Connection Between the Passivity Index and Conic Systems

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Abstract

This report covers open-loop conditions that guarantee closed-loop stability of a feedback interconnection of two systems. The traditional solutions for this problem are passivity theory and the small gain theorem. These approaches can be shown to be part of a larger design framework. Two such larger frameworks are the conic systems framework and the passivity index framework. This report shows that these two actually reduce to the same open-loop conditions on the two systems in the interconnection. This connection is shown for the purpose of applying this generalized framework to switched systems. The recent application of passivity to switched systems gives one method of guaranteeing that the feedback interconnection of two switched systems is stable. However, a more general framework is presented in this report that is based on passivity theory but can show stability for the feedback interconnection of systems that aren't necessarily passive or even stable.

1 Introduction

When studying the stability of feedback systems, the two tools that are most widely used are the passivity theorem and the small gain theorem. It has been shown previously [1] that neither is more general than the other. Additionally, that reference showed that if a given feedback interconnection satisfies one theorem, the interconnection with an appropriate loop transformation satisfies the other. Both the passivity theorem and the small gain theorem are useful in practice. However, there is a much larger class of interconnected systems that can be shown to be stable.

One such framework is the generalization of passivity to passivity indices [2]. The concept is that some systems aren't passive but are by some measure "nearly" passive. This means that a static feedback or feed-forward gain could be applied to the system to force it to be passive. Quantifying the type and magnitude of this gain is one measure of the level of passivity of a system. Controllers can be designed with a complementary "excess" of passivity to compensate for the shortage in the given plant.

Another framework is the conic systems framework [3]. A conic system is one whose input u and output y are restricted to some conic sector of the $U \times Y$ inner product space. This is a general concept that includes both passivity and $\mathcal{L}_2$ gain as special cases. Conic theory provides a result that can assess stability of an interconnection based on the upper and lower bounds of the conic sectors of each system. Conceptually, a larger sector for a given plant can be compensated by a smaller sector for a synthesized controller.

It is important to note that these frameworks are mainly for design purposes. When analyzing a given feedback interconnection, typically the complete description of both systems is used to assess stability. In the case of synthesizing a controller for a given plant, often key properties of the given plant are extracted and a controller is designed using these properties. The passivity indices are one measure of stabilizability of an interconnection. Another is the boundaries of the conic sectors of a given interconnection. This report will show that these two frameworks provide the same information for control design.

All the material and frameworks mentioned up to this point have been well studied in their application to traditional, continuously-varying systems. However, recently there has been a redefinition of passivity that can apply to switched systems. This framework extends the intuitive design methodology of passivity to systems with switching dynamics. As the third section of this report will show, much less restrictive conditions for closed loop stability can be found. This report will introduce one set of conditions that generalize passivity for switched systems. This set is the redefinition of conic theory for switched systems.

This report is focused on open-loop system conditions that are sufficient to show closed-loop stability. The following section of this paper will cover some mathematical preliminaries as well as review the concepts of the small gain theorem and passivity theory. The third section will cover more general conditions for closed-loop stability. This includes presenting the concepts of passivity indices and conic systems. The two frameworks will be shown to reduce to the same conditions on a given system. The fourth section of this report will be on the generalization of these conditions to switched systems. In doing so, these frameworks can be shown to apply to systems with switching dynamics just as well as traditional, continuously-varying systems.

2 Background Theory

This section will provide some background material. A few mathematical preliminaries will be given followed by a short review of the small gain theorem and passivity theory.

Recall that the truncation of a signal, $x_T(t)$, is another signal that is equal to the original, $x(t)$, before a time T and zero after.

$$x_T(t) = \begin{cases} x(t) & \text{for } t \leq T \\ 0 & \text{for } t > T \end{cases} \tag{1}$$

The signals of interest in this paper are in the space ($\mathcal{L}_{2e}$) of all functions whose truncations have finite integrals:

$$\mathcal{L}_{2e} = \left\{ x(t) \middle| \int_0^T x(t)dt < \infty, \ \forall T \right\}. \tag{2}$$

When assessing stability, the set of allowable system inputs U is equal to $\mathcal{L}_{2e}$. The system G is an operator that maps an input $u(t)$ onto an output $y(t) = Gu(t)$. We restrict our attention to systems G that map inputs in U to outputs in $Y = \mathcal{L}_{2e}$.

2.1 Small Gain Theorem

One set of open-loop conditions that guarantee closed-loop stability is the small gain theorem. It is an intuitive assessment of the stability of an interconnection (Fig. 1) based on the $\mathcal{L}_2$ gain of each system. If the open-loop gain of the system is less than one, a given signal will be attenuated in every closed-loop path through the interconnection. The attenuation of given inputs and disturbances is sufficient for closed-loop stability. The following definitions are well studied and come directly from [4].

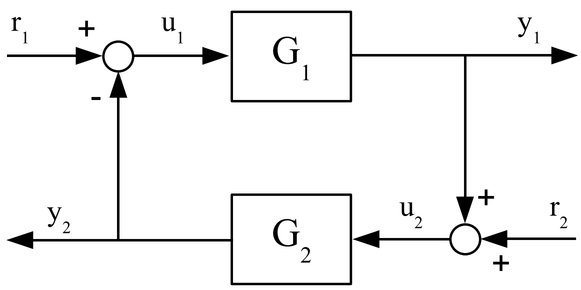

Figure 1: The general feedback interconnection of two systems.

