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"Internal Models over Rings"

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INTRODUCTION

The role of internal models in the regulation of linear, time invariant, lumped continuous time systems is well understood. Recent work has shown the applicability of internal models to systems described over rings in the solution of the robust regulation and robust asymptotic tracking problems [1,2]. We show, using a description of systems over rings, that internal models are necessary in regulation even when robust regulation is not required. Furthermore, the existence of internal models in the regulation of plants, where the measured and controlled variables are not necessarily the same, is examined in this algebraic setting. To this effect, a general regulation problem is defined including a stability condition over a desirable region of the complex plane; we call this problem RPIS over $\mathbb{R}_g(s)$. A complete solution of RPIS over $\mathbb{R}_g(s)$ is given; two sets of solvability conditions are derived, a parameterization of controllers is presented, and the structure of the controller that solves RPIS over $\mathbb{R}_g(s)$ is completely characterized. The results presented here are related to "internal descriptions" of the plant and controller; thus, maintaining insight into the problem. The RPIS over $\mathbb{R}_g(s)$ treatment presented here is an extension of the work presented in [3,4], and gives simpler conditions than other recent ones in [5] and [6]. Furthermore, our approach, which characterizes the

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controller's structure, provides a transparent and direct treatment of internal models.

Factorizations of transfer function matrices over a desirable ring, $\mathbb{R}_g(s)$, are used to represent the systems, that is, a given transfer function matrix is modeled as the *ratio* of two rational matrices with entries in $\mathbb{R}_g(s)$. Let $\mathbb{R}_g(s)$ be a nonempty subset of $\mathbb{R}_p(s)$, the ring of proper rational functions with real coefficients, consisting of the proper rational functions that have all their poles in S_g. S_g corresponds to the *good* region of the complex plane, so that S_g is symmetric with respect to the real axis and contains at least one real point. For a description of the properties of $\mathbb{R}_g(s)$ see [7-9]. Let $M(\mathbb{R}_g(s))$denote the set of all matrices with entries in $\mathbb{R}_g(s)$, regardless of dimensions. The background to develop the theory can be found in [1,2,10,11]. We will develop the theory in the context of linear time-invariant continuous and discrete systems, but it can be easily extended to consider linear distributed continuous and discrete systems (for the appropriate algebraic tools see [1-Chapter 8, 2,10,11]).

INTERNAL MODELS

The study of internal models in multivariable systems started in the early 1970's (for example, see [12-15]). In these papers, the researchers investigated the necessary controller structure required to achieve robust regulation with internal stability. The main result is known as the Internal Model Principle (IMP). The IMP states that the robust regulation problem with internal stability is solvable if and only if feedback of the controlled variables is used and the controller includes a replication of the exogenous

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system dynamics in its denominator. The IMP is implicit in the results presented in [16].

In 1977, a characterization of internal models in the frequency domain appeared in [17,18]. In particular, Bengtsson, in [17], gave a definition of internal models without the robustness requirement. In this case the internal model is a property of the loop gain, that is, the transfer function matrix of the cascade connection of the plant and controller. In this way, the regulation problem is solved utilizing any available structure in the plant.

The basic idea of internal models in regulation can be expressed as follows: *"for regulation a controller must create in the closed loop an appropriate model of the dynamic structure of the exogenous system."* When robustness is required, the poles of T_{wd}, the exosystem's transfer function matrix, must be present as poles of the controller's transfer function matrix with appropriate redundancy. When robustness is not required, the poles of T_{wd} appear as poles of the loop gain, at least in the case when the controlled and measured variables of the plant are the same.

The internal model part of our work builds on the results presented in [17]. We present internal model definitions in our algebraic setting for the regulation problem with no robustness requirement, and show that a necessary and sufficient condition for regulation, when the controlled and measured variables of the plant are the same, is that the cascade connection of the plant and controller contain an internal model of the exogenous system.

The following definition can be considered to be an extension of the internal model definition given in [17].

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Definition 1. Let R(s), V(s) be arbitrary proper rational matrices with the same number of rows. Let $R = \tilde{Q}_r'^{-1}\tilde{P}_r'$ and $V = \tilde{Q}_v'^{-1}\tilde{P}_v'$ be left coprime (l.c.) $\mathbb{R}_g$-factorizations, where $\tilde{Q}_r'$, $\tilde{P}_r'$, $\tilde{Q}_v'$, $\tilde{P}_v' \in M(\mathbb{R}_g(s))$, and $\tilde{Q}_r'$ and $\tilde{Q}_v'$ are square, nonsingular and biproper. Then R(s) contains an internal model of V(s) if $\tilde{Q}_r'\tilde{Q}_v'^{-1} \in M(\mathbb{R}_g(s))$.

