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Stability of Networked Passive Switched Systems

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***Abstract*—In this paper the problem of controlling nonlinear switched systems over a network with time-varying delay is addressed. The solution presented is an extension of results from the control of continuously-varying passive systems over a network using the wave variable transformation. Background material is presented on passivity and the wave variable transformation. The concept of passivity for switched systems is also covered in this paper. Stability results are then shown for passive switched systems connected over a network with time-varying delay.**

I. INTRODUCTION

Passive systems theory is a widely used tool for the analysis and synthesis of nonlinear systems [1], [2]. Passivity is a characterization of system behavior based on energy. Passive systems store and dissipate energy without generating their own. This implies that passive systems are Lyapunov stable and minimum phase. Passivity can be used to assess the stability of a single system, but it is more restrictive than directly showing Lyapunov stability. A real benefit is that when two passive systems are combined in negative feedback, the resulting interconnection is passive and stable. These results provide open-loop conditions to guarantee closed-loop stability.

Although passivity is often used to assess stability of feedback systems, these results do not hold when systems are interconnected over a delayed network. The delays cause the energy being sent over the network to be different than the energy received at the other side of the network. Typically the interconnection becomes unstable even for small delays.

There are many applications where systems must be controlled over a delayed network. One solution to the delay problem is covered in [3] and [4]. This framework uses passivity theory and the wave variable transformation [5]. The approach was first used to guarantee stability in telemanipulation systems over networks with constant time delays [6]. It has been expanded in many works such as [7] and [8]. In general, it can be applied to any system that is passive or can be made passive with a local controller. The wave variable transformation is used to map the generalized power variables, the ones used to show passivity, to wave variables. After being transformed to wave variables, the energy exchanged with the network is decoupled between waves going out over the network and waves coming in from the network. The decoupling makes the delayed channel lossless; no energy is added or removed by a channel with constant delays. Later, this work was expanded to apply to networks with time-varying delay [9]. The approach for treating time-varying delays is used in the current paper.

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The framework for control of passive systems over a network applies to non-switched systems. The present paper expands the framework to switched systems. This allows for a much larger class of systems to be controlled over a network. The extension presented here uses a definition of passivity for switched systems in [10]. The present paper applies the concept of passive switched systems to networked passive systems using a modified wave variable transformation.

The remainder of the paper is organized as follows. Background material on passivity theory and the wave variable transformation for networks is covered in Section 2. The constant time delay case is covered first, and then the time-varying delay case is presented with a modified wave variable transformation. Section 3 covers a previously presented definition of passive switched systems. Section 4 presents new material on interconnecting passive switched systems over a network with time-varying delay. Section 5 contains concluding remarks.

II. BACKGROUND MATERIAL

A. Passivity Theory

Passivity is a characterization of system behavior based on a generalized notion of energy. A passive system is one that stores and dissipates energy without generating its own. For a thorough background on passivity, refer to [11] and [12].

In this section of the paper, passivity will be applied to continuously-varying systems of the form,

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x). \end{aligned} \tag{1}$$

Showing passivity is typically done by finding a storage function $V(x)$ that represents a notion of internal energy. V is required to be positive definite; that is, it is strictly positive for all arguments not equal to zero, $V(x) > 0$ for $x \neq 0$, and equal to zero only for the zero argument, $V(0) = 0$. This function is used to show passivity in the following definition.

Definition 1. *Consider a nonlinear system (1). This system is passive if there exists a positive definite storage function $V(x)$ such that*

$$\int_{t_1}^{t_2} (u^T y - \epsilon y^T y) dt \geq V(x(t_2)) - V(x(t_1)), \tag{2}$$

for $\epsilon \geq 0$. If $\epsilon > 0$, the system is said to be output strictly passive (OSP).

This definition uses the system input u and output y. In this paper, these will be referred to as power variables even when their product is not a traditional notion of power.