Definition 1. *A system is finite gain $\mathcal{L}_2$ stable with gain γ if, $\forall u \in U$, γ is the smallest value such that there exists a β to satisfy:*

$$||y||_2 \leq \gamma \, ||u||_2 + \beta. \tag{3}$$

Theorem 1. *Consider the feedback connection (Fig. 1) of two $\mathcal{L}_2$ systems,*

$$||y_{1T}||_{\mathcal{L}_2} \leq \gamma_1 \, ||u_{1T}||_{\mathcal{L}_2} + \beta_1$$
$$||y_{2T}||_{\mathcal{L}_2} \leq \gamma_2 \, ||u_{2T}||_{\mathcal{L}_2} + \beta_2$$

If $\gamma_1\gamma_2 < 1$, the feedback connection is finite gain $\mathcal{L}_2$ stable.

This can be used as a design tool whenever a given plant has an arbitrarily large, finite $\mathcal{L}_2$ gain, γ_p. A controller can then be designed to have $\mathcal{L}_2$ gain less than $\frac{1}{\gamma_p}$ so that the product of the gains is bounded above by one.

2.2 Passive Systems Theory

Passivity theory is an intuitive analysis of dynamical systems based on the dissipation of an abstract "energy." Systems exchange energy with the external environment through their inputs and outputs. A passive system is one that can store and dissipate the energy supplied from the environment without generating its own. In many common examples, the energy has physical significance. However, passivity is general enough to apply to any input-output representation with a generalized notion of energy.

Passivity theory is similar to Lyapunov stability theory but is more restrictive. For example, passive systems aren't simply Lyapunov stable but are also minimum phase and have low relative degree. This makes Lyapunov stability theory more relevant for the stability analysis of a single system. However, passivity is a property that is preserved when systems are combined in parallel or feedback. These additional properties make passivity an important design tool for interconnected systems.

For a traditional continuously-varying system (non-switched) it is assumed that a given system is a general nonlinear system of the following form,

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u). \end{aligned} \tag{4}$$

Definition 2. *A system (4) is dissipative if there exists an energy storage function $V(x)$ such that the energy stored in the system is always bounded above by the energy supplied $\omega(u, y)$ to the system over any finite time interval ($T \in [0, \infty)$),*

$$\int_0^T \omega(u, y) dt \geq V(x(T)) - V(x(0)). \tag{5}$$

Definition 3. *A system (4) is passive if it is dissipative with respect to the following supply rate,*

$$\omega(u, y) = u^T y \tag{6}$$

Definition 4. *A system (4) is finite-gain $\mathcal{L}_2$ stable if it is dissipative with respect to the following supply rate, where γ is an upper bound on the $\mathcal{L}_2$-gain of the system,*

$$\omega(u, y) = \gamma^2 u^T u - y^T y \tag{7}$$

However, passivity is inherently an input-output property of systems. It is independent of the internal realization of a system. The definition of passivity can be given with no dependence on the state or energy stored, and can be written

$$\int_0^T u^T y dt \geq -\beta, \tag{8}$$

where $\beta \geq 0$ represents initial stored energy. When it is assumed that the system has zero initial conditions, the definition can be further simplified to the following.

$$\int_0^T u^T y dt \geq 0. \tag{9}$$

This inequality is a statement that the energy supplied to the system is always nonnegative. Equivalently, the system absorbs energy without supplying energy to the environment. This is precisely the definition of a positive real system for the transfer function representation of an LTI system.

Passivity provides many properties that can be used to design stable systems. For example, passive systems are Lyapunov stable. Additionally, the parallel and feedback interconnections of passive systems are again passive and so Lyapunov stable. When other stability results are desired, such as $\mathcal{L}_2$ stability, it is possible to use more restrictive conditions to get these results.

Definition 5. *A system is output strictly passive (OSP) if it satisfies the following inequality for* $\epsilon > 0$*:*

$$\int_0^T u^T y dt \geq \epsilon \int_0^T y^T y dt. \tag{10}$$

Definition 6. *A system is input strictly passive (ISP) if it satisfies the following inequality for* $\delta > 0$*:*

$$\int_0^T u^T y dt \geq \delta \int_0^T u^T u dt. \tag{11}$$

It can quickly be shown that OSP systems are $\mathcal{L}_2$ stable. Additionally, the feedback interconnection of two systems that are either both ISP or both OSP is $\mathcal{L}_2$ stable.

3 Passivity Indices and Conic Systems Theory

Passivity theory provides many results for the stability of a single system and for interconnected systems. However, the results that passivity provides are not the most general. For example, there are many feedback systems that are $\mathcal{L}_2$ stable by the small gain theorem. Feedback systems (Fig. 1) that are stable by the passivity theorem or by the small gain theorem can be combined into a larger framework. This section will be concerned with presenting two such frameworks, the passivity index framework and the conic systems framework. It will be shown that these two actually reduce to the same condition on a given system.

3.1 Passivity Indices

The concept of a passivity index was first presented by [5] and further expanded by [6]. Later it was covered in a text by Sepulchre, et al. [7]. The most thorough presentation so far is the text by Bao and Lee [2]. In these works, the authors cover two indices applied to traditional continuously-varying systems. The first index is based on the range of positive feedback gains that will passivate a system. The other is the range of feed-forward gains that will passivate a system. The two are independent in the sense that knowing one index does not provide any information about the other except that the other index must exist. Both indices are necessary to characterize the level of passivity in a system.