From the definition we see that R(s) contains an internal model of V(s) if and only if $\tilde{Q}_r'=\tilde{D}_r'\tilde{Q}_v'$ where $\tilde{D}_r' \in M(\mathbb{R}_g(s))$, that is, R(s) contains a copy of the *bad* poles (in $\Omega = \mathbb{C}\backslash S_g$) of V(s) with appropriate structure, in the form of a right divisor of its denominator matrix.

A second definition of internal models when the transfer function matrices have the same number of columns is given below.

Definition 2. Let R(s) and V(s) be arbitrary proper rational matrices with the same number of columns. Let $R = P_r'Q_r'^{-1}$ and $V = P_v'Q_v'^{-1}$ be right coprime (r.c.) $\mathbb{R}_g$-factorizations. Then R(s) contains an internal model of V(s) if and only if $Q_v'^{-1}Q_r' \in M(\mathbb{R}_g(s))$, that is, Q_v' is a left divisor of Q_r'.

As an illustration of the applicability of these definitions consider the system $\Sigma(P,C)$ in Figure 1,

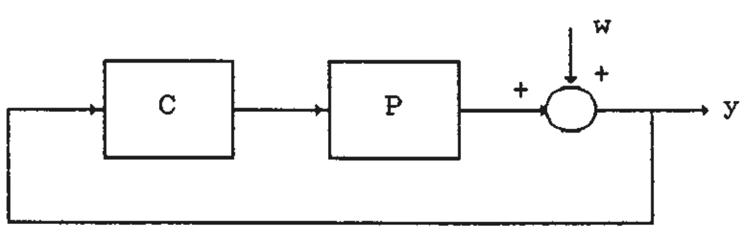

Figure 1. Basic system $\Sigma(P,C)$ configuration.

where P and C are the transfer function matrices of the controllable and observable plant and controller, respectively; the vector y is the vector of output variables and the vector w contains the unmeasurable disturbances at

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the output of the plant. It is assumed that w can be modeled as the output of a causal, linear, time-invariant finite dimensional system described by

$$w = T_{wd}d, \tag{1}$$

where d is a bounded vector, and T_{wd} is antistable, that is, all the poles of T_{wd} are in Ω. One interpretation for d is as the vector of initial conditions of the exogenous system. Let $T_{wd}=\tilde{D}_w'^{-1}\tilde{N}_w'$ and $PC=\tilde{D}_{pc}'^{-1}\tilde{N}_{pc}'$ be l.c. $\mathbb{R}_g$-factorizations. Assume that the compensated system $\Sigma(P,C)$ is well defined ($|I+PC|\neq 0$), all its input-output maps are proper, and that it is $\mathbb{R}_g$-stable, that is, all its eigenvalues are in S_g. It can be shown that $\Sigma(P,C)$ is $\mathbb{R}_g$-stable if and only if $(\tilde{D}_{pc}'+\tilde{N}_{pc}')$ is unimodular over $\mathbb{R}_g(s)$. The regulation requirement is satisfied if T_{cd}, the transfer function matrix from d to y_c, is stable. In our setting, regulation over $\mathbb{R}_g(s)$ requires $T_{cd} \in M(\mathbb{R}_g(s))$. Now, regulation over $\mathbb{R}_g(s)$ is satisfied if and only if ($\Leftrightarrow$)

$$T_{cd} = (I + PC)^{-1}T_{wd} \in M(\mathbb{R}_g(s)) \tag{2}$$

$$\Leftrightarrow \quad T_{cd} = (\tilde{D}_{pc}'+\tilde{N}_{pc}')^{-1}\tilde{D}_{pc}'T_{wd} \in M(\mathbb{R}_g(s))$$

$$\Leftrightarrow \quad T_{cd} = (\tilde{D}_{pc}'+\tilde{N}_{pc}')^{-1}\tilde{D}_{pc}'\tilde{D}_w'^{-1}\tilde{N}_w' \in M(\mathbb{R}_g(s))$$

$$\Leftrightarrow \quad T_{cd} = (\tilde{D}_{pc}'+\tilde{N}_{pc}')^{-1}\tilde{D}_{pc}'\tilde{D}_w'^{-1} \in M(\mathbb{R}_g(s))$$

$$\Leftrightarrow \quad T_{cd} = \tilde{D}_{pc}'\tilde{D}_w'^{-1} \in M(\mathbb{R}_g(s))$$

$$\Leftrightarrow \quad \tilde{D}_{pc}' = \tilde{D}'\tilde{D}_w' \qquad \tilde{D}' \in M(\mathbb{R}_g(s)) \tag{3}$$

The last statement shows that PC contains an internal model of T_{wd}.

