geq 0 \ \ implies \ \ x = 0, \tag{4}$$

and the system is zero-state detectable (ZSD) if for any $x \in X$

$$y(t) = h(\phi(t, t_0, x, 0)) = 0, \quad \forall t \geq t_0 \geq 0 \ \ implies \ \ lim_{t \to \infty} \phi(t, t_0, x, 0) = 0. \tag{5}$$

With the definition of zero-state detectability (ZSD), the relation between passivity and Lyapunov stability can be established.

Theorem 1 (Passivity and Stability [7]). *Let a system H (as represented in (1)) be passive with a $\mathcal{C}^1$ storage function $V(x)$ and $h(x,u)$ be $\mathcal{C}^1$ in u for all x. Then the following properties hold:*

1) *If $V(x)$ is positive definite, then the equilibrium $x = 0$ of H with $u = 0$ is Lyapunov stable.*

2) *If H is ZSD, then the equilibrium $x = 0$ of H with $u = 0$ is Lyapunov stable.*
3) *If in addition to either Condition 1 or Condition 2, $V(x)$ is radially unbounded (i.e., $V(x) \to \infty$ as $\|x\| \to \infty$), then the equilibrium $x = 0$ in the above condition is globally stable (GS).*

From Theorem 1, we can see that if a system is passive, and if its storage function is positive and radially unbounded, we can choose its storage function as the Lyapunov function and stability results under zero inputs will follow. To extend the passivity-based stability conditions to more general cases for both passive and non-passive systems, we need to introduce passivity indices which are defined in terms of an excess or shortage of passivity.

Definition 3 (Excess/Shortage of Passivity [7]). *Let $H : u \mapsto y$. System H is said to be:*

- *Input Feed-forward Passive (IFP) if it is dissipative with respect to supply rate $\omega(u, y) = u^T y - \nu u^T u$ for some $\nu \in \mathbb{R}$, denoted as OFP(ν).*
- *Output Feedback Passive (OFP) if it is dissipative with respect to the supply rate $\omega(u, y) = u^T y - \rho y^T y$ for some $\rho \in \mathbb{R}$, denoted as OFP(ρ).*

A positive ν or ρ means that the system has an excess of passivity. In this case, the system is said to be ***strictly input passive*** *or* ***strictly output passive*** *respectively.*

III. Diagonal stability and its connection with the secant criterion

The connection between diagonal stability and the secant criterion has been shown in [1]-[2]. We briefly introduce their results as follows.

Definition 4 (Diagonal Stability [8]). *A matrix $A := (a_{ij})$ is said to belong to the class of Hurwitz diagonally stable matrix if there exists $D > 0$ diagonal such that*

$$A^T D + DA < 0 \tag{6}$$

Theorem 2 (Secant Criterion [2]). *A matrix of the form:*

$$A = \begin{bmatrix} -\alpha_1 & 0 & \cdots & 0 & -\beta_n \\ \beta_1 & -\alpha_2 & \ddots & & 0 \\ 0 & \beta_2 & -\alpha_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_{n-1} & -\alpha_n \end{bmatrix} \qquad \alpha_i > 0, \quad \beta_i > 0, \quad i = 1, \cdots, n \tag{7}$$

is diagonally stable, that is, it satisfies (6) for some diagonal matrix $D > 0$, if and only if the secant criterion

$$\frac{\beta_1 \cdots \beta_n}{\alpha_1 \cdots \alpha_n} < sec(\pi/n)^n = \frac{1}{cos(\frac{\pi}{n})^n} \tag{8}$$

holds; here we assume $n > 2$.

In the subsequent sections, we will show how we use this secant criterion to quantify the shortage/excess of passivity for cascaded systems.

IV. The shortage of passivity in cascaded output strictly passive systems

Proposition 1 . *Consider the cascade interconnection shown in Fig.1, where $n \geq 2$. If each block is OFP(ρ_i) with $\rho_i \in \mathbb{R}^+$, namely there exists a $\mathcal{C}^1$ storage function V_i for each subsystem, such that*

$$\dot{V}_i \leq -\rho_i y_i^T y_i + u_i^T y_i \ . \tag{9}$$

Then for some $\nu \in \mathbb{R}^+$, such that

$$\nu > \frac{cos(\frac{\pi}{n+1})^{n+1}}{\rho_1 \rho_2 \ldots \rho_n} \ , \tag{10}$$

the cascaded system admits a storage function of the form

$$V = \sum_{i=1}^{n} d_i V_i, \quad d_i > 0 \tag{11}$$

and the cascaded interconnection is Input Feed-forward Passive with passivity index -ν (so the system with positive feed-forward νI will be passive.)

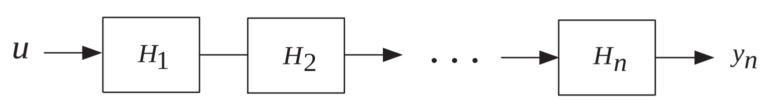


Fig. 1: Cascaded Interconnection

Proof . To show that the cascaded interconnection is IFP(-ν), we need to show that the storage function (11) satisfies:

$$\dot{V} \leq \nu u^T u + u^T y_n \tag{12}$$

for some $\nu > 0$. Since $V = \sum_{i=1}^{n} d_i V_i$, and $\dot{V}_i \leq -\rho_i y_i^T y_i + u_i^T y_i$, this is equivalent to showing that

$$\sum_{i=1}^{n} d_i(-\rho_i y_i^T y_i + u_i^T y_i) - \nu u^T u - u^T y_n \leq 0 \ . \tag{13}$$

Define

$$A = \begin{bmatrix} -1 & 0 & \cdots & 0 & -\frac{1}{\nu} \\ 1 & -\rho_1 & \ddots & & 0 \\ 0 & 1 & -\rho_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -\rho_n \end{bmatrix} \qquad \rho_i > 0, \quad \nu > 0 \tag{14}$$

and $D = diag\{\nu, d_1, d_2, \ldots, d_n\}$. Then it can be seen that the left-hand side of (13) is equal to

$$[u^T \ y^T] DA \begin{pmatrix} u \\ y \end{pmatrix} \ . \tag{15}$$

where $y = [y_1^T, \ldots, y_n^T]^T$. According to Theorem 2, if

$$\nu > \frac{cos(\frac{\pi}{n+1})^{n+1}}{\rho_1 \rho_2 \ldots \rho_n} \ , \tag{16}$$

then there exists a diagonal matrix $D > 0$ such that

$$DA + A^T D < 0 \ , \tag{17}$$

Then

$$[u^T \ y^T] DA \begin{pmatrix} u \\ y \end{pmatrix} = \frac{1}{2} [u^T \ y^T] DA \begin{pmatrix} u \\ y \end{pmatrix} + \frac{1}{2} [u^T \ y^T] DA \begin{pmatrix} u \\ y \end{pmatrix} \tag{18}$$