When used individually, the indices are designed so that each is the largest such gain that will passivate the system and any smaller gain will also render the system passive. When a system has a positive value for an index, this is termed an *excess* of that particular form of passivity. Likewise a negative value for that index is termed a *shortage*. This means that passive systems have a positive or zero value for both indices. Non-passive systems can have either one index negative while the other positive or neither index exists.

Definition 7. *The output feedback passivity index (OFP) is the largest gain that can be placed in positive feedback with a system such that the interconnected system is passive (Fig. 2). This index is denoted OFP(ρ). This definition is the same as the following dissipative inequality:*

$$\int_0^T u^T y dt \geq V(x(T)) - V(x(0)) + \rho \int_0^T y^T y dt. \tag{12}$$

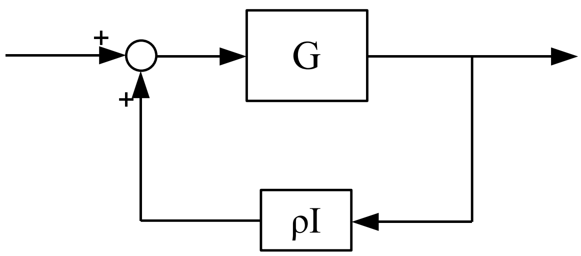

Figure 2: This block diagram shows the physical significance of the OFP index. The feedback of a gain ρ compensates for an excess or shortage of stability in a given system.

Definition 8. *The input feed-forward passivity index (IFP) is the largest gain that can be put in a negative parallel interconnection with a system such that the interconnected system is passive (Fig. 3). This index is denoted IFP(ν). This definition is the same as the following dissipative inequality:*

$$\int_0^T u^T y dt \geq V(x(T)) - V(x(0)) + \nu \int_0^T u^T u dt. \tag{13}$$

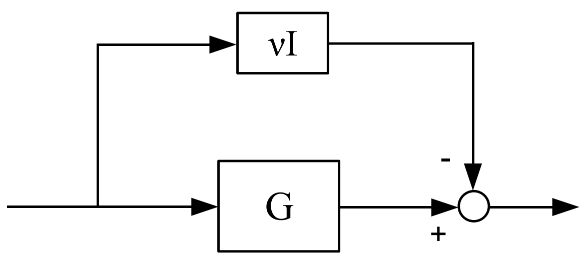

Figure 3: This block diagram shows the physical significance of the IFP index. The feed-forward of a gain ν compensates for an excess or shortage of the minimum phase property in a given system.

When applying the two indices simultaneously (Fig. 4), a system is said to have OFP(ρ) and IFP(ν). Applying both indices gives the following dissipative inequality:

$$\int_0^T \left[(1+\rho\nu)u^T y - \rho y^T y - \nu u^T u\right] dt \geq V(x(T)) - V(x(0)). \tag{14}$$

When considering a system simply as an input-output relation with zero initial conditions, this inequality can be simplified to the following:

$$(1+\rho\nu)\int_0^T u^T y dt \geq \rho \int_0^T y^T y dt + \nu \int_0^T u^T u dt. \tag{15}$$

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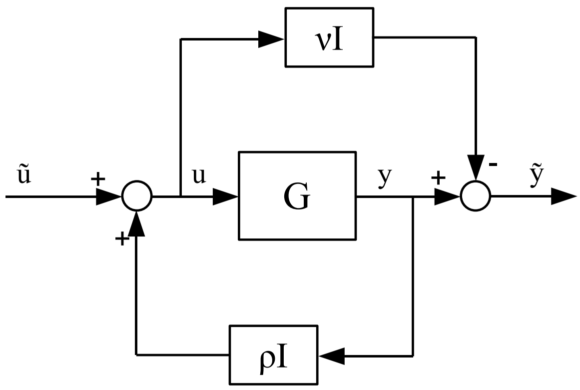

Figure 4: This block diagram shows how the indices are defined when both are applied. Note that there may be a dependence between ρ and ν when applying both indices.

It should be noted that certain combinations of ρ and ν make the system dynamics trivial ($\tilde{y}(t) = 0,\ \forall t$). To avoid this case, the following assumption is made. It is a technical assumption to avoid trivial dynamics. Whenever the condition would not be met, one of the indices can be chosen and reduced by some small value.

A1. It is assumed that $\rho\nu \neq 1$.

When using both indices, there are cases where the indices are dependent, such as when one index is positive and the other is negative. In these cases, the system isn't always made passive when a gains smaller than the maximum that passivate are interconnected appropriately. However, it is still possible to take the positive index as being equal to zero and proceed to design the controller with this fixed index.

It should be noted that it isn't always possible to passivate a system using these two loop transformations. When the OFP index doesn't exist for a system, that system is unstable and can be made passive with negative feedback if the system is of low relative degree and is minimum phase. Likewise, when a system lacks IFP it is non-minimum phase and can be made passive with positive feed-forward if the system is stable. This means that, in general, a system that is both unstable and non-minimum phase cannot be made passive with any combination of feedback and feed-forward gains. In this case, neither index exists. This means, it is possible to have any combination of positive and negative OFP and IFP indices except both being negative. The passivity index methods covered in this report can be applied whenever both indices exist.

The indices can be used as a means to passivate both systems in an existing feedback connection (Fig. 1). This direct application requires the ability to fully design the interface of the dynamic system. Often, this isn't true, or when it is, this framework is typically not the best method of control. However, the indices can be used as a powerful design tool when they aren't applied directly. The knowledge of the indices for a given system can be used to design a feedback controller that compensates for a shortage of passivity in the given system. How this is done is summarized in the following theorem [8].