When two passive systems are interconnected in negative feedback the resulting system is passive. This property of

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passivity, feedback invariance, makes it a strong tool for the analysis and synthesis of interconnected systems. This property is essential for the networked control systems framework presented in this paper.

The subset of passive systems that are OSP is important because these systems are $\mathcal{L}_2$ stable. They also form feedback interconnections that are $\mathcal{L}_2$ stable. These results can be extended to internal stability with an appropriate detectability assumption. One such definition, asymptotic zero-state detectability, is given. Systems that are $\mathcal{L}_2$ stable can be shown to be asymptotically stable if they are also asymptotically zero-state detectable.

Definition 2. *[10] Consider an unforced nonlinear system,*

$$\begin{array}{rcl} \dot{x} & = & f(x) \\ y & = & h(x). \end{array} \tag{3}$$

This system is asymptotically zero-state detectable (ZSD) if $\forall \epsilon > 0, \exists \delta > 0$ *such that* $||y(t+s)|| < \delta$ *for some* $t \geq 0$, $\Delta > 0$ *and* $0 \leq s \leq \Delta$ *implies* $||x(t)|| < \epsilon$.

This form of detectability for nonlinear systems is less restrictive than zero-state detectability. It can be used to show that OSP systems are asymptotically stable. It also can be applied to general systems where the output approaches zero asymptotically.

B. Network Structure

The networked control structure used in this paper is given in Fig. 1. Typically, G_1 is a given passive plant and G_2 is a designed passive controller. For this initial work, the delays in the network are assumed to be constant but the two delays T_1 and T_2 can be different. The signal relationships are given as,

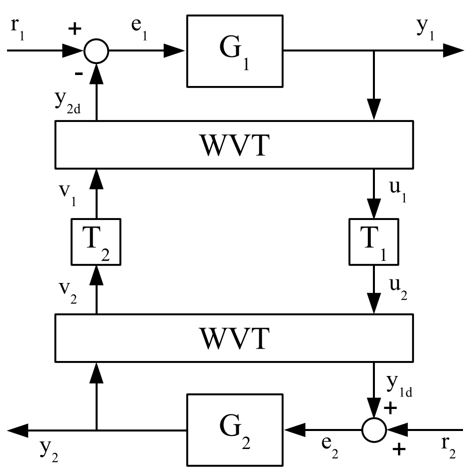

Fig. 1. This figure shows the network structure for control of passive plants using the wave variable transformation. The blocks T_1 and T_2 are the network time delays. The two blocks labeled WVT are the transformations to wave variables on each side of the network.

$$e_1 = r_1 - y_{2d} \tag{4}$$

$$e_2 = r_2 + y_{1d}. \tag{5}$$

The network is modeled as a constant delay in each direction,

$$u_2(t) = u_1(t - T_1) \tag{6}$$

$$v_1(t) = v_2(t - T_2). \tag{7}$$

The wave variable transformation (WVT) is defined as in [5]. The linear transformation to wave variables is

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \frac{1}{\sqrt{2b}} \begin{bmatrix} bI & I \\ bI & -I \end{bmatrix} \begin{bmatrix} y_1 \\ y_{2d} \end{bmatrix} \tag{8}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \frac{1}{\sqrt{2b}} \begin{bmatrix} bI & I \\ bI & -I \end{bmatrix} \begin{bmatrix} y_{1d} \\ y_2 \end{bmatrix}, \tag{9}$$

where b is the impedance of the channel and can be chosen in the synthesis of a controller. With the inputs and outputs of the two wave variable transformation blocks as defined in the figure, the transformation is actually implemented as

$$\begin{bmatrix} u_1 \\ y_{2d} \end{bmatrix} = \begin{bmatrix} -I & \sqrt{2b}I \\ -\sqrt{2b}I & bI \end{bmatrix} \begin{bmatrix} v_1 \\ y_1 \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} v_2 \\ y_{1d} \end{bmatrix} = \begin{bmatrix} I & -\sqrt{\frac{2}{b}}I \\ \sqrt{\frac{2}{b}}I & -\frac{1}{b}I \end{bmatrix} \begin{bmatrix} u_2 \\ y_2 \end{bmatrix}. \tag{11}$$