Since

$$[u^T \ y^T] DA \begin{pmatrix} u \\ y \end{pmatrix} = [u^T \ y^T] (DA)^T \begin{pmatrix} u \\ y \end{pmatrix} = [u^T \ y^T] A^T D \begin{pmatrix} u \\ y \end{pmatrix} \tag{19}$$

we have

$$[u^T \ y^T] DA \begin{pmatrix} u \\ y \end{pmatrix} = \frac{1}{2} [u^T \ y^T] (A^T D + DA) \begin{pmatrix} u \\ y \end{pmatrix} < 0 \tag{20}$$

and thus

$$\dot{V} \leq \nu u^T u + u^T y_n \ . \tag{21}$$

This shows that the cascaded system is IFP(-ν). ■

V. The shortage of passivity in cascaded input strictly passive systems

Proposition 2 . *Consider the cascaded interconnection shown in Fig.1, where $n \geq 2$. If each block is IFP(ν_i) with $\nu_i \in \mathbb{R}^+$, namely there exists a $\mathcal{C}^1$ storage function V_i for each subsystem, such that*

$$\dot{V}_i \leq -\nu_i u_i^T u_i + u_i^T y_i \ . \tag{22}$$

Then for some $\rho \in \mathbb{R}^+$, such that

$$\rho > \frac{cos(\frac{\pi}{n+1})^{n+1}}{\nu_1 \nu_2 \dots \nu_n} \ , \tag{23}$$

the cascaded system admits a storage function of the form given by

$$V = \sum_{i=1}^{n} d_i V_i, \quad d_i > 0 \tag{24}$$

such that the cascaded interconnection is Output Feedback Passive with passivity index -ρ (so the system with negative feedback $-\rho I$ is passive.)

Proof . To show that the cascaded interconnection is OFP(-ρ), we need to show that the storage function (24) satisfies:

$$\dot{V} \leq \rho y_n^T y_n + u^T y_n \tag{25}$$

since $V = \sum_{i=1}^{n} d_i V_i$ and $\dot{V}_i \leq -\nu_i u_i^T u_i + u_i^T y_i$, this is equivalent to showing that

$$\sum_{i=1}^{n} d_i(-\nu_i u_i^T u_i + u_i^T y_i) - \rho y_n^T y_n - u^T y_n \leq 0 \ . \tag{26}$$

Define

$$A = \begin{bmatrix} -\nu_1 & 0 & \cdots & 0 & -\frac{1}{\rho} \\ 1 & -\nu_2 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -\nu_n & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix} \qquad \rho > 0, \quad \nu_i > 0 \tag{27}$$

and $D = diag\{\rho, d_1, d_2, \dots, d_n\}$. Then it can been seen that the left-hand of (26) is equal to

$$[u^T \ y^T] DA \begin{pmatrix} u \\ y \end{pmatrix} \tag{28}$$

where $y = [y_1^T, \dots, y_n^T]^T$. According to Theorem 1, if

$$\rho > \frac{cos(\frac{\pi}{n+1})^{n+1}}{\nu_1 \nu_2 \dots \nu_n} \ , \tag{29}$$

then there exists a diagonal matrix $D > 0$ such that

$$DA + A^T D < 0 \ , \tag{30}$$

Then

$$[u^T \ y^T]DA \begin{pmatrix} u \\ y \end{pmatrix} = \frac{1}{2}[u^T \ y^T](A^T D + DA) \begin{pmatrix} u \\ y \end{pmatrix} < 0 \tag{31}$$

and thus

$$\dot{V} \leq \rho y_n^T y_n + u^T y_n \ . \tag{32}$$

This shows that the cascaded system is OFP(-ρ). ■

VI. Stabilization via unity ouput feedback for cascaded output strictly / input strictly passive systems

In the previous sections we have shown that a cascade of Input Feed-forward Passive systems lacks Output Feedback passivity if its subsystems' IFP indices satisfy the secant criterion, and we can quantify this shortage of passivity by using a passivity index; we also show that a cascade of Output Feed-back Passive systems lacks Input Feed-forward passivity in the same way, and this shortage of passivity can also be quantified by using a passivity index. The advantage of this quantification of passivity for these particular cascaded systems as discussed in the previous sections is that it provides us with a way to render those cascaded systems passive by applying a controller in the feed-forward or feedback path with proper excess of passivity. This is a classical result on passivity-based design for feedback control system as shown in [7]. Here, we are more interested in the conditions under which those particular cascaded systems could be stabilized directly via output feedback.

Proposition 3 . *Consider the feedback interconnection shown in Fig.2, where $n > 2$, and suppose each block is OFP(ρ_i), for some $\rho_i \in \mathbb{R}^+$. If*

$$\frac{1}{\rho_1 \rho_2 \dots \rho_n} < \frac{1}{cos(\frac{\pi}{n})^n} \ , \tag{33}$$

and the input to the cascaded system $u = 0$, then the closed-loop system admits a Lyapunov function which is a weighted sum of each subsystem's storage function given by:

$$V = \sum_{i=1}^{n} d_i V_i, \quad d_i > 0 \tag{34}$$

and

$$\dot{V} = \sum_{i=1}^{n} d_i \dot{V}_i \leq -\varepsilon |y|^2 \tag{35}$$

for some $\varepsilon > 0$ *, where* $y = [y_1^T, y_2^T, \ldots, y_n^T]^T$*. Moreover, if each subsystem is ZSD, then the equilibrium* $x_i = 0$ *of each subsystem is Lyapunov stable.*

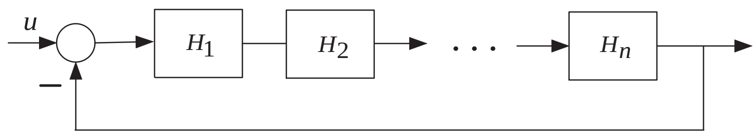


Fig. 2: Stabilized Via Output Feedback

Proof . Fig.2 is a cyclic structure and since $u = 0$, we have

$$\begin{aligned} u_1 &= -y_n \\ u_2 &= y_1 \\ &\vdots \\ u_n &= y_{n-1} \end{aligned} \tag{36}$$

Moreover, since each subsystem is OFP(ρ_i) we have

$$\begin{aligned} \dot{V}_1 &\leq -y_n^T y_1 - \rho_1 y_1^T y_1 \\ \dot{V}_2 &\leq y_1^T y_2 - \rho_2 y_2^T y_2 \\ &\vdots \\ \dot{V}_n &\leq y_{n-1}^T y_n - \rho_n y_n^T y_n. \end{aligned} \tag{37}$$

If we take the time derivative of the Lyapunov function candidate $V = \sum_{i=1}^{n} d_i V_i$, we will get

$$\dot{V} \leq d_1(-y_n^T y_1 - \rho_1 y_1^T y_1) + \sum_{i=2}^{n} d_i (y_{i-1}^T y_i - \rho_i y_i^T y_i) \tag{38}$$

