# Optimal Control of Switched Hybrid Systems: A Brief Survey

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**Abstract**

## I. Introduction

Switched systems are a particular class of hybrid systems consisting of several subsystems and a switching law specifying the active subsystems at each time instant. Examples of switched systems can be found in chemical engineering, automotive systems, and electrical circuit systems, to name a few.

The problem of determining optimal control laws for hybrid systems and in particular for switched systems, has been extensively investigated in recent years and many results may be found in the control and computer science literature. It has attracted researchers from various fields in science and engineering, due to the problems' significance in theory and applications. The results are both theoretical and computational. The available theoretical results usually extend the classical maximum principle or the dynamic programming approach to switched systems. The computational results take advantage of efficient nonlinear optimization techniques and high-speed computers to develop efficient numerical methods for the optimal control of switched systems.

This paper surveys the recent progress in computational methods for optimal control problems of switched systems. Such problems are difficult to solve, due to switchings of subsystem dynamics. The recent decade has seen some breakthroughs in theoretical results as well as the development of efficient computational methods, however there are no theoretical or computational results applicable to general optimal control problems for all kinds of switched systems. The existing literature results are often based on different models and differ in problem formulation and approaches. Therefore, this report is an attempt to summarize recent results that use different problem formulations and explore the underlying relations among them.

The report is organized as follows. In Section II, a brief overview of theoretical results on optimal control of hybrid systems is presented and the general optimal control problem formulation of switched systems is given. Section III reviews the existing optimal control methodologies for switched systems with continuous control input. The optimal control problems of continuous-time and discrete-time switched systems are discussed separately. Section IV focuses mainly on the results on optimal control of autonomous switched systems. Section V concludes the report.

## II. General Optimal Control Problems of Switched Dynamic Systems

### A. Optimal control of hybrid systems

The problem of determining optimal control laws for hybrid systems has been extensively investigated in the recent years and many results can be found in the control and computer science literature. For the theoretical point of view, numerous results on necessary conditions for optimality have appeared for different models of hybrid systems [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]. However, most of the results consider general problems and it is not always possible to develop tractable algorithms to numerically compute the optimal solution. Here we present some important theoretical results.

For continuous-time hybrid systems, Branicky and Mitter [2] compare several algorithms for optimal control, while Branicky *et al.* [1] discuss general necessary conditions for the existence of optimal control laws for hybrid systems by using dynamic programming. They established a general hybrid framework for the optimal control problem, proved the existence of optimal (relaxed or chattering) controls and near-optimal (precise or nonchattering) controls, and derived generalized quasi-variational inequalities (GQVI's) that the associated value function is expected to satisfy.

Necessary optimality conditions for a trajectory of a hybrid system are derived using the maximum principle by Sussmann [3] and Piccoli [4], who consider a fixed sequence of finite length. Several versions of hybrid maximum principles are proposed. A similar approach is used by Riedinger *et al.* [5], who only consider the attention to linear quadratic cost functionals but considering both autonomous and controlled switches.

Hedlund and Rantzer [6], [7] use convex dynamic programming (CDP) to approximate hybrid optimal control laws and to compute lower and upper bounds of the optimal cost, while the case of piecewise-affine systems is discussed by Rantzer and Johansson [8]. A MATLAB toolbox [13] is developed to solve hybrid optimal control problems via CDP. For determining the optimal feedback control law these techniques require the discretization of the state space in order to solve the corresponding Hamilton-Jacobi-Bellman equations.

Shaikh and Caines [9] consider a finite-time hybrid optimal control problem and give necessary optimality conditions for a fixed sequence of modes using the maximum principle. In [10] these results are extended to non-fixed sequences by using a suboptimal result based on the Hamming distance permutations of an initial given sequence. Finally, in [11], [12], the authors derive a feedback law for a finite time LQR problem by integrating the computation of the optimality zones in to the hybrid maximum principle algorithms class.

### *B. Optimal control of switched systems: problem formulation*

In order to find ways to numerically compute the optimal control in hybrid systems, many researchers have been focusing on a particular class of hybrid systems models, the switched systems. A switched system may be obtained from a hybrid system by neglecting the details of the discrete behavior and instead considering all possible switching patterns from a certain class. The discrete behavior in switched systems is "simplified" to "switching", which in general represents discontinuity in vector fields. There are many definitions of switched systems and here we adopt the definition in [14].

A switched system consists of several subsystems and a switching law. A switching takes places when a certain event signal is received. An event signal may be an external signal (generated exogenously) or an internal signal generated when an internal condition for the states, inputs and/or time evolution is satisfied. In the sequel, we call a switching triggered by an external event an *externally forced switching* (EFS) and a switching triggered by an internal event an *internally forced switching* (IFS).