The energy stored in the network is the sum of the energy going into the network minus the energy coming out of the network.

$$V_N = \frac{1}{2}\int_{t_0}^{t} (u_1^T u_1 + v_2^T v_2 - u_2^T u_2 - v_1^T v_1) d\tau. \tag{12}$$

When the system delays are constant, this expression can be simplified to show that the energy in the network is positive.

$$V_N = \frac{1}{2}\int_{t-T_1}^{t} u_1^T u_1 d\tau + \frac{1}{2}\int_{t-T_2}^{t} v_2^T v_2 d\tau \geq 0 \tag{13}$$

The quantity V_N is always nonnegative. This can be used to show that the network is a passive system. By the definition of energy stored in the network (12), it can be seen that the energy on the G_1 side of the network bounds the energy on the G_2 side.

$$\frac{1}{2}\int_{t_0}^{T} (u_1^T u_1 - v_1^T v_1) d\tau \geq \frac{1}{2}\int_{t_0}^{T} (v_2^T v_2 - u_2^T u_2) d\tau \tag{14}$$

$$\Longrightarrow \int_{t_0}^{T} y_1^T y_{2d} d\tau \geq \int_{t_0}^{T} y_2^T y_{1d} d\tau \tag{15}$$

This fact can be used to show stability of the overall system.

Theorem 1. *Consider two passive systems of the form (1) where* $f_i(0,0) = 0$ *and* $h_i(0) = 0$ *for* $i = 1, 2$. *These two systems are interconnected over a delayed network using the wave variable transformation (Fig. 1). If the delays in the network are constant, the interconnected system is* $\mathcal{L}_2$ *stable.*

Additionally, if the two systems are asymptotically zero-state detectable, the overall system is asymptotically stable for $r(t) = 0$.

C. Compensating for time-varying delay

The proposed architecture applies to constant time delays. However, typical communication channels include time-varying delays. One method of solving this problem is to introduce a modified wave variable transformation in order to compensate for time-varying delays in the system. This solution is a simple modification of the solution presented in [9]. It is assumed that the delay in the network is measurable in real-time and that the maximum rate of change of each delay is bounded, $\frac{dT_i}{dt} \leq 1$.

The time-varying delays are compensated by time-varying gains in the transformation. Essentially, each received wave variable is scaled before the transformation is applied. The scaling is defined as

$$\hat{u}_2(t) = f_1(t)u_1(t - T_1(t)) \tag{16}$$

$$\hat{v}_1(t) = f_2(t)v_2(t - T_2(t)). \tag{17}$$

These time-varying gains can be can be incorporated into the wave variable transformation,

$$\begin{bmatrix} u_1 \\ y_{2d} \end{bmatrix} = \begin{bmatrix} -f_2(t)I & \sqrt{2b}I \\ -\sqrt{2b}f_2(t)I & bI \end{bmatrix} \begin{bmatrix} v_1 \\ y_1 \end{bmatrix} \tag{18}$$

$$\begin{bmatrix} v_2 \\ y_{1d} \end{bmatrix} = \begin{bmatrix} f_1(t)I & -\sqrt{\frac{2}{b}}I \\ \sqrt{\frac{2}{b}}f_1(t)I & -\frac{1}{b}I \end{bmatrix} \begin{bmatrix} u_2 \\ y_2 \end{bmatrix}. \tag{19}$$

Now this wave variable transformation can replace the previous one (10-11). The architecture in Fig. 1 can still be used for the time-varying delay case. The gains f_1 and f_2 are time-varying and are chosen such that

$$f_1^2(t) \leq 1 - \frac{dT_1}{d\tau} \tag{20}$$

$$f_2^2(t) \leq 1 - \frac{dT_2}{d\tau}. \tag{21}$$

If the gains are chosen to satisfy the above inequalities with equality, the channel remains lossless.