Theorem 2. *Consider the interconnection (Fig. 1) of two systems where each subsystem has the dynamics previously given (4). Assume that the two systems in the interconnection have OFP(ρ_i) and IFP(ν_i). The interconnection is $\mathcal{L}_2$ stable if the following matrix is positive definite:*

$$A = \begin{bmatrix} (\rho_1 + \nu_2)I & \frac{1}{2}(\rho_2\nu_2 - \rho_1\nu_1)I \\ \frac{1}{2}(\rho_2\nu_2 - \rho_1\nu_1)I & (\rho_2 + \nu_1)I \end{bmatrix} > 0 \tag{16}$$

The proof of the theorem is omitted because it is very similar to the proof of Theorem 3.

Using the passivity indices provides more information about a system other than the simple assessment of passive or not passive. In the analysis of an existing interconnection, typically the full system description is used to assess stability so the indices aren't as useful. The index is more useful as a passive control design tool to create stable feedback loops even when the systems in the loop aren't passive.

3.2 Conic Systems

Passivity indices are an extension of passive theory. This is because the indices allow for the feedback stability results provided by passivity theory to be applied to systems that aren't necessarily passive. An extension like this was done much earlier in the study of conic systems. Conic systems theory is an analysis tool, based on operator theory, that assesses the input-output behavior of system. This was first presented by Zames [3]. He showed that this was a general framework that contained both the passivity theorem and the small gain theorem as special cases. In this respect, it is a very general assessment of input-output stability of a feedback loop.

Definition 9. *A conic system is one whose input $u(t) \in U$ and output $y(t) \in Y$ are constrained to a conic region of their product space $U \times Y$ (Fig. 5).*

This conic sector can be defined in two ways. The first is by the slope of the center of the cone c and its radius r. The second is to define it by the slope of the upper bound b and lower bound a. In this case, $b > 0$, these definitions can be related by the expressions, $b = c + r$ and $a = c - r$ or $c = \frac{a+b}{2}$ and $r = \frac{b-a}{2}$. When using the first definition, the shaded sector of the figure can be defined as

$$||y_T - cu_T||_2 \leq r\,||u_T||_2\,. \tag{17}$$

This expression is valid for "interior" conic systems. For the purposes of this report, we will define these to be systems where $b > 0$ and $b > a$. There is no sign restriction on a. Writing this definition

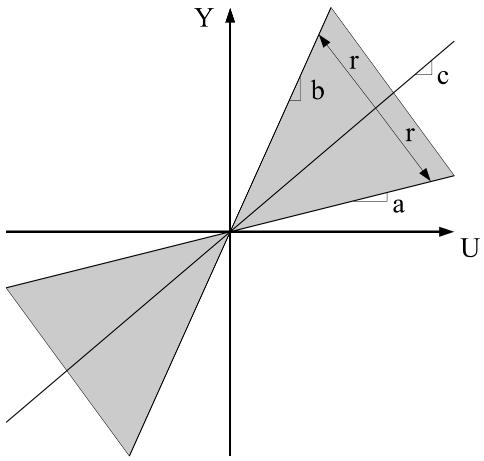

Figure 5: This is the graphical representation of a conic system. For all time instants, the inner product of input and output lies in the shaded cone. This figure shows the case of $b > 0$, but a can be positive (as drawn) or negative.

out in terms of integrals and making the substitutions noted above, the following derivation can be found.

$$
\begin{aligned}
\int_0^T (y - cu)^T (y - cu) dt &\le r^2 \int_0^T u^T u dt \\
\int_0^T (y - (\frac{a+b}{2})u)^T (y - (\frac{a+b}{2})u) dt &\le (\frac{b-a}{2})^2 \int_0^T u^T u dt \\
\int_0^T \left[y^T y - (a+b) u^T y + (\frac{a+b}{2})^2 u^T u \right] dt &\le (\frac{b-a}{2})^2 \int_0^T u^T u dt \\
\int_0^T y^T y dt + ab \int_0^T u^T u dt &\le (a+b) \int_0^T u^T y dt \\
\frac{1}{b} \int_0^T y^T y dt + a \int_0^T u^T u dt &\le (1 + \frac{a}{b}) \int_0^T u^T y dt
\end{aligned}
$$

This result is consistent with other recent work [9].

In the case when $b < 0$, a similar definition can be formulated. In this case, the shaded region is defined as the region outside of the cone defined by c and r (Fig. 6). Now the relations become $b = c - r$ and $a = c + r$ which leads to $c = \frac{a+b}{2}$ and $r = \frac{a-b}{2}$. The shaded region is defined by the following expression,

$$||y_T - cu_T||_2 \ge r \, ||u_T||_2 \, . \tag{18}$$

This expression is valid for “exterior” conic systems. For the purposes of this report, we will define these to be systems where $b < 0$ and $b < a$. Again, there is no sign restriction on a. It should be noted that this definition of the values a and b in exterior conic systems is reversed from Zames' work where $a < 0$ and $a < b$. The definitions presented here allow for a single proof for all cases of signs of a and b while Zames proves these cases separately. Writing this definition out in terms of