Define

$$A = \begin{bmatrix} -\rho_1 & 0 & \cdots & 0 & -1 \\ 1 & -\rho_2 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & -\rho_{n-1} & 0 \\ 0 & \cdots & 0 & 1 & -\rho_n \end{bmatrix} \qquad \rho_i > 0 \tag{39}$$

and $D = diag\{d_1, d_2, \ldots, d_n\}$. Notice that the right-hand side of (38) is equal to $y^T DAy$, where $y = [y_1^T, \ldots, y_n^T]^T$. According to Theorem 1, if

$$\frac{1}{\rho_1 \ldots \rho_n} < \frac{1}{cos(\frac{\pi}{n})^n} \tag{40}$$

then there exists some diagonal matrix $D > 0$ such that matrix A is diagonally stable, and we will have

$$\dot{V} \le y^T DAy = \frac{1}{2} y^T (A^T D + DA) y \le -\varepsilon |y|^2 \tag{41}$$

for some $\varepsilon > 0$, this implies that $\lim_{t\to\infty} y_i(t) = 0$, for $i = 1, \ldots, n$. If each subsystem H_i is ZSD, then $\lim_{t\to\infty} y_i(t) = 0$ implies $\lim_{t\to\infty} x_i(t) = 0$, so each subsystem's equilibrium $x_i = 0$ is Lyapunov stable. ■

Remark 1: Instead of unity output feedback, if we add a gain $K > 0$ in the feedback loop, then it is easy to show that condition (33) in Proposition 3 becomes:

$$\frac{K}{\rho_1 \rho_2 \ldots \rho_n} < \frac{1}{cos(\frac{\pi}{n})^n}.$$

Remark 2: One should notice that if there is no dynamics cancelation in the cascaded systems (i.e., for a cascade of linear systems, this means there is no zero-pole cancelation between each interconnected subsystem), then the state X of the entire cascaded system is just the cascade of each subsystem's state: $X = [x_1^T, \ldots, x_n^T]^T$. In this case, $\lim_{t\to\infty} x_i(t) = 0, \quad i = 1, \ldots, n$ implies $\lim_{t\to\infty} X(t) = 0$, thus by direct unity output feedback, the closed-loop system is Lyapunov stable with $u = 0$.

Remark 3: We should look at the special case when there are only two subsystems in the cascaded interconnection since we exclude this case($n = 2$) in our proposition. Consider the cascade interconnection via unity output feedback as shown in Fig. 3. In this case, assume that

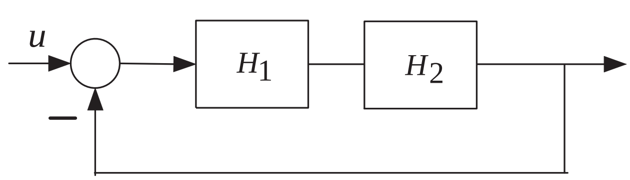


Fig. 3: Special Case When $n = 2$

$u = 0$, H_1 is OFP(ρ_1) and H_2 is OFP(ρ_2), so we have

$$\begin{aligned} \dot{V}_1 &\leq u_1^T y_1 - \rho_1 y_1^T y_1 = -y_2^T y_1 - \rho_1 y_1^T y_1 \\ \dot{V}_2 &\leq u_2^T y_2 - \rho_2 y_2^T y_2 = y_2^T y_1 - \rho_2 y_2^T y_2 \end{aligned} \tag{42}$$

If we consider the storage function for the closed-loop system to be given by:

$$V = d_1 V_1 + d_2 V_2 \quad d_1 > 0, \quad d_2 > 0 \tag{43}$$

we will have

$$\begin{aligned} \dot{V} &= d_1 \dot{V}_1 + d_2 \dot{V}_2 \\ &\leq d_1(-y_2^T y_1 - \rho_1 y_1^T y_1) + d_2(y_2^T y_1 - \rho_2 y_2^T y_2) \ . \end{aligned} \tag{44}$$

If we choose $d_1 = d_2$, we obtain

$$\dot{V} \leq -d_1 \rho_1 y_1^T y_1 - d_2 \rho_2 y_2^T y_2. \tag{45}$$

So, in this case, if $\rho_1 > 0$ and $\rho_2 > 0$, which means both H_1 and H_2 are output strictly passive, then we can simply choose the sum of their storage function as the potential Lyapunov function for the closed-loop system when we directly apply the unity output feedback to the cascaded interconnection of H_1 and H_2; if in addition, both H_1 and H_2 are ZSD and there is no dynamics cancelation for the cascaded interconnection of them, then the closed-loop system with $u = 0$ is Lyapunov stable.

We have examined the conditions under which the cascaded interconnection of a class of output feedback passive systems can be directly stabilized via output feedback. Now, let's examine whether we can obtain similar results for a cascade of input feed-forward passive systems.

Proposition 4 . *Consider the feedback interconnection shown in Fig. 2, where $n > 2$, and suppose that $u = 0$ and each block is IFP(ν_i) for some $\nu_i \in \mathbb{R}^+$. If*

$$\frac{1}{\nu_1 \nu_2 \ldots \nu_n} < \frac{1}{cos(\frac{\pi}{n})^n}, \tag{46}$$

then the close-loop system admits a Lyapunov function which is a weighted sum of each sub-system's storage function given by:

$$V = \sum_{i=1}^{n} d_i V_i, \quad d_i > 0 \tag{47}$$

and

$$\dot{V} = \sum_{i=1}^{n} d_i \dot{V}_i \leq -\varepsilon |y|^2 \tag{48}$$

for some $\varepsilon > 0$, where $y = [y_1^T, y_2^T, \ldots, y_n^T]^T$. Moreover, if each subsystem is ZSD, then the equilibrium $x_i = 0$ of each subsystem is Lyapunov stable.

Proof . Again, since Fig.2 is a cyclic structure and thus we have

$$\begin{aligned} u_1 &= -y_n \\ u_2 &= y_1 \\ &\vdots \\ u_n &= y_{n-1} \end{aligned} \tag{49}$$

Moreover, since each subsystem is IFP(ν_i) we have

$$\begin{aligned} \dot{V}_1 &\le -y_n^T y_1 - \nu_1 y_n^T y_n \\ \dot{V}_2 &\le y_1^T y_2 - \nu_2 y_1^T y_1 \\ &\vdots \\ \dot{V}_n &\le y_{n-1}^T y_n - \nu_n y_{n-1}^T y_{n-1} \end{aligned} \tag{50}$$

If we take the derivative of the Lyapunov function candidate $V = \sum_{i=1}^n d_i V_i$, we get

$$\dot{V} \le d_1(-y_n^T y_1 - \nu_1 y_n^T y_n) + \sum_{i=2}^n d_i(y_{i-1}^T y_i - \nu_i y_{i-1}^T y_{i-1}) \tag{51}$$