The energy in the channel is now given as

$$V_N = \frac{1}{2}\int_{t_0}^{t} (u_1^T u_1 + v_2^T v_2 - \hat{u}_2^T \hat{u}_2 - \hat{v}_1^T \hat{v}_1) d\tau. \tag{22}$$

Substituting in $\hat{u}_2$ and $\hat{v}_1$ as defined above (16-17) gives

$$V_N = \frac{1}{2}\int_{t_0}^{t} (u_1^T u_1 + v_2^T v_2 - f_1^2 u_1^T(\tau - T_1)u_1(\tau - T_1) - f_2^2 v_2^T(\tau - T_2)v_2(\tau - T_2))d\tau.$$

The two time-varying terms in the above integral can be bounded by a constant term. The following derivation uses the substitution $s = \tau - T_1(t)$.

$$\begin{aligned} &\int_{t_0}^{t} f_1^2 u_1^T(\tau - T_1(\tau))u_1(\tau - T_1(\tau))d\tau \\ &\quad \leq \int_{t_0}^{t} (1 - \frac{dT_1}{d\tau})u_1^T(\tau - T_1)u_1(\tau - T_1)d\tau \\ &\quad = \int_{t_0 - T_1(t)}^{t - T_1(t)} u_1^T(s)u_1(s)ds \\ &\quad \leq \int_{t_0}^{t} u_1^T(s)u_1(s)ds \end{aligned}$$

A similar derivation can be done to show that,

$$\int_{t_0}^{t} f_2^2 v_2^T(\tau - T_2(\tau))v_2(\tau - T_2(\tau))d\tau \leq \int_{t_0}^{t} v_2^T(s)v_2(s)ds$$

Applying these two inequalities to equation (22) shows that V_N is always nonnegative despite time-varying delay. This leads to the result that the energy on the G_1 side of the network bounds the energy on the G_2 side, as in the constant time-delay case,

$$\int_{t_0}^{T} y_1^T y_{2d} dt \geq \int_{t_0}^{T} y_2^T y_{1d} dt. \tag{23}$$

From this inequality, a stability result like Theorem 1 can be shown even when there are time-varying delays in the system.

III. Passive Switched Systems

The concept of passivity is applicable to a wide range of systems. Recently there have been several generalizations of passivity to switched systems [10], [13], [14]. These approaches can be summarized as requiring two fundamental conditions. The primary condition is that each subsystem is passive when it is active. The second condition varies between these works. In each, it is a condition that is sufficient to ensure that energy added due to switching is finite. If this more general condition was able to be guaranteed directly, then showing that each active subsystem is passive is sufficient to show the system is a passive switched system. However, this condition typically depends on the switching signal chosen, and in general it can't be shown for arbitrary switching. The notion of passivity for switched systems adopted in this paper is from [10].

In the present paper, passivity is applied to switched systems of the form,

$$\begin{aligned} \dot{x} &= f_\sigma(x, u) \\ y &= h_\sigma(x), \end{aligned} \tag{24}$$

where it is assumed that $f(0,0) = 0$ and $h(0) = 0$ for all subsystems. The switching signal $\sigma(t)$ indicates the current active subsystem out of the set $\Sigma = \{1, ..., m\}$, i.e. $\sigma : \mathbb{R}^+ \to \Sigma$. A single switching instant is denoted t_{i_k}, which is the k^{th} time that the i^{th} subsystem becomes active. This system becomes inactive at time t_{i_k+1} and becomes active again at time $t_{i_{(k+1)}}$. The values of i are a subset of $\mathbb{N}$ from 1 to m, and k take on values in $\mathbb{N}$ that is allowed to be infinite. To avoid Zeno behavior, it is assumed that on any finite time interval, t_0

to arbitrary time T, the system switches a finite number of times K where K typically depends on the time T chosen. To avoid trivial asymptotic analysis, it is assumed that the system switches an infinite number of times on the infinite time horizon. Passivity for switched systems is defined as follows.