RPIS and Internal Models in a General System

Problem Formulation

Consider the system $\Sigma(S_p,S_c)$ in Figure 2,

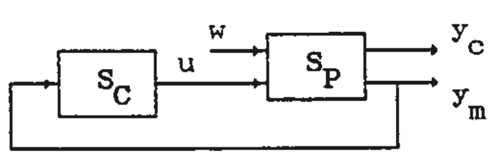

Figure 2. The compensated system $\Sigma(S_p,S_c)$.

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where S_P and S_C denote the plant and controller, respectively. For a recent overview of the regulation problem with internal stability for $\Sigma(S_P,S_C)$ see [19]. Assume that S_P and S_C are controllable and observable. Let an input-output description of the plant be

$$\begin{bmatrix} y_m \\ y_c \end{bmatrix} = P \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}, \tag{4}$$

where P, $P_{ij} \in M(\mathbb{R}_g(s))$; the vectors y_c, y_m contain the variables of the plant to be controlled and the ones that are measured, respectively; the vector w contains all the variables that affect the plant, but are not manipulated by the controller (for example, nonmeasurable disturbances and initial conditions); and u is the vector of control inputs. This general plant model is used because it unifies the study of plants where the controlled and measured variables are not necessarily the same ($y_c \neq y_m$), and where an exogenous signal w is present. Let the control u be given by

$$u = -Cy_m. \tag{5}$$

We also assume that the controlled system $\Sigma(S_P,S_C)$ is well-defined, that is, $|I+P_{11}C| = |I+CP_{11}| \neq 0$, and that all possible input-output maps are proper. As in the first part of the paper it is assumed that w is the output of a linear system with input-output relation given by

$$w = T_{wd}d, \tag{6}$$

where d is a bounded vector.

We call RPIS over $\mathbb{R}_g(s)$ the problem of finding a linear compensator S_C that makes

(1) $T_{cd} \in M(\mathbb{R}_g(s))$, and

(2) $\Sigma(S_P,S_C)$ $\mathbb{R}_g$-stable.

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<u>Solvability Conditions</u>

A complete treatment of $\mathbb{R}_g$-stability for the compensated system $\Sigma(S_P,S_C)$ is presented in [7-9]. Some of the results in [7-9] are presented here for clarity. A system is $\mathbb{R}_g$-*stable* if all its eigenvalues are in S_g. Because of the general plant model it is necessary to characterize the class of plants that can be stabilized over S_g. S_P is stabilizable over $\mathbb{R}_g(s)$ if and only if S_P is *stabilizable over* $\mathbb{R}_g(s)$ from u and *detectable over* $\mathbb{R}_g(s)$ from y_m (for example, see [7-9]). With the following definitions we will be able to state a useful set of conditions to determine whether S_P is stabilizable over $\mathbb{R}_g(s)$ or not. Let

$$P_{11} = N'_1 D'^{-1}_1 = \tilde{D}'^{-1}_1 \tilde{N}'_1 \tag{7}$$

$$C = \tilde{D}'^{-1}_c \tilde{N}'_c \tag{8}$$

be coprime factorizations over $\mathbb{R}_g(s)$, and let a doubly coprime factorization of P_{11} be

$$U'\ U'^{-1} = \begin{bmatrix} x'_1 & x'_2 \\ -\tilde{N}'_1 & \tilde{D}'_1 \end{bmatrix} \begin{bmatrix} D'_1 & -\tilde{x}'_2 \\ N'_1 & \tilde{x}'_1 \end{bmatrix}, \tag{9}$$

where $x'_1,\ x'_2,\ \tilde{x}'_1,\ \tilde{x}'_2 \in M(\mathbb{R}_g(s))$. Then S_P is stabilizable, usually stated as S_P is $\mathbb{R}_g$-admissible, if and only if

$$P_{21}D'_1,\ \tilde{D}'_1 P_{12},\ P_{22}-P_{21}D'_1 x'_2 P_{12} \in M(\mathbb{R}_g(s)). \tag{10}$$