Define

$$A = \begin{bmatrix} -\nu_2 & 0 & \cdots & 0 & -1 \\ 1 & -\nu_3 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & -\nu_n & 0 \\ 0 & \cdots & 0 & 1 & -\nu_1 \end{bmatrix} \qquad \nu_i > 0 \tag{52}$$

and $D = diag\{d_1, d_2, \ldots, d_n\}$. Notice that the right-hand side of (51) is equal to $y^T DAy$, where $y = [y_1^T, \ldots, y_n^T]^T$. According to Theorem 1, if

$$\frac{1}{\nu_1 \ldots \nu_n} < \frac{1}{cos(\frac{\pi}{n})^n} \tag{53}$$

then there exists some diagonal matrix $D > 0$ such that matrix A is diagonally stable, and we will have

$$\dot{V} \le y^T DAy = \frac{1}{2} y^T (A^T D + DA) y \le -\varepsilon |y|^2 \tag{54}$$

for some $\varepsilon > 0$, this implies that $\lim_{t\to\infty} y_i(t) = 0$, for $i = 1, \ldots, n$. In addition, If each subsystem H_i is ZSD, then $\lim_{t\to\infty} y_i(t) = 0$ implies $\lim_{t\to\infty} x_i(t) = 0$, so each subsystem's equilibrium $x_i = 0$ is Lyapunov stable. ■

Remark 4: Instead of unity output feedback, if we add a gain $K > 0$ in the feedback loop, then it is easy to show that condition (46) in Proposition 4 becomes:

$$\frac{1}{K\nu_1\nu_2\ldots\nu_n} < \frac{1}{cos(\frac{\pi}{n})^n}.$$

Remark 5: Let's look at the special case when there are only two subsystems in the cascaded interconnection since we exclude this case($n = 2$) in the above proposition. Again, consider the cascaded interconnection via unity output feedback as shown in Fig. 3. In this case, assume that $u = 0$, H_1 is IFP(ν_1) and H_2 is IFP(ν_2), so we have

$$\begin{aligned} \dot{V}_1 &\leq u_1^T y_1 - \nu_1 u_1^T u_1 = -y_2^T y_1 - \nu_1 y_2^T y_2 \\ \dot{V}_2 &\leq u_2^T y_2 - \nu_2 u_2^T u_2 = y_2^T y_1 - \nu_2 y_1^T y_1 \end{aligned} \tag{55}$$

If we consider the storage function for the closed-loop system as given by:

$$V = d_1 V_1 + d_2 V_2 \quad d_1 > 0, \quad d_2 > 0 \tag{56}$$

we will have

$$\begin{aligned} \dot{V} &= d_1\dot{V}_1 + d_2\dot{V}_2 \\ &\leq d_1(-y_2^T y_1 - \nu_1 y_2^T y_2) + d_2(y_2^T y_1 - \nu_2 y_1^T y_1) \end{aligned} \tag{57}$$

if we choose $d_1 = d_2$, we can obtain

$$\dot{V} \leq -d_1\nu_1 y_2^T y_2 - d_2\nu_2 y_1^T y_1. \tag{58}$$

So, in this case, if $\nu_1 > 0$ and $\nu_2 > 0$, which means both H_1 and H_2 are input strictly passive, then again we can simply choose the sum of their storage functions as the potential Lyapunov function for the closed-loop system when we directly apply the unity output feedback to the cascaded interconnection of H_1 and H_2; similarly, if in addition, both H_1 and H_2 are ZSD and there is no dynamics cancelation for the cascaded interconnection of them, then the closed-loop system with $u = 0$ is Lyapunov stable.

VII. THE SHORTAGE/EXCESS OF PASSIVITY FOR CASCADED SYSTEMS WITH SIMULTANEOUS IFP INDICES AND OFP INDICES

In the previous sections, we have shown the lack of Input Feed-forward passivity for a cascade of Output Strictly Passive systems and lack of Output Feedback passivity for a cascade of Input Strictly Passive systems. One should notice that in both cases, each individual subsystem in the cascaded interconnection is passive. In this section, we would like to extend our previous results to the more general case when the interconnected subsystem may be passive / non-passive. First, we need to review the concept of dissipative systems introduced by Willems [9].

Definition 4 (Supply Rate [9]). *The supply rate $\omega(t) = \omega(u(t), y(t))$ is a real valued function defined on $U \times Y$, such that for any $u(t) \in U$ and $x_0 \in X$ and $y(t) = h(\phi(t, t_0, x_0, u))$, $\omega(t)$ satisfies*

$$\int_{t_0}^{t_1} |\omega(t)| dt < \infty \tag{59}$$

for all $t_1 \geq t_0 \geq 0$.

Definition 5 (Dissipative Systems [9]). *System H with supply rate $\omega(t)$ is said to be dissipative if there exists a $\mathcal{C}^1$ nonnegative real function $V(x) : X \to \mathbb{R}^+$, called the storage function, such that for all $x_0 \in X$ and $u \in U$,*

$$\dot{V}(x) \leq \omega(u(t), y(t)). \tag{60}$$

We can see that passive system is a special case of dissipative system, with the supply rate given by $\omega(u(t), y(t)) = u^T(t)y(t)$.

With the concepts of supply rate and dissipative systems, we are ready to present the following proposition.

Proposition 5 . *Consider the cascaded interconnection shown in Fig.1, where $n \geq 2$, and let each block be dissipative with respect to the supply rate given by $\omega_i(u_i, y_i) = u_i^T y_i - \rho_i y_i^T y_i - \nu_i u_i^T u_i$, that is there exists a $\mathcal{C}^1$ storage function V_i for each subsystem, such that*

$$\dot{V}_i \leq u_i^T y_i - \rho_i y_i^T y_i - \nu_i u_i^T u_i \tag{61}$$

here, ν_i and ρ_i are not necessarily all positive.
If

$$1 + \nu_1 > 0, \quad 1 + \rho_n > 0, \quad \nu_i + \rho_{i-1} > 0, \quad i = 2, \ldots, n \tag{62}$$

then for some

$$\delta > \frac{cos(\frac{\pi}{n+1})^{n+1}}{(\nu_1+1)(\nu_2+\rho_1)\dots(\nu_n+\rho_{n-1})(\rho_n+1)} \tag{63}$$

the cascaded system admits a storage function of the form given by

$$V = \sum_{i=1}^{n} d_i V_i, \quad d_i > 0 \tag{64}$$

such that the cascaded interconnection has simultaneous OFP(-δ) and IFP(-δ).