Definition 3. *[10] A switched system (24) is passive if there exist positive definite storage functions $V_i(x)$ and cross supply rates $\omega_j^i(u, y, x, t)$ such that the following conditions hold.*

1) *Each subsystem i is passive while active, i.e. for $\epsilon \geq 0$, $t_{i_k} \leq t_1 \leq t_2 \leq t_{i_k+1}$ and $\forall i, k$,*

$$\int_{t_1}^{t_2} (u^T y - \epsilon_i y^T y)dt \geq V_i(x(t_2)) - V_i(x(t_1)). \quad (25)$$

2) *Each subsystem j is dissipative when it is inactive, i.e. $\forall j \neq i$, and for $t_{i_k} \leq t_1 \leq t_2 \leq t_{i_k+1}$,*

$$\int_{t_1}^{t_2} \omega_j^i(u, y, x, t)dt \geq V_j(x(t_2)) - V_j(x(t_1)). \quad (26)$$

3) *For all i and j there exist absolutely integrable functions $\phi_j^i(t)$ and some input $u^*(t)$ such that, $\forall t \geq t_0$,*

$$\omega_j^i(u^*, y, x, t) \leq \phi_j^i(t), \forall j \neq i. \quad (27)$$

A system is considered an output strictly passive (OSP) switched system if it is passive with all $\epsilon_i > 0$.

In the previous definition, for each subsystem i there is a single $V_i(x)$. However, it is often convenient to index the storage functions, as the time indices are indexed, by V_{i_k}. This notation denotes the storage function for the i^{th} subsystem over the k^{th} time it is active. Of course the storage function doesn't change for the same subsystem over different active time intervals, i.e. $V_{i_{k_1}} = V_{i_{k_2}}$ for all k_1, k_2. Although the notation V_{i_k} seems to imply that there are an infinite number of storage functions, there are actually only m unique storage functions.

Passive switched systems are Lyapunov stable. Asymptotic stability can be shown when negative output feedback is applied or when the system is an output strictly passive switched system.

Theorem 2. *[10] Consider a switched system that is output strictly passive. If all of the subsystems are asymptotically zero-state detectable, then the switched system is asymptotically stable.*

By itself, this result is only an indirect method of showing asymptotic stability. There are more direct methods of showing asymptotic stability in the literature (for example, see [15] and [16] and the references therein). However, when using Theorem 2 in conjunction with Lemma 1, open-loop conditions for asymptotic stability of the feedback interconnection of two switched systems are derived.

Lemma 1. *The negative feedback interconnection of two output strictly passive switched systems is again an output strictly passive switched system.*

These two results can be applied to the feedback interconnection of two switched systems. The switched systems must be OSP and have all subsystems be asymptotically zero-state detectable. When each of these switched systems meets the two open-loop conditions, the resulting interconnected system is OSP and asymptotically stable. These open-loop conditions for closed loop stability can be applied to networked control systems in the following section.

IV. Main Results

The main results of this paper are presented incrementally in the following lemmas and in Theorem 3. Lemma 2 shows how the network structure including the wave variable transformation preserves the OSP nature of the active subsystems. Lemma 3 shows how the definition of passive switched systems implies that the energy accumulated due to switching is a finite quantity for arbitrary switching. Lemma 4 expands upon Lemma 3 to show that the $\mathcal{L}_2$ norm of the output is also finite. Finally, Theorem 3 shows that two OSP switched systems in this network structure produce a compensated system that is asymptotically stable.