It will be convenient to denote these expressions as

$$P'_1 := P_{21}D'_1,\ P'_2 := \tilde{D}'_1 P_{12},\ P'_3 := P_{22} - P_{21}D'_1 x'_2 P_{12}. \tag{11}$$

<u>Remark</u>: If S_P is $\mathbb{R}_g$-admissible then P_{11} contains an internal model of P_{12} and P_{21}. Furthermore, it is shown in [9] that $P'_3 \in M(\mathbb{R}_g(s))$ implies that P_{12} and P_{21} contain an internal model of P_{22}. Similar conditions are presented in [20], where the term "containment" is used.

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If the plant is $\mathbb{R}_g$-admissible, the compensated system is well defined ($|I+P_{11}C|=|I+CP_{11}|\neq 0$), and every input-output map is proper, then $\Sigma(S_P,S_C)$ is $\mathbb{R}_g$-stable if and only if

$$\tilde{D}_c'D_1' + \tilde{N}_c'N_1' \text{ is unimodular over } \mathbb{R}_g(s). \tag{12}$$

A characterization of all $\mathbb{R}_g$-stabilizing C is given by

$$C = (x_1'-K'\tilde{N}_1')^{-1}(x_2'+K'\tilde{D}_1'), \tag{13}$$

where $K' \in M(\mathbb{R}_g(s))$ is such that $|x_1'-K'\tilde{N}_1'|\neq 0$ and $(x_1'-K'\tilde{N}_1')$ is biproper. This parameterization of C_y can be used to combine the regulation and $\mathbb{R}_g$-stability conditions into one. The characterization in (13) can be used to simplify T_{cw}, the transfer function matrix from w to y_c. Note that

$$T_{cw} = P_{22}-P_{21}C(I + P_{11}C\ )^{-1}P_{12} \tag{14}$$

and substituting (11) and (13) into (14) gives

$$T_{cw} = P_3' - P_1'K'P_2'. \tag{15}$$

The regulation over $\mathbb{R}_g(s)$ condition is satisfied if and only if $T_{cd}=T_{cw}T_{wd} \in M(\mathbb{R}_g(s))$, which can be written using (15) as

$$T_{cd} = (P_3' - P_1'K'P_2')\tilde{D}_w'^{-1}\tilde{N}_w' \in M(\mathbb{R}_g(s)), \tag{16}$$

where $T_{wd}=\tilde{D}_w'^{-1}\tilde{N}_w'$ is a l.c. factorization over $\mathbb{R}_g(s)$. So RPIS over $\mathbb{R}_g(s)$ is solvable if and only if there exists $K' \in M(\mathbb{R}_g(s))$ such that (16) is satisfied. Let

$$P_2'\tilde{D}_w'^{-1} = N_{2w}'D_{2w}'^{-1} = \tilde{D}_{2w}'^{-1}\tilde{N}_{2w}' \tag{17}$$

$$P_3'\tilde{D}_w'^{-1} = N_{3w}'D_{3w}'^{-1}, \tag{18}$$

where (N_{2w}',D_{2w}'), $(\tilde{D}_{2w}',\tilde{N}_{2w}')$, (N_{3w}',D_{3w}') are coprime factorizations. Then there exist x_{1w}', $x_{2w}' \in M(\mathbb{R}_g(s))$, satisfying

$$x_{1w}'D_{2w}' + x_{2w}'N_{2w}' = I. \tag{19}$$

The solvability conditions are given in the following theorem.

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Theorem 1. The RPIS over $\mathbb{R}_g(s)$ for the system $\Sigma(S_P, S_C)$ is solvable if and only if either (i) or (ii) is satisfied.

(i) There exist K', $V' \in M(\mathbb{R}_g(s))$ so that

$$V'\tilde{D}'_w + P'_1K'P'_2 = P'_3 \tag{20}$$

(ii) (a) $D'_{1w} \in M(\mathbb{R}_g(s))$, where $D'_{1w}=D'^{-1}_{3w}D'_{2w}$ and (21)

(b) there exist K', $R' \in M(\mathbb{R}_g(s))$ so that

$$P'_1K' - R'\tilde{D}'_{2w} = N'_{3w}D'_{1w}x'_{2w}. \tag{22}$$

The theorem is proven in [9]. A similar condition to the one in (20) has appeared in [1], for the plant considered in Figure 1. The conditions in (ii) demonstrate once again that the solvability condition of the regulation problem with internal stability depends on a *skew-prime* equation.