Proof . To show that the cascaded interconnection has simultaneous IFP(-δ) and OFP(-δ) , we need to show that the storage function $V(x)$ satisfies:

$$\dot{V} \leq u^T y_n + \delta u^T u + \delta y_n^T y_n \tag{65}$$

which is equivalent to showing that

$$\sum_{i=1}^{n} d_i \dot{V}_i - \delta u^T u - \delta y_n^T y_n - u^T y_n \leq 0. \tag{66}$$

Define

$$A = \begin{bmatrix} -\nu_1 - 1 & 0 & \cdots & 0 & -\frac{1}{\delta} \\ 1 & -\nu_2 - \rho_1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & -\nu_n - \rho_{n-1} & 0 \\ 0 & \cdots & 0 & 1 & -1 - \rho_n \end{bmatrix} \qquad \delta > 0 \tag{67}$$

and $D = diag\{\delta, d_1, \dots, d_n\}$. Notice that the left-hand side of (66) is equal to

$$[u^T \ y^T] DA \begin{pmatrix} u \\ y \end{pmatrix}. \tag{68}$$

where $y = [y_1^T, \dots, y_n^T]^T$. According to Theorem 1, if (62)-(63) are satisfied, then there exists a diagonal matrix $D > 0$ such that

$$DA + A^T D < 0 \ , \tag{69}$$

it results in

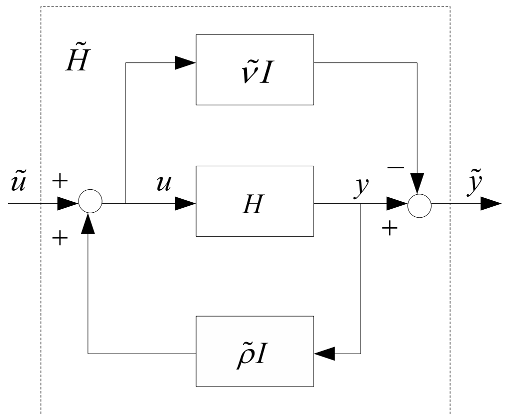
$$[u^T \ y^T] DA \begin{pmatrix} u \\ y \end{pmatrix} = \frac{1}{2} [u^T \ y^T](A^T D + DA) \begin{pmatrix} u \\ y \end{pmatrix} < 0 \tag{70}$$

and thus

$$\dot{V} \leq \delta u^T u + \delta y_n^T y_n + u^T y_n. \tag{71}$$

This shows that the cascade has simultaneous IFP(-δ) and OFP(-δ). ■

Remark 6: In this case, we can show that if $0 < \delta < \frac{1}{2}$, then the entire cascaded system (if we denote it as H) can be rendered passive with both feed-forward $\tilde{\nu}I$ and output feedback $\tilde{\rho}I$ as shown in the figure below, where $\tilde{\nu} = \tilde{\rho} = \frac{-1-\sqrt{1-4\delta^2}}{2\delta}$.



Remark 7: We can also show that

1) if

$$1-\nu_1 < 0, \quad 1-\rho_n < 0, \quad \nu_i+\rho_{i-1} > 0, \quad i = 2,\ldots,n \tag{72}$$

and for some

$$\delta > \frac{cos(\frac{\pi}{n+1})^{n+1}}{(\nu_1-1)(\nu_2+\rho_1)\ldots(\nu_n+\rho_{n-1})(\rho_n-1)}, \tag{73}$$

the cascaded system has simultaneous IFP(δ) and OFP(δ), which means the cascaded system is passive;

2) if

$$1-\nu_1 < 0, \quad 1+\rho_n > 0, \quad \nu_i+\rho_{i-1} > 0, \quad i = 2,\ldots,n \tag{74}$$

and for some

$$\delta > \frac{cos(\frac{\pi}{n+1})^{n+1}}{(\nu_1-1)(\nu_2+\rho_1)\ldots(\nu_n+\rho_{n-1})(\rho_n+1)}, \tag{75}$$

the cascaded system has simultaneous IFP(δ) and OFP($-\delta$), which shows the cascaded system lacks output feedback passivity;

3) if

$$1+\nu_1 > 0, \quad 1-\rho_n < 0, \quad \nu_i+\rho_{i-1} > 0, \quad i = 2,\ldots,n \tag{76}$$

and for some

$$\delta > \frac{cos(\frac{\pi}{n+1})^{n+1}}{(\nu_1+1)(\nu_2+\rho_1)\dots(\nu_n+\rho_{n-1})(\rho_n-1)}, \tag{77}$$

the cascaded system has simultaneous IFP($-\delta$) and OFP(δ), which shows the cascaded system lacks input feed-forward passivity.

Remark 8: From Proposition 5 and the above discussion in Remark 4, we can see that if the cascaded system has non-passive subsystems, then the cascaded interconnection could either be passive or non-passive, and this depends on the passivity indices of its subsystems.

VIII. Stabilization via ouput feedback for cascaded systems with simultaneous IFP indices and OFP indices

We have shown that passivity quantification for the cascaded system which has both passive and non-passive components is non-deterministic in general, and this quantification largely depends on each subsystem's passivity indices. Now, we want to show under what conditions such cascaded system can be stabilized directly via output feedback.

Proposition 6 . *Consider the feedback interconnection shown in Fig. 2, where $n > 2$. If each block is dissipative with respect to the supply rate given by $\omega_i(u_i, y_i) = u_i^T y_i - \rho_i y_i^T y_i - \nu_i u_i^T u_i$, that is there exists a $\mathcal{C}^1$ storage function V_i for each subsystem, such that*

$$\dot{V}_i \leq -\rho_i y_i^T y_i - \nu_i u_i^T u_i + u_i^T y_i \tag{78}$$

here ν_i and ρ_i are not necessarily all positive.
If

$$\rho_n + \nu_1 > 0, \quad \nu_i + \rho_{i-1} > 0, \quad i = 2\dots, n \tag{79}$$

and

$$\frac{1}{(\rho_1+\nu_2)(\rho_2+\nu_3)\dots(\rho_{n-1}+\nu_n)(\nu_1+\rho_n)} < \frac{1}{cos(\frac{\pi}{n})^n} \tag{80}$$

then the close-loop system with $u = 0$ admits a Lyapunov function which is a weighted sum of each subsystem's storage function given by:

$$V = \sum_{i=1}^{n} d_i V_i, \quad d_i > 0 \tag{81}$$

and

$$\dot{V} = \sum_{i=1}^{n} d_i \dot{V}_i \leq -\varepsilon |y|^2 \tag{82}$$

for some $\varepsilon > 0$*, where* $y = [y_1^T, y_2^T, \dots, y_n^T]^T$*. Moreover, if each subsystem is ZSD, then the equilibrium* $x_i = 0$ *of each subsystem is Lyapunov stable.*

Proof . Again since Fig.2 is a cyclic structure and we assume $u = 0$, we have

$$\begin{aligned} u_1 &= -y_n \\ u_2 &= y_1 \\ &\vdots \\ u_n &= y_{n-1} \end{aligned} \tag{83}$$

Moreover, since each subsystem is dissipative with respect to the supply rate given by $\omega_i(u_i, y_i) = u_i^T y_i - \rho_i y_i^T y_i - \nu_i u_i^T u_i$, we have

$$\begin{aligned} \dot{V}_1 &\leq -y_n^T y_1 - \nu_1 y_n^T y_n - \rho_1 y_1^T y_1 \\ \dot{V}_2 &\leq y_1^T y_2 - \nu_2 y_1^T y_1 - \rho_2 y_2^T y_2 \\ &\vdots \\ \dot{V}_n &\leq y_{n-1}^T y_n - \nu_n y_{n-1}^T y_{n-1} - \rho_n y_n^T y_n \end{aligned} \tag{84}$$