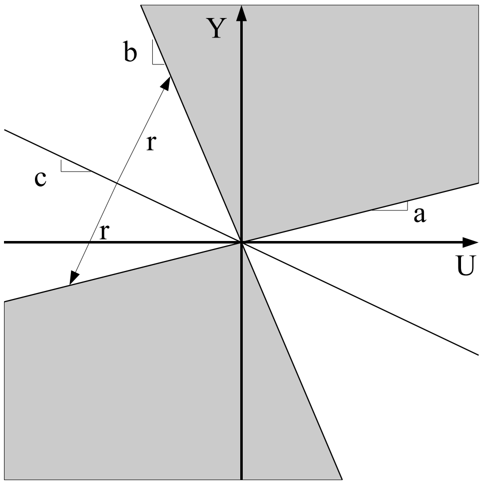

Figure 6: This is a conic system when $b < 0$. For all time instants, the inner product of input and output lies in the shaded cone. Again, a can be positive (as drawn) or negative.

integrals and making the substitutions noted above, the following derivation can be made.

$$
\begin{aligned}
\int_0^T (y - cu)^T (y - cu) dt &\geq r^2 \int_0^T u^T u dt \\
\int_0^T (y - (\frac{a+b}{2})u)^T (y - (\frac{a+b}{2})u) dt &\geq (\frac{a-b}{2})^2 \int_0^T u^T u dt \\
\int_0^T \left[y^T y - (a+b) u^T y + (\frac{a+b}{2})^2 u^T u \right] dt &\geq (\frac{a-b}{2})^2 \int_0^T u^T u dt \\
\int_0^T y^T y dt + ab \int_0^T u^T u dt &\geq (a+b) \int_0^T u^T y dt \\
\frac{1}{b} \int_0^T y^T y dt + a \int_0^T u^T u dt &\leq (1 + \frac{a}{b}) \int_0^T u^T y dt
\end{aligned}
$$

After dividing through by $b < 0$, we find the last line of this derivation is the same expression that is derived in the case when $b > 0$. At this point, the expression is valid for all signs of a and b. With the definitions presented in this report, the following definition can be made for all a and b.

Definition 10. *A system is constrained to the cone defined by the lines with slopes b and a (Fig. 5-6) if and only if the following inequality holds,* $\forall T \in [0, \infty)$*:*

$$
(1 + \frac{a}{b}) \int_0^T u^T y dt \geq \frac{1}{b} \int_0^T y^T y dt + a \int_0^T u^T u dt. \tag{19}
$$

The definition of conic systems becomes useful in the study of interconnected systems.

Theorem 3. *Consider the interconnection (Fig. 1) of two conic systems (4). Assume that system one is constrained to the cone* $[a_1, b_1]$ *and system two to the cone* $[a_2, b_2]$*. The interconnection is*

$\mathcal{L}_2$ stable if the following matrix is positive definite:

$$A = \begin{bmatrix} (a_2 + \frac{1}{b_1})I & \frac{1}{2}(\frac{a_1}{b_1} - \frac{a_2}{b_2})I \\ \frac{1}{2}(\frac{a_1}{b_1} - \frac{a_2}{b_2})I & (a_1 + \frac{1}{b_2})I \end{bmatrix} > 0. \tag{20}$$

Proof. Denote the evaluation of the state at a time T by $x(T) = x_T$. The initial state is x_0 and the switching times are x_{t_0}, x_{t_1}, etc. The conic bounds on the two systems ensure that the following inequalities hold,

$$\int_{x_0}^{T} \left[(1 + \frac{a_1}{b_1})u_1^T y_1 - \frac{1}{b_1} y_1^T y_1 - a_1 u_1^T u_1\right] dt \geq V_1(x_T) - V_1(x_0)$$
$$\int_{x_0}^{T} \left[(1 + \frac{a_2}{b_2})u_2^T y_2 - \frac{1}{b_2} y_2^T y_2 - a_2 u_2^T u_2\right] dt \geq V_2(x_T) - V_2(x_0)$$

Define the total energy stored in the interconnection to be $V(x) = V_1(x) + V_2(x)$. Summing the two inequalities and using the interconnection relationships gives

$$\int_{x_0}^{T} -y^T A y + r^T B y + r^T C r dt \geq V(x_T) - V(x_0),$$

with the matrices defined as follows,

$$A = \begin{bmatrix} (a_2 + \frac{1}{b_1})I & \frac{1}{2}(\frac{a_1}{b_1} - \frac{a_2}{b_2})I \\ \frac{1}{2}(\frac{a_1}{b_1} - \frac{a_2}{b_2})I & (a_1 + \frac{1}{b_2})I \end{bmatrix}, \quad B = \begin{bmatrix} (1 + \frac{a_1}{b_1})I & 2a_1 I \\ -2a_2 I & (1 + \frac{a_2}{b_2})I \end{bmatrix},$$

$$\text{and} \quad C = \begin{bmatrix} -a_1 I & 0 \\ 0 & -a_2 I \end{bmatrix}.$$

Note that A is positive definite by assumption. The following definitions can be made to bound this expression:

$$a = \sqrt{\lambda_{min}(A^T A)}, \quad b = ||B||_2, \quad \text{and} \quad c = ||C||_2,$$

where $||\cdot||_2$ denotes the largest singular value of the matrix and a is the smallest singular value of A. Now, a simplified upper bound can be found:

$$\begin{aligned} V(x_T) - V(x_0) &\leq -a\,||y_T||_2^2 + b\,||r_T||_2\,||y_T||_2 + c\,||r_T||_2^2 \\ &= -\frac{1}{2a}(a\,||y_T||_2 - b\,||r_T||_2)^2 + \frac{k^2}{2a}\,||r_T||_2^2 - \frac{a}{2}\,||y_T||_2^2 \\ &\leq \frac{k^2}{2a}\,||r_T||_2^2 - \frac{a}{2}\,||y_T||_2^2, \end{aligned}$$

where $k^2 = b^2 + 2ac$. Note that this is equivalent to:

$$V(x_T) - V(x_0) \leq \int_{t_0}^{T} \left(\frac{k^2}{2a} r^T r - \frac{a}{2} y^T y\right) dt$$

$$\frac{2}{a}(V(x_T) - V(x_0)) \leq \int_{t_0}^{T} \left(\frac{k^2}{a^2} r^T r - y^T y\right) dt$$