The first lemma of this section, Lemma 2, shows how the wave variable transform and the network interconnections preserve the output strictly passive nature of the active subsystems. For this result, it should be noted that the set of switching instants of the overall system is the union of the sets of switching instants of the two systems in the interconnection. This means that, if the systems G_1 and G_2 have m_1 and m_2 subsystems, the total number of subsystems in the interconnection can be up to $m = m_1 \cdot m_2$. Each subsystem has storage functions V_i^1 for G_1 and V_i^2 for G_2. Note that the loop signals can be stacked to make the vectors $e = [e_1^T e_2^T]^T$, $r = [r_1^T r_2^T]^T$, and $y = [y_1^T y_2^T]^T$.

Lemma 2. *Consider the architecture in Fig. 1 with measurable time-varying delays and the modified wave variable transformation (18-19). If each system G_1 and G_2 is an OSP switched system then each active subsystem of the switched system $r \to y$ is OSP.*

Proof. Since the mapping $e \to y$ is OSP, V_i^1 and V_i^2 exist for G_1 and G_2, respectively, that satisfy

$$\int_{t_1}^{t_2} e_1^T y_1 dt \geq \int_{t_1}^{t_2} \epsilon_i^1 y_1^T y_1 dt + V_i^1(x_1(t_2)) - V_i^1(x_1(t_1)) \quad (28)$$

$$\int_{t_1}^{t_2} e_2^T y_2 dt \geq \int_{t_1}^{t_2} \epsilon_i^2 y_2^T y_2 dt + V_i^2(x_2(t_2)) - V_i^2(x_2(t_1)), \quad (29)$$

for $t_{i_k} \leq t_1 \leq t_2 \leq t_{i_k+1}$, $\forall i, k$. Using the wave variable transformation (18-19,23) and the signal relations in the loop (4-5), the following derivation holds.

$$\int_{t_0}^{T} y_1^T y_{2d} dt \geq \int_{t_0}^{T} y_2^T y_{1d} dt$$

$$\int_{t_0}^{T} y_1^T (r_1 - e_1) dt \geq \int_{t_0}^{T} y_2^T (e_2 - r_2) dt$$

$$\int_{t_0}^{T} (y_1^T r_1 + y_2^T r_2) dt \geq \int_{t_0}^{T} (y_1^T e_1 + y_2^T e_2) dt.$$

Define a new energy storage function $V_i(x) = V_i^1(x_1) + V_i^2(x_2)$. Applying (28) and (29) to the above inequality gives the following result.

$$\int_{t_0}^{T} y^T r dt \geq \int_{t_0}^{T} y^T e dt \geq \epsilon_i \int_{t_0}^{T} y^T y dt + V_i(x(T)) - V_i(x(t_0)).$$

where $\epsilon_i = min\{\epsilon_i^1, \epsilon_i^2\}$. This shows that each active subsystem i of the mapping $r \to y$ is OSP with storage function V_i. $\square$

This lemma verified that the network structure using the wave variable transformation preserves the OSP behavior of each active subsystem.

As explained before, the differing definitions of passivity for switched systems have two main conditions. The first is that the active subsystems are passive. The second is a condition that is sufficient to show that the energy added due to switching is finite. The following lemma will demonstrate how a switched system being passive guarantees that the energy added to the system due to switching is bounded.

Lemma 3. *If a system $e \to y$ is a passive switched system, then the energy added at switching instants is finite for arbitrary switching.*

Proof. The first line of the following derivation is the energy added at switching instants t_{i_k} from initial time t_0 to arbitrary time T for a particular subsystem i. It can be assumed that K switches occur on this interval where K depends on T. Denote the number of times that subsystem i is active on this interval by K_i.

$$\begin{aligned}
&\sum_{i_k=1}^{K-1} [V_{i_k}(x(t_{i_k})) - V_{i_k-1}(x(t_{i_k}))] \\
&= \sum_{i=1}^{m} \sum_{k=1}^{K_i} \left[V_{i_{k+1}}(x(t_{i_{k+1}})) - V_{i_k}(x(t_{i_k+1}))\right] + \\
&\qquad \sum_{i=1}^{m} \left[V_{i_1}(x(t_{i_1})) - V_{i_{K_i}}(x(t_{i_{K_i}}))\right] \\
&\leq \sum_{i=1}^{m} \sum_{k=1}^{K_i} \left[V_{i_{k+1}}(x(t_{i_{k+1}})) - V_{i_k}(x(t_{i_k+1}))\right] + \sum_{i=1}^{m} V_{i_1}(x(t_{i_1}))
\end{aligned}$$