If we take the time derivative of the Lyapunov function candidate $V = \sum_{i=1}^{n} d_i V_i$, we obtain

$$\dot{V} \leq d_1(-y_n^T y_1 - \nu_1 y_n^T y_n - \rho_1 y_1^T y_1) + \sum_{i=2}^{n} d_i (y_{i-1}^T y_i - \nu_i y_{i-1}^T y_{i-1} - \rho_i y_i^T y_i) \tag{85}$$

Define

$$A = \begin{bmatrix} -\nu_2 - \rho_1 & 0 & \cdots & 0 & -1 \\ 1 & -\nu_3 - \rho_2 & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & -\nu_n - \rho_{n-1} & 0 \\ 0 & \cdots & 0 & 1 & -\nu_1 - \rho_n \end{bmatrix} \tag{86}$$

and $D = diag\{d_1, d_2, \dots, d_n\}$. Notice that the right-hand side of (85) is equal to $y^T D A y$, where $y = [y_1^T, \dots, y_n^T]^T$. According to Theorem 1, if (79)-(80) are satisfied, then there exists some diagonal matrix $D > 0$ such that matrix A is diagonal stable, and we will have

$$\dot{V} \leq y^T D A y = \frac{1}{2} y^T (A^T D + DA) y \leq -\varepsilon |y|^2 \tag{87}$$

for some $\varepsilon > 0$, this implies that $\lim_{t\to\infty} y_i(t) = 0$, for $i = 1, \dots, n$. Moreover, if each subsystem is ZSD, then $\lim_{t\to\infty} y_i(t) = 0$ implies $\lim_{t\to\infty} x_i(t) = 0$ for $i = 1, \dots, n$. So each subsystem's equilibrium $x_i = 0$ is Lyapunov stable. ■

Remark 9: Instead of unity output feedback, if we add a gain $K > 0$ in the feedback loop, then it is easy to show that condition (80) in Proposition 6 becomes:

$$\frac{K}{(\rho_1+\nu_2)(\rho_2+\nu_3)\dots(\rho_{n-1}+\nu_n)(K^2\nu_1+\rho_n)} < \frac{1}{cos(\frac{\pi}{n})^n}.$$

Remark 10: Let's look at the special case when there are only two subsystems in the cascaded interconnection ($n = 2$). Again, consider the cascaded interconnection via unity output feedback as shown in Fig. 3. In this case, assume that $u = 0$, the storage functions for H_1 and H_2 satisfy:

$$\begin{aligned} \dot{V}_1 &\le u_1^T y_1 - \nu_1 u_1^T u_1 - \rho_1 y_1^T y_1 = -y_2^T y_1 - \nu_1 y_2^T y_2 - \rho_1 y_1^T y_1 \\ \dot{V}_2 &\le u_2^T y_2 - \nu_2 u_2^T u_2 - \rho_2 y_2^T y_2 = y_2^T y_1 - \nu_2 y_1^T y_1 - \rho_2 y_2^T y_2. \end{aligned} \tag{88}$$

If we consider the storage function for the closed-loop system as given by:

$$V = d_1 V_1 + d_2 V_2, \qquad d_1 > 0,\ d_2 > 0, \tag{89}$$

then we have

$$\begin{aligned} \dot{V} &= d_1 \dot{V}_1 + d_2 \dot{V}_2 \\ &\le d_1(-y_2^T y_1 - \nu_1 y_2^T y_2 - \rho_1 y_1^T y_1) + d_2(y_2^T y_1 - \nu_2 y_1^T y_1 - \rho_2 y_2^T y_2), \end{aligned} \tag{90}$$

if we choose $d_1 = d_2$, then we get

$$\dot{V} \le -d_1(\rho_1+\nu_2) y_1^T y_1 - d_1(\nu_1+\rho_2) y_2^T y_2. \tag{91}$$

So, in this case, if $\nu_1 + \rho_2 > 0$ and $\nu_2 + \rho_1 > 0$, then again we can simply choose the sum of storage functions for H_1 and H_2 as the potential Lyapunov function for the closed-loop system when we directly apply the unity output feedback to their cascaded interconnection; moreover, if both H_1 and H_2 are ZSD and there is no dynamics cancelation for the cascaded interconnection of them, then the closed-loop system with $u = 0$ is Lyapunov stable.

For the remaining of this section, we provide several simple examples as illustrations for Proposition 6.

Example 1 . Consider the feedback interconnection shown in Fig. 4., where H_1, H_2 and H_3 are simple SISO linear systems given by

$$H_1 : \begin{cases} \dot{x}_1 = -9x_1 + u_1 \\ y_1 = x_1 + \dfrac{1}{9}u_1 \end{cases} \tag{92}$$

$$H_2 : \begin{cases} \dot{x}_2 = -\dfrac{1}{2}x_2 + u_2 \\ y_2 = x_2 - \dfrac{1}{2}u_2 \end{cases} \tag{93}$$

and

$$H_3 : \begin{cases} \dot{x}_3 = -\dfrac{1}{6}x_3 + u_3 \\ y_3 = x_3 + 6u_3 \end{cases} \tag{94}$$

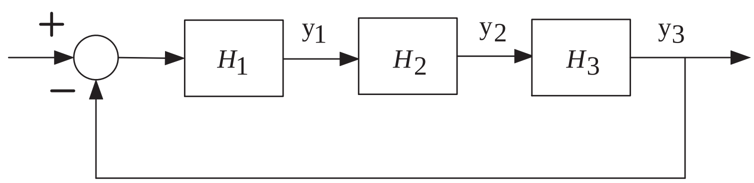


Fig. 4: Example

It is easy to show that H_1 admits a storage function given by $V_1 = \frac{1}{6}x_1^2$, such that H_1 is passive with the supply rate $\omega_1(u_1, y_1) = u_1y_1 - \frac{2}{27}u_1^2 - 3y_1^2$; so H_1 has simultaneous IFP index $\nu_1 = \frac{2}{27}$ and OFP index $\rho_1 = 3$. H_2 admits a storage function given by $V_2 = x_2^2$, and H_2 is dissipative with respect to the supply rate $\omega_2(u_2, y_2) = u_2y_2 + \frac{3}{4}u_2^2 - y_2^2$; so H_2 has simultaneous IFP index $\nu_2 = -\frac{3}{4}$ and OFP index $\rho_2 = 1$. H_3 admits a storage function given by $V_3 = \frac{1}{6}x_3^2$, and H_3 is passive with the supply rate given by $\omega_3(u_3, y_3) = u_3y_3 - 4u_3^2 - \frac{1}{18}y_3^2$; so the IFP index is $\nu_3 = 4$ while the OFP index is $\rho_3 = \frac{1}{18}$. Then we have:

$$\begin{aligned} \nu_1 + \rho_3 &= \frac{2}{27} + \frac{1}{18} = \frac{7}{54} > 0 \\ \nu_2 + \rho_1 &= -\frac{3}{4} + 3 = 2\frac{1}{4} > 0 \\ \nu_3 + \rho_2 &= 4 + 1 = 5 > 0 \end{aligned} \tag{95}$$

and

$$(\nu_1+\rho_3)(\nu_2+\rho_1)(\nu_3+\rho_2)=\frac{35}{24}>[cos(\frac{\pi}{3})]^3=\frac{1}{8} \tag{96}$$

According to Proposition 6, the cascade of H_1, H_2 and H_3 could be stabilized via direct output feedback, the simulation result is shown in Fig. 5. Since each subsystem is ZSD, the equilibrium at the origin should be Lyapunov stable, this is shown in Fig. 6.