The remaining steps are to rearrange, remove the $V(x_T)$ since it is positive, and to apply the fact that for any values α and β $\sqrt{\alpha^2+\beta^2} \leq \alpha+\beta$:

$$\begin{aligned} ||y_T||_2^2 &\leq \frac{k^2}{a^2}\,||r_T||_2^2 + \frac{2}{a}V(x_0) \\ ||y_T||_2 &\leq \frac{k}{a}\,||r_T||_2 + \sqrt{\frac{2}{a}V(x_0)}. \end{aligned}$$

This shows that the loop interconnection is $\mathcal{L}_2$ stable with gain less than or equal to $\frac{k}{a}$. $\square$

Passive systems are a special case of conic systems. They are ones that are restricted to the cone that is precisely the first and third quadrants ($a = 0$ and $b \to \infty$). The inequality (19) can be used to show the expected inequality is the special case of passivity.

$$\begin{aligned} (1+\frac{a}{b})\int_0^T u^T y dt &\geq \frac{1}{b}\int_0^T y^T y dt + a\int_0^T u^T u dt \\ \int_0^T u^T y dt &\geq 0 \end{aligned}$$

That last line is precisely the passive inequality (9). Conic systems theory shows that a system that is not passive but still constrained to any cone, that does not span the entire $U \times Y$ space, can be compensated by a system that is contained within a more restrictive conic sector.

Another special case of conic systems is $\mathcal{L}_2$ stable systems. These are ones that have a cone centered at $c = 0$ and have a finite radius ($r < \infty$). The conic bounds become $b = r$ and $a = -r$. Making these substitutions into the conic definition (19) gives the following derivation.

$$\begin{aligned} (1+\frac{a}{b})\int_0^T u^T y dt &\geq \frac{1}{b}\int_0^T y^T y dt + a\int_0^T u^T u dt \\ 0 &\geq \frac{1}{r}\int_0^T y^T y dt - r\int_0^T u^T u dt \\ r^2\int_0^T u^T u dt &\geq \int_0^T y^T y dt \\ r\,||u||_2 &\geq ||y||_2 \end{aligned}$$

The final line in that derivation is the expected definition for a finite gain $\mathcal{L}_2$ stable system with gain r and offset $\beta = 0$.

3.3 Connecting the Two Frameworks

As mentioned previously, the passivity index framework and the conic systems framework are actually the same input-output condition for a given system.

To show this connection, the definition of conic systems (19) is used with the following substitutions, $a = \nu$ and $b = \frac{1}{\rho}$, to give

$$(1+\nu\rho)\int_0^T u^T y dt \geq \rho\int_0^T y^T y dt + \nu\int_0^T u^T u dt.$$

This expression is precisely the dissipative inequality (15) for a system with passivity indices ρ and ν. This demonstration shows that the passivity index framework is the same dissipative condition as in the conic systems framework.

4 Extension to Switched Systems

Sufficient conditions for a feedback interconnection to be stable have been found based on the passivity theorem, the small gain theorem, conic systems theory, and passivity index theory. While the first two are straightforward to apply, the latter two frameworks are more general. Neither of these two general frameworks has been applied to switched systems. The development that follows can be seen as a generalization of both frameworks to systems with switching behavior.

4.1 Definition of Passivity for Switched Systems

As stated previously, passivity provides many benefits over general stability theory, especially in the analysis and synthesis of interconnected systems. The benefits of passivity have been further extended when applied to switched systems. Passivity can be applied to systems with naturally hybrid dynamics or systems with switching controllers. For example, many passive controllers can be designed for a given passive system based on varying criteria then switching decisions can be made in real-time by a supervisory controller.

There have been a few definitions of passivity proposed for switched systems [10],[11],[12]. The most comprehensive definition in the literature so far is the work of Zhao and Hill [12]. In this paper, passivity for switched systems is defined and used to show some of the expected stability results and that passivity is preserved when two systems are combined in negative feedback. The switched systems in question have the following state dynamics with a finite number of subsystems, $\sigma(t) \in \{1, ..., m\}$,

$$\begin{array}{rcl} \dot{x} & = & f_\sigma(x,u) \\ y & = & h_\sigma(x,u). \end{array} \tag{21}$$

For the remainder of this paper, assume that a given system switches a finite number of times on any finite time interval. The *switching signal*, $\sigma(t)$, is a function of time that takes on the value of the index of the subsystem $\{1, ..., m\}$ that is active at each time instant. Consider the k^{th} time switching to the i^{th} subsystem. The switching signal has the value i from time t_{i_k} up to time t_{i_k+1}. The next time system i becomes active is at time $t_{i_{k+1}}$.