An important special case is when $y_c=y_m$ and there are exogenous signals acting on the output of the plant as shown in Figure 1. In this case, the regulation condition is satisfied when

$$T_{cd} = (I + P_{11}C_y)^{-1}P_{12}T_{wd} \in M(\mathbb{R}_g(s)), \tag{23}$$

where $P_{11}=P$. In order to combine both the regulation and $\mathbb{R}_g$-stability requirements, substitute the characterization of $\mathbb{R}_g$-stabilizing controllers C_y in (13) in T_{cd}, giving

$$T_{cd} = (\tilde{x}'_1 - N'_1K')\tilde{D}'_1P_{12}T_{wd} \tag{24}$$

So RPIS over $\mathbb{R}_g(s)$ is solvable if and only if there exists $K' \in M(\mathbb{R}_g(s))$ such that (24) is satisfied. The solvability conditions follow from Theorem 1; they are given in Corollary 1.1.

Corollary 1.1: RPIS over $\mathbb{R}_g(s)$ for the system depicted in Figure 1 is solvable if and only if there exist e'_1, $e'_2 \in M(\mathbb{R}_g(s))$ satisfying

$$\tilde{N}'_1e'_1 + e'_2\tilde{D}'_{2w} = I. \tag{25}$$

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Structure of the Controller

In this section we are going to characterize the structure of the controllers that solve RPIS over $\mathbb{R}_g(s)$. Using this characterization it will be possible to comment on internal models. First, write the characterization of all $\mathbb{R}_g$-stabilizing controllers in (13) as

$$[\tilde{D}_c' \quad \tilde{N}_c'] = [I \quad K']U'. \tag{26}$$

In order for (26) to characterize the set of all C that solve RPIS over $\mathbb{R}_g(s)$ K' must satisfy additional conditions. These additional conditions are obtained by characterizing the general solution to one of the solvability conditions. In particular, we consider (22). The general solution of (22) is the sum of a particular and the homogeneous solution. Let

$$K' = K_p' + K_h' \tag{27}$$

where K_p' (K_h') denotes a particular (the homogeneous) solution. Substitution of (27) in (26) gives a characterization of all C that solve RPIS over $\mathbb{R}_g(s)$, whenever it is solvable,

$$\tilde{D}_c' = \bar{x}_1' - K_h'\tilde{N}_1' \tag{28}$$

$$\tilde{N}_c' = \bar{x}_2' + K_h'\tilde{D}_1', \tag{29}$$

where $\bar{x}_1'=x_1'-K_p'\tilde{N}_1'$, $\bar{x}_2'=x_2'+K_p'\tilde{D}_1' \in M(\mathbb{R}_g(s))$. A characterization of a large class of solutions for K_h' is given by

$$K_h' = W'\tilde{D}_{2w}'. \tag{30}$$

See [9] for a discussion on the generality of solutions in (30). Substituting (30) in (29) and solving for $[I \quad W'\tilde{D}_{2w}']$ gives the set of structural conditions that must be satisfied by C

$$\tilde{D}_c'D_1' + \tilde{N}_c'N_1' = I \tag{31}$$

$$\tilde{D}_c'(-\bar{\bar{x}}_2') + \tilde{N}_c'\bar{\bar{x}}_1' = W'\tilde{D}_{2w}'. \tag{32}$$

Notice that (31) is the $\mathbb{R}_g$-stability condition and that (32) corresponds to the RPIS over $\mathbb{R}_g(s)$ requirement. A characterization of controllers that