Example 2 . We have shown in Example 1 that when H_1 and H_3 are both passive with positive passivity indices while H_2 is dissipative but lacks input feed-forward passivity ($\nu_2 < 0$), we can stabilize the cascaded system via unity output feedback if the passivity indices of each subsystems satisfies the criterion in Proposition 6; now let's examine whether the stability results would still be hold if H_2 is dissipative but lacks output feedback passivity ($\rho_2 < 0$). In this case, choose:

$$H_2:\begin{cases}\dot{x}_2=\frac{1}{2}x_2+u_2\\ y_2=x_2+\frac{1}{2}u_2\end{cases} \tag{97}$$

it is easy to show that H_2 admits the same storage function given by $V_2 = x_2^2$ while the passivity indices are given as $\nu_2 = \frac{3}{4}$ and $\rho_2 = -1$. Then we have:

$$\begin{aligned}\nu_1+\rho_3&=\frac{2}{27}+\frac{1}{18}=\frac{7}{54}>0\\ \nu_2+\rho_1&=\frac{3}{4}+3=3\frac{3}{4}>0\\ \nu_3+\rho_2&=4-1=3>0\end{aligned} \tag{98}$$

and

$$(\nu_1+\rho_3)(\nu_2+\rho_1)(\nu_3+\rho_2)=\frac{35}{24}>[cos(\frac{\pi}{3})]^3=\frac{1}{8} \tag{99}$$

The simulation result is shown in Fig. 7. Again, since each subsystem is ZSD, the equilibrium at the origin should be Lyapunov stable, this is shown in Fig. 8.

Example 3 . Let's examine further when H_2 lacks both input feed-forward passivity and output feedback passivity ($\rho_2 < 0$ and $\nu_2 < 0$). Choose

$$H2:\begin{cases}\dot{x}_2=\frac{1}{2}x_2+u_2\\ y_2=x_2-\frac{1}{2}u_2\end{cases} \tag{100}$$

it is easy to show that H_2 admits the same storage function given by $V_2 = x_2^2$ while the passivity indices are given as $\nu_2 = -\frac{5}{12}$ and $\rho_2 = -\frac{1}{3}$. Then we have:

$$\begin{aligned} \nu_1 + \rho_3 &= \frac{2}{27} + \frac{1}{18} = \frac{7}{54} > 0 \\ \nu_2 + \rho_1 &= -\frac{5}{12} + 3 = \frac{31}{12} > 0 \\ \nu_3 + \rho_2 &= 4 - \frac{1}{3} = \frac{11}{3} > 0 \end{aligned} \tag{101}$$

and

$$(\nu_1 + \rho_3)(\nu_2 + \rho_1)(\nu_3 + \rho_2) = 1.2779 > [cos(\frac{\pi}{3})]^3 = \frac{1}{8} \tag{102}$$

The simulation result is shown in the Fig. 9, and since each subsystem is still ZSD, the equilibrium at the origin is Lyapunov stable, this is shown in Fig. 10.