Definition 11. *A given switched system (21) is passive if the following condition is satisfied. There must exist storage functions $V_i(x)$ such that each subsystem is passive while active,*

$$\int_{t_1}^{t_2} u^T(t)y(t)dt \geq V_i(x(t_2)) - V_i(x(t_1)), \forall i = 1, ..., m. \tag{22}$$

In the paper [12], the authors showed that this definition meets many of the expected stability results for passive systems. Specifically, they showed that strictly passive systems are asymptotically stable and that output strictly passive systems are $\mathcal{L}_2$ stable. Additionally, it was shown that the negative feedback interconnection of two passive switched systems is still passive.

Again, for the stability analysis of a given switched system, Lyapunov stability theory using multiple Lyapunov functions is the less restrictive option [13]. However, the passive switched system definition simplifies control design synthesis because any passive controller can be interconnected with any passive plant and the feedback loop is stable for any switching signal.

4.2 Design Using Conic Theory for Switched Systems

Typically, passive systems are controlled by designing a passive feedback controller. The feedback invariance that passivity provides gives the immediate result that this interconnected system (Fig. 1) is passive and stable. This approach is valid when the systems in the loop have switching behavior or when they are continuously-varying. In many cases, when the system to be controlled is not passive, the passivity indices can be used to design a feedback system that is still stable.

The following result serves as a design guideline for using conic theory. It is very general in that it can be applied to nonlinear switched systems that may or may not be passive or even stable. The main restriction with using these methods is that each of the subsystems in the switched system must be contained within a cone.

The main result of this paper is the application of conic theory to switched systems. The switched systems of interest are of the form 21, with initial condition $x(0) = x_0$. The functions are assumed to satisfy $f_i(0,0) = 0$ and $h_i(0,0) = 0, \forall i$. The signals $x(t) \in \mathbb{R}^n$ and $u(t), y(t) \in \mathbb{R}^p$.

In the switched system there are a finite number of subsystems m. The switching signal $\sigma(t)$ tracks the active subsystem, $\sigma : \mathbb{R}_+ \to \{1, 2, ..., m\}$. There are two methods of indicating switching instances used in this paper. The first is to index them purely chronologically where t_0 represents the initial time when the zeroth subsystem is active, t_1 is the time of the first switch, and $\{t_0, t_1, ..., t_i, ...\}$ is the set of all switching times. The second method indexes the switching times t_{i_k}, which represents the k_{th} time that the i_{th} subsystem becomes active. This subsystem becomes inactive at time t_{i_k+1} and becomes active again at time $t_{i_{k+1}}$. The following assumption is required for the switching to be well defined.

A2. It is assumed that the system switches a finite number of times on any finite interval. Specifically, the number of switches on the interval $[t_0, T]$ can be bounded by a constant $K = K_T$ that is dependent on T.

In the switched system framework, typically the switched signal is measureable in real time or known a priori. With either of these assumptions, many controllers can be designed, one for each subsystem of the given plant, and the controller can switch with the given plant. These controllers are designed to satisfy *Theorem 1* so that each subsystem of the interconnection is $\mathcal{L}_2$ stable. It remains to be shown that the overall switched system is $\mathcal{L}_2$ stable.

Theorem 4. *A switched system (21) is formed as the feedback interconnection of two other switched systems. If each subsystem of the interconnected system satisfies the conditions of Theorem 1 (i.e. each subsystem is $\mathcal{L}_2$ stable with gain $\gamma_i = \frac{k_i}{a_i}$) and the accumulated energy at switching instants is bounded,*

$$V_0(x(t_0)) + \sum_{i_k=1}^{\infty} [V_{i_k}(x(t_{i_k})) - V_{i_k-1}(x(t_{i_k}))] \leq \beta^2,$$

then the overall switched system is also $\mathcal{L}_2$ stable.

Proof. Each subsystem i is $\mathcal{L}_2$ stable. Consider the times t_1 and t_2 such that $t_{i_k} \leq t_1 \leq t_2 \leq t_{i_k+1}$. The following inequality holds $\forall i$,

$$\int_{t_1}^{t_2} y^T y dt \leq \gamma_i^2 \int_{t_1}^{t_2} r^T r dt + V_i(x(t_1)) - V_i(x(t_2)). \tag{23}$$

To verify that the system is $\mathcal{L}_2$ stable, the system output must be analyzed from the initial time t_0 to arbitrary time T. Note that $t_0 \leq t_1 \leq ... \leq t_k \leq T$.

$$\begin{aligned}
\int_{t_0}^{T} y^T y dt &= \sum_{i_k=1}^{K} \int_{t_{i_k-1}}^{t_{i_k}} y^T y dt + \int_{t_K}^{T} y^T y dt \\
&\leq \sum_{i_k=0}^{K} \left[\gamma_{i_k}^2 \int_{t_{i_k}}^{t_{i_k+1}} r^T r dt + V_{i_k}(x(t_{i_k})) - V_{i_k}(x(t_{i_k+1})) \right] + \\
&\quad \gamma_{i_K}^2 \int_{t_K}^{T} r^T r dt + V_{i_K}(x(t_K)) - V_{i_K}(x(T)) \\
&\leq \gamma^2 \int_{t_0}^{T} r^T r dt + V_0(x(t_0)) + \sum_{i_k=0}^{K} [V_{i_k+1}(x(t_{i_k+1})) - V_{i_k}(x(t_{i_k+1})] \\
&\leq \gamma^2 \int_{0}^{T} r^T r dt + \beta^2 \\
||y_T||_2 &\leq \gamma\, ||r_T||_2 + \beta
\end{aligned}$$

The above holds with $\gamma = \max_i\{\gamma_i\}$ and β is defined previously. Again, this proof uses the facts that $V_{i_k} \geq 0, \forall i$ and for values α and β $\sqrt{\alpha^2 + \beta^2} \leq \alpha + \beta$. $\square$

To use this result, consider the interconnection of two systems that are allowed to have switched nonlinear dynamics. Assume that either system is a given plant that is not passive but that is contained within a switching cone across all time. The cones of the given subsystems can be used to determine bounds on the designed cones for the other system in the loop. This system is a control system that can be designed to be contained within a complementary switching cone to guarantee stability of the loop.