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solve RPIS over $\mathbb{R}_g(s)$ can be obtained from (31) and (32) and is written below

$$\tilde{D}_c' = \bar{x}_1' - W'\tilde{D}_{2w}'\tilde{N}_1' \tag{33}$$

$$\tilde{N}_c' = \bar{x}_2' + W'\tilde{D}_{2w}'\tilde{D}_1'. \tag{34}$$

Notice that the set characterized by (33) and (34) is nonempty if RPIS over $\mathbb{R}_g(s)$ is solvable. Suppose that $\tilde{D}_{2w}' \in \mathrm{Ker}(W')$, then a compensator that solves RPIS over $\mathbb{R}_g(s)$ is described by

$$\tilde{D}_c' = \bar{x}_1' = x_1' - K_p'\tilde{N}_1' \tag{35}$$

$$\tilde{N}_c' = \bar{x}_2' = x_2' + K_p'\tilde{D}_1'. \tag{36}$$

The characterization of controllers given in (33) and (34) depends on the particular choice of W'. One way to characterize the part of the controller that is independent of the choice of W' is by defining

$$G_d' = \text{a g.c.r.d.}^1\ (\bar{x}_1', \tilde{D}_{2w}'\tilde{N}_1') \tag{37}$$

$$G_n' = \text{a g.c.r.d.}\ (\bar{x}_2', \tilde{D}_{2w}'\tilde{D}_1'), \tag{38}$$

then

$$\tilde{D}_c' = \bar{D}_c'G_d', \quad \tilde{N}_c' = \bar{N}_c'G_n' \tag{39}$$

$$\Rightarrow \qquad C = G_d'^{-1}\bar{C}G_n', \tag{40}$$

where $\bar{C}=\bar{D}_c'^{-1}\bar{N}_c'$. The significance of G_d' and G_n' is that they are introduced solely because of the regulation over $\mathbb{R}_g(s)$ requirement, while $\bar{C}$ must satisfy both structural conditions: (31) and (32). Note that $|G_d'||G_n'|$ divides $|\tilde{D}_{2w}'|$ and that $\bar{C}$ satisfies the $\mathbb{R}_g$-stability condition (31) if and only if

$$|G_d'|\ |G_n'| \text{ and } |\tilde{D}_{2w}'| \text{ are associates}^2. \tag{41}$$

If (41) is satisfied and if RPIS is solvable then

$$\tilde{D}_c' = \bar{D}_c'G_d', \quad \tilde{N}_c' = \bar{N}_c'G_n' \tag{42}$$

---

[1] g.c.r.d. denotes greatest common right divisor.

[2] Two elements a, b $\in \mathbb{R}_g(s)$ are associates if they differ by u, a unit in $\mathbb{R}_g(s)$, that is, a = ub.

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is a necessary and sufficient condition for regulation over $\mathbb{R}_g(s)$. In this case the controller must contain the exogenous system dynamics in G_d' and/or G_n'.

In the special case when $y_m=y_c$ as in the system depicted in Figure 1 the structure of the controller is well understood. The controller that solves the RPIS over $\mathbb{R}_g(s)$ in this case is of the form

$$C = G_d'^{-1}\bar{C}, \tag{43}$$

where G_d' satisfies (41) and $G_n'=I$ [21,22]. This controller structure has been associated to the existence of an internal model in the cascade connection PC. Note that G_d' contains the poles of T_{wd} that are not poles of the plant with similar structure.

For the general plant, $y_c \neq y_m$, the controller that solves RPIS over $\mathbb{R}_g(s)$ is given by (40). This was derived using the class of homogeneous solutions (30). Observe that there may be additional solutions to the homogeneous equation which though result in a controller of the form in (40) [19]. When (41) is satisfied, then the appropriate transmission zeros in T_{cw} are introduced via poles of C (in $G_d'^{-1}$) and zeros of C (in G_n'), in addition to some appropriate structure on the plant. But, this is not necessary. So, for the general plant internal models as defined here are not necessary [4]. In general, the role of the controller in regulation is to introduce appropriate transmission zeros in T_{cw} [23]. These transmission zeros are introduced by the imposition of structural relations ((31) and (32)) on the numerator and denominator matrices of C. In the case $y_c \neq y_m$ this does not necessarily translate into transmission zeros of T_{cw} being poles or zeros of C as discussed above.

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CONCLUSION

We have presented two definitions for internal models over rings that pertain to the regulation problem with internal stability without a robustness requirement. The applicability of the internal model definitions was illustrated with two compensated systems. When the controlled and measured variables of the plant are the same, the necessity of internal models as defined here is directly established. When the controlled and measured variables differ, the controller C does not need to contain explicit information about the dynamics of the exosystem, but the numerator and denominator matrices of C must satisfy the structural conditions in (31)-(32). Research to generalize the definitions of internal models presented here and establish the conditions under which they will be necessary for RPIS over $\mathbb{R}_g(s)$ is ongoing.

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