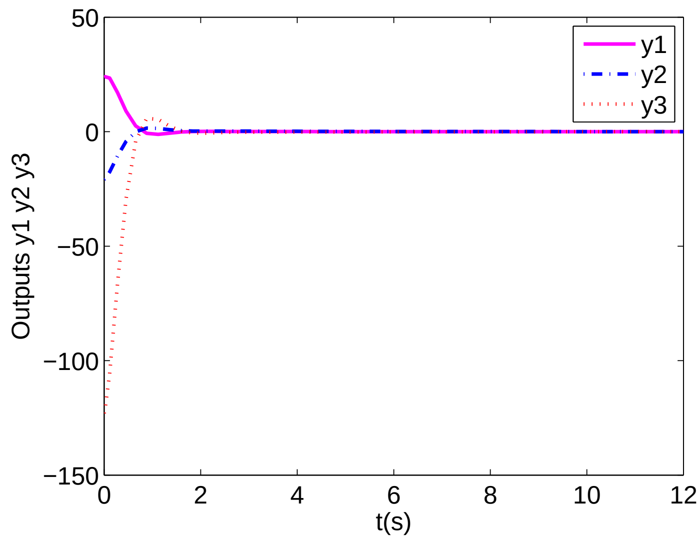


Fig. 5: Outputs of each subsystems in Example 1

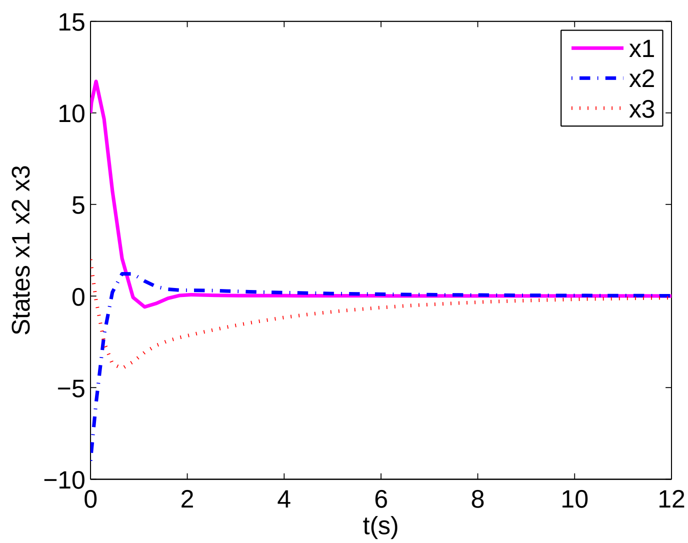


Fig. 6: States of each subsystems in Example 1

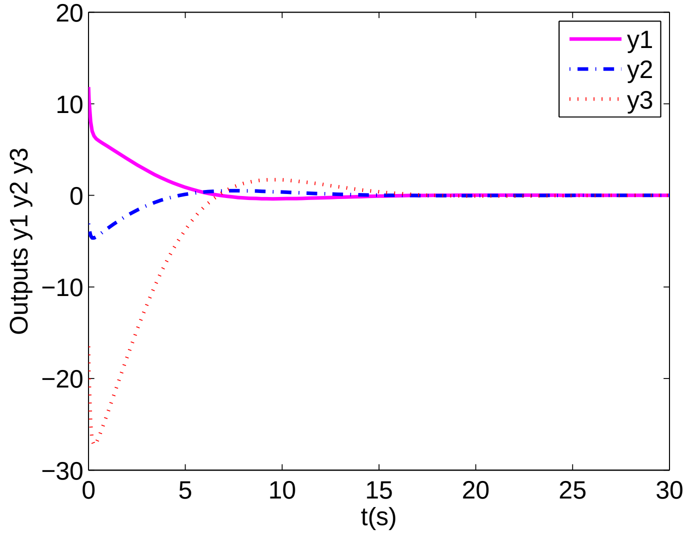


Fig. 7: Outputs of each subsystems in Example 2

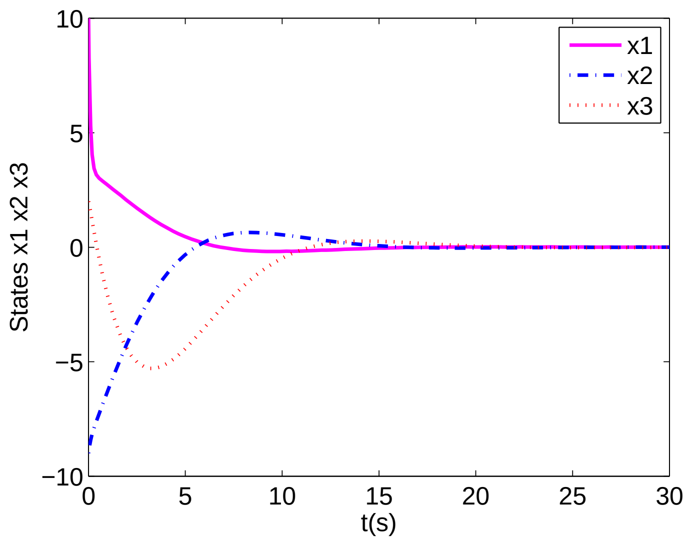


Fig. 8: States of each subsystems in Example 2

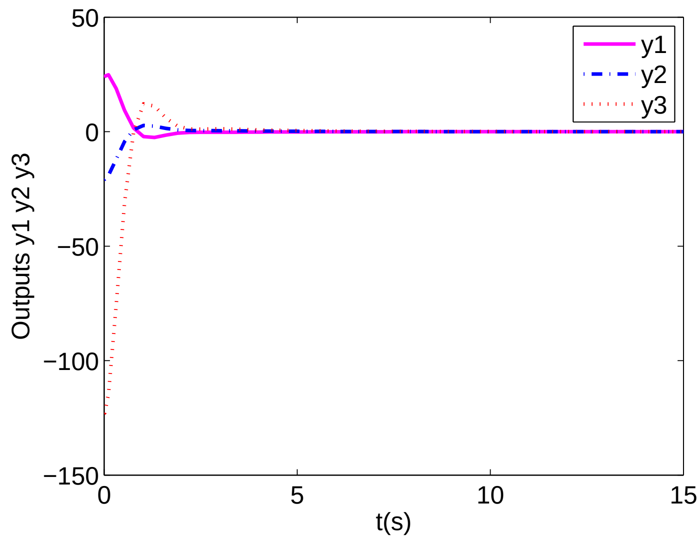


Fig. 9: Outputs of each subsystems in Example 3

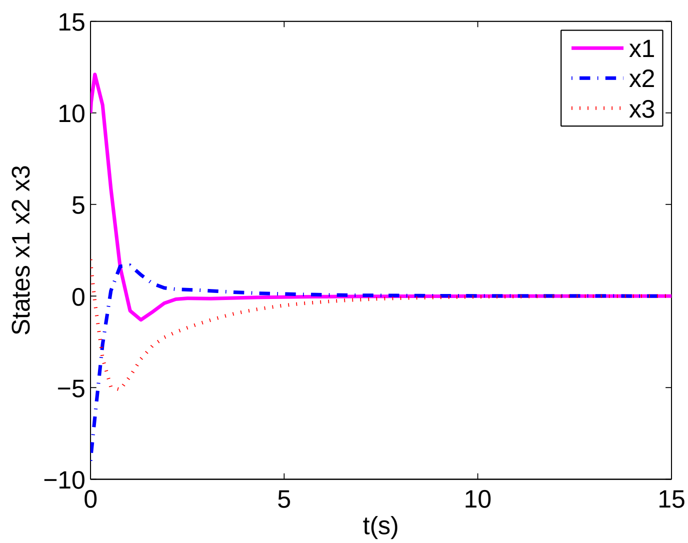


Fig. 10: States of each subsystems in Example 3

IX. CONCLUSION

In this report, we revisit the results in [1]-[2] from the perspective of passivity indices. For cascaded strictly output / strictly input passive systems, if we know each subsystem's passivity indices , then we can quantify the shortage of passivity for the cascaded systems. This actually enables us to design a proper feed-forward or a feedback system which has excessive passivity to compensate the shortage of passivity for the cascade interconnection and render it passive, as claimed by the passivation results in [7].

We further show that the cascaded strictly output passive/strictly input passive systems can be stabilized by output feedback if the product of passivity indices of each subsystem satisfies the secant criterion. We also extended our results to the more general case when taking both IFP indices and OFP indices into consideration, since in this case, each interconnected subsystem is not required to be passive. We have shown that in this case, passivity quantification for the cascaded interconnection is in general, depends on its subsystems' individual passivity indices. However, we are still able to stabilize the cascaded system via output feedback if their passivity indices satisfy the conditions developed in Proposition 6.

REFERENCES

[1] Sontag, E.D., "Passivity gains and the secant condition for stability", Systems and Control Letters, 55, 177-183, 2006.

[2] Murat Arcak and Eduardo D. Sontag, "Diagonal stability of a class of cyclic systems and its connection with the secant criterion", Automatica 42 (2006), 1531-1537.

[3] Tyson, J.J. and Othmer, H.G., "The dynamics of feedback control circuits in biochemical pathways ". In R. Rosen and F. M. Snell (Eds.), *Progress in theoretical biology*, Vol.5, pp. 1-62, 1978. New York: Academic Press.

[4] Thron, C.D., "The secant condition for instability in biochemical feedback control-Parts I and II ". *Bulletin of Mathematical Biology*, 53, 383-424.

[5] Jie Bao and Peter L.Lee, *Process Control: The Passive Systems Approach*, Advances in Industrial Control, Springer-Verlag London, 2007.

[6] C. I. Byrnes, A. Isidori, and J.C. Willems, "Passivity, feedback equivalence and the global stabilization of minimum phase nonlinear systems ", *IEEE Transactions on Automatic Control*, v 36, n 11, p 1228-40, Nov. 1991.

[7] Sepulchre, Rodolphe and Jankovic, Mrdjan and Kokotovic, Petar, *Constructive Nonlinear Control*, Springer-Verlag, 1997.

[8] Eugenius Kaszkurewicz and Amit Bhaya, *Matrix Diagonal Stability in Systems and Computation*,Birkhäuser Boston, 1999.

[9] Jan C. Willems, "Dissipative dynamical systems part I: General theory", Archive for Rational Mechanics and Analysis, Springer Berlin / Heidelberg, Volume 45, Number 5, January, 1972.