4.3 Example

The following example illustrates how this design methodology can be used to design stable loops even when a given subsystem is unstable or non-minimum phase. This example was chosen to be simple to follow so it is a LTI system.

Example 1. Consider the negative feedback interconnection of two systems (Fig. 1). G_1 is a given dynamic system with two switching subsystems (21). The first subsystem is unstable with indices, $\rho_1 = -6$ and $\nu_1 = 0$.

$$\begin{aligned}
f_{1P}(x,u) &= \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
h_{1P}(x,u) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x
\end{aligned}$$

The second subsystem is non-minimum phase with indices, $\rho_2 = 0$ and $\nu_2 = -\frac{1}{2}$.

$$\begin{aligned} f_{2P}(x,u) &= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ h_{2P}(x,u) &= \begin{bmatrix} -1 & 1 \end{bmatrix} x \end{aligned}$$

Controllers can be designed for the two subsystems independently. A controller for the first subsystem must have an IFP index greater than 6 and a positive OFP to satisfy Theorem 3. A controller can be designed with a proportional gain of at least 6 and a single pole that is in the open left-half plane.

$$\begin{aligned} f_{1c}(x,u) &= -px + u \\ h_{1c}(x,u) &= x + K_P \end{aligned}$$

The values chosen for this example are $K_P = 8$ and $p = \frac{1}{10}$. The resulting controller has OFP index $\frac{1}{18}$ and IFP index 8. Using the indices, the following matrix is found for the first interconnected system. This matrix being positive definite guarantees that the first subsystem is $\mathcal{L}_2$ stable.

$$A_1 = \frac{1}{18} \begin{bmatrix} 36 & 4 \\ 4 & 1 \end{bmatrix} > 0 \tag{24}$$

For the second controller, a phase lead controller was designed with a gain K. The pole must be less than $-\frac{1}{2}$. For this example, the pole location was chosen to be -1 ($p = 1$) and the zero and gain were chosen so that $z = \frac{1}{2}$ and $K = 1$.

$$\begin{aligned} f_{2c}(x,u) &= -px + u \\ h_{2c}(x,u) &= K(z-p)x + K \end{aligned}$$

The second controller has OFP index 1 and IFP index $\frac{1}{2}$. This results in the following A matrix for the second interconnected system.

$$A_2 = \frac{1}{4} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} > 0 \tag{25}$$

At this point, the control system is a switched system containing two stable subsystems. Of course the switching signal must be analyzed to verify that the switching behavior preserves stability. In this case, Theorem 3 is satisfied with arbitrary switching. This is because the two systems are linear and their matrix pencil is always Hurwitz.

For this example, the switching was made to be exponentially distributed (with average switching time 0.1 seconds) to represent that switching was equally likely at any time. This example was simulated with a zero input ($r = 0$) to show asymptotic stability. The figure below shows the response of the three system states. Note that two of the states are from the given plant while the third one is of the controller.

This example shows how Theorem 3 can be used to design stable control systems. Of course the example is a simple linear, time-invariant case, however, the theorem and proof is valid for general nonlinear switched systems.

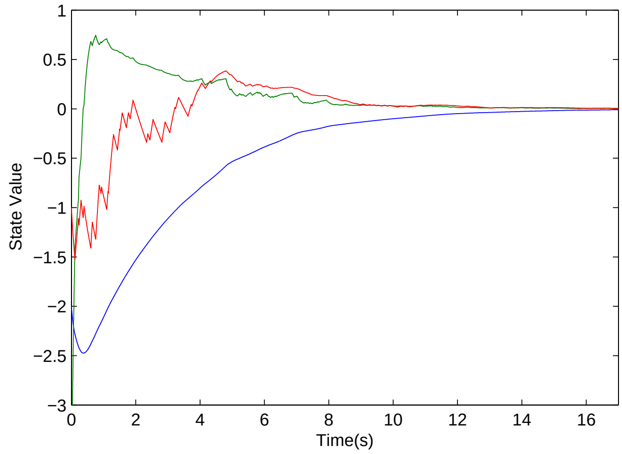

Figure 7: The convergence of the state in the given example.

5 Conclusion

This report focused on the varying conditions that can be imposed on two systems to give stability of their feedback interconnection. These conditions start with the well known results of the small gain theorem and the passivity theorem. Although these results are intutitive, they do not provide the most general results for the stability of their interconnection. These have been generalized in one of two ways. One method is the classic conic systems theory that is based on whether the inner product of the input and output of the two systems can be restricted. Another method is passivity index theory which is based on loop transformations of linear systems. It was shown that these two frameworks reduce to the same input-output condition on a given system. The remainder of this report was on the extension of these general frameworks to switched systems. This extension, based on the recent redefinition of passivity for switched systems, would provide the powerful control design methods for systems with switching behavior. It was shown that the design method still applies and an example was provided that illustrates how the main theorem can be used.

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