By the definition of passive switched systems, there exist absolutely integrable functions ϕ_j^i to bound the energy accumulated by the j subsystem while the i subsystem is active. For a particular switching sequence, a set of piecewise continuous functions can be defined to indicate the function ϕ_j^i that is valid at each time for the j^{th} inactive subsystem,

$$\phi_j(t) = \begin{cases} \phi_j^i(t) & \forall i \neq j \\ 0 & i = j \end{cases}$$

Since each ϕ_j^i is absolutely integrable, then each ϕ_j is also absolutely integrable. The energy accumulated by each subsystem i can be bounded.

$$\sum_{k=1}^{K_i} \left[V_{i_{k+1}}(x(t_{i_{k+1}})) - V_{i_k}(x(t_{i_k+1}))\right] \leq \sum_{k=1}^{K_i} \int_{t_{i_k+1}}^{t_{i_{k+1}}} \phi_i(t) dt$$

This leads to a bound on the energy added due to switching.

$$\begin{aligned}
&\sum_{i_k=1}^{K-1} [V_{i_k}(x(t_{i_k})) - V_{i_k-1}(x(t_{i_k}))] \\
&\qquad \leq \sum_{i=1}^{m} \left[\int_{t_0}^{\infty} \phi_i(t) dt + V_{i_1}(x(t_{i_1}))\right] < \infty
\end{aligned}$$

Each of the terms in this finite sum is a finite quantity so the energy is bounded. This upper bound is independent of the choice of T. Taking the limit as $T \to \infty$ shows that the energy is bounded for all time. $\square$

This result shows that the energy added due to switching is finite for arbitrary switching sequence. This meets the generalized second condition required of a passive switched system discussed earlier. This result will be used in the following lemma to show that the $\mathcal{L}_2$ norm of the system output y is bounded.

Lemma 4. *If a system $e \to y$ is an OSP switched system and r is defined as in Fig. 1, then the $\mathcal{L}_2$ norm of the output is finite for $r = 0$.*

Proof. The $\mathcal{L}_2$ norm of y is taken from initial time t_0 to arbitrary time T. As assumed earlier, over this time interval there are K switches and K_i switches to the i^{th} subsystem. In the following derivation, note that $t_0 \leq t_1 \leq ... \leq t_k \leq T$. Lemma 2 is invoked to upper bound $e^T y$ by $r^T y$ and then $r(t) = 0$ is applied.

$$\begin{aligned}
\int_{t_0}^{T} y^T y dt &= \sum_{i_k=1}^{K} \int_{t_{i_k-1}}^{t_{i_k}} y^T y dt + \int_{t_K}^{T} y^T y dt \\
&\leq \frac{1}{\epsilon} \sum_{i_k=0}^{K} \left[\int_{t_{i_k}}^{t_{i_k+1}} e^T y dt + V_{i_k}(x(t_{i_k})) - V_{i_k}(x(t_{i_k+1}))\right] \\
&\leq \frac{1}{\epsilon} \sum_{i_k=0}^{K} \left[\int_{t_{i_k}}^{t_{i_k+1}} r^T y dt + V_{i_k}(x(t_{i_k})) - V_{i_k}(x(t_{i_k+1}))\right] \\
&\leq \frac{1}{\epsilon} \sum_{i_k=0}^{K-1} [V_{i_k+1}(x(t_{i_k+1})) - V_{i_k}(x(t_{i_k+1}))] + \\
&\qquad \frac{1}{\epsilon} \sum_{i=1}^{m} \left[V_{i_1}(x(t_{i_1})) - V_{i_{K_i}}(x(t_{i_{K_i}}))\right] \\
&\leq \frac{1}{\epsilon} \sum_{i_k=0}^{K-1} [V_{i_k+1}(x(t_{i_k+1})) - V_{i_k}(x(t_{i_k+1}))] + \\
&\qquad \frac{1}{\epsilon} \sum_{i=1}^{m} V_{i_1}(x(t_{i_1}))
\end{aligned}$$

In the last line of the derivation above, there are two summations. The second summation is the sum of the initially stored energy across all subsystems. Since initially stored energy is finite, this sum is finite. The first summation can be shown to be finite by applying Lemma 3. Again, this bound is independent of the time T chosen earlier. As we take the limit as $T \to \infty$, the bound still holds. This shows that the $\mathcal{L}_2$ norm of the output is finite and bounded above by the sum of the initially stored energy and the energy added due to the switching sequence. □

Remark. Lemma 4 assumes $r(t) = 0$ to show asymptotic stability. However, this result is also valid for $r(t)$ such that $r^T y < \epsilon y^T y$.

The main result of the paper will now be presented. It assumes that each system in feedback is an output strictly passive switched system. It employs the wave variable transformation to guarantee stability despite time-varying delays. The proof is based on the lemmas presented.

Theorem 3. *Consider two systems, G_1 and G_2, each an OSP switched system with all subsystems asymptotically zero-state detectable. These two systems are interconnected over a network with measurable time-varying delays using the modified wave variable transformation as in Fig. 1. Then this system is asymptotically stable.*

Proof. By applying Lemma 4, the quantity

$$\int_{t_0}^{\infty} y^T y dt$$

is finite. It will be shown by contradiction that $y(t)$ approaches zero asymptotically. Assume that it doesn't. Since $y(t)$ is continuous for each subsystem, there is at least one subsystem where there exists a $\delta > 0$ and an infinite sequence of intervals such that $y(t) \geq \delta$. However, this implies that the $\mathcal{L}_2$ norm of $y(t)$ is not finite, which contradicts Lemma 4. This implies $y(t)$ approaches zero asymptotically.

With the asymptotically zero-state detectable assumption, the fact that $y(t)$ approaches zero asymptotically implies that the state $x(t)$ also approaches zero asymptotically. Therefore, the overall interconnected system is asymptotically stable. □

The theorem shows how the proposed architecture (Fig. 1) can be used to guarantee stability for an interconnection of two output strictly passive switched systems. This approach requires that the wave variable transformation can be added. In a given network, it may not be possible to change the network interface. The addition would require additional computation at each side of the network which may not be available.

This theorem is more applicable as a synthesis tool. This approach assumes that a given plant is an output strictly passive (switched or non-switched) system. The controller must be designed to be an output strictly passive switched system with asymptotically zero-state detectable subsystems. It is allowed to be switched or non-switched as long as it meets the definition of an output strictly passive switched system given in this paper. The resulting interconnection is an asymptotically stable system despite time-varying delays in the network.

This design framework is very general. It can be applied to nonlinear switched systems that are connected over a realistic network with time-varying delay. The main limitation to this setup is that the systems must be output strictly passive. Although many realistic systems are passive, systems with unstable or non-minimum phase dynamics can't be controlled using this framework. This framework would be more applicable if it could tolerate a larger class of systems that aren't necessarily passive.

V. Conclusion

This paper presented a stability result for networked passive switched systems with time-varying delay. When the systems in the loop are output strictly passive switched systems with asymptotically zero-state detectable subsystems, the control loop is shown to be stable despite time-varying delays. The development is based on a previous framework for control design of networked passive systems with delay. The extension allows for the passive network approach to apply to a much larger class of systems. The nonlinear switched system model is general and allowing for time-varying delay is a realistic network model. This approach decouples the design of the network interface from the design of the controller. With this decoupling, the approach is an intuitive approach to stabilize a large class of networked switched systems.

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