**Definition 1** (General definition of switched systems)**.** [14] A switched system is a 3-tuple $\mathcal{S} = (\mathcal{D},\, \mathcal{F},\, \mathcal{L})$ where

- $\mathcal{D} = (I,\, E)$ is a directed graph indicating the discrete mode structure of the system. $I = 1, 2, \cdots, M$ is the set of indices for subsystems. $E$ is a subset of $I \times I - (i,i)|i \in I$ which contains the valid events. If an event $e = (i_1, i_2)$ takes place, the system switches from subsystem $i_1$to $i_2$. Furthermore $E = E_E \cup E_I$ ($E_E$ and $E_I$ may not be disjoint) where $E_E$ is the external event set and $E_I$ is the internal event set.

- $\mathcal{F} = \{f_i : X_i \times U_i \to \mathbb{R}^n | i \in I\}$where $f_i$ describes the vector field for the $i$-th subsystem $\dot{x} = f_i(x, u)$. Here $X_i \subseteq \mathbb{R}^n$, $U_i \subseteq \mathbb{R}^m$ are the state and control constraint sets for the $i$-th subsystem, respectively.
- $\mathcal{L} = \mathcal{L}_E \cup \mathcal{L}_I$provides logic constraints that relate the continuous state and mode switchings. Here $L_E = \{\Lambda_e | \Lambda_e \subseteq \mathbb{R}^n, \emptyset \neq \Lambda_e \subseteq X_{i_1} \cap X_{i_2}, e = (i_1, i_2) \in E_E\}$ corresponds to the external events; only when $x \in \Lambda_e$ for $e = (i_1, i_2)$, an externally forced switching (EFS) from subsystem $i_1$ to $i_2$ is possible. Also here $L_I = \{\Gamma_e | \Gamma_e \subseteq \mathbb{R}^n, \emptyset \neq \Gamma_e \subseteq X_{i_1} \cap X_{i_2}, e = (i_1, i_2) \in E_I\}$ corresponds to the internal events; when the state trajectory intersects $\Gamma_e$, $e = (i_1, i_2)$, at subsystem $i_1$, the event $e = (i_1, i_2)$ must be triggered and the system is internally forced to switch (IFS) to subsystem $i_2$.

**Definition 2.** [14] A switching sequence $\sigma$ in $[t_0, t_f]$ is a timed sequence $\sigma = ((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K))$, where $0 \leq K < \infty$, $t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f$, and $i_k \in I$ for $0 \leq k \leq K$.

*Remark* 3. Given a switched system, the overall exogenous control input is a pair $(\sigma_E, u)$. Along with the evolution of $x(t)$, an IFS sequence $\sigma_I$ will be generated implicitly. $\sigma_E$ and $\sigma_I$ then lead to the overall $\sigma$. For a switched system in Definition 1, the continuous state does not exhibit jumps at switching instants. However, we note that some methods reported here can be extended to problems with jumps.

*Remark* 4. A variety of particular models can be defined to address different aspects of the general switched systems in Definition 1. Based on the types of switching, we have

1) Switched systems with *state-dependent switchings*,
2) Switched systems with *state-independent switchings*.

Based on the types of subsystems, we have

1) *Continuous-time (discrete-time) switched systems* if subsystems are continuous (discrete) time systems,
2) *Switched linear (nonlinear) systems* if subsystems are linear (nonlinear) systems.

If the continuous control input $u$ is absent from the model, we call it an *autonomous switched system*.

Although in principle general optimal control problems can be formulated for switched systems with both EFS and IFS, results would be difficult to obtain. Typically the original problem can be divided into two important classes of problems which can be solved individually, i.e., optimal control problem with EFS only (EFS Problems), and problems for systems with IFS only (IFS) problem. Most of the literature in this report address one of these two classes of problems.

**Problem 5** (EFS problem)**.** [14] Consider a switched system $\mathcal{S}$ with EFS only. Find an admissible control pair $(\sigma_E, u)$ ($u$ is piecewise continuous) such that $x$ departs from a given initial state $x(t_0) = x_0$ at the given initial time $t_0$ and meets the terminal manifold defined by $\psi\left(x\left(t_f\right), t_f\right) = 0$ where $\psi$ is a vector function and

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))dt + \sum_{1 \leq k \leq K} \delta(x(t_k), i_{k-1}, i_k) \tag{1}$$

is minimized (here $K$ is the number of switchings in $\sigma_E$).

**Problem 6** (IFS problem)**.** [14] Consider a switched system $\mathcal{S}$ with IFS only. Find an admissible control $u(t)$ ($u$ is piecewise continuous) such that $x$ departs from a given initial state $x(t_0) = x_0$ at the given initial time $t_0$ and meets he terminal manifold defined by $\psi\left(x\left(t_f\right), t_f\right) = 0$ where $\psi$ is a vector function and

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))dt + \sum_{1 \leq k \leq K} \delta(x(t_k), i_{k-1}, i_k) \tag{2}$$

is minimized (here $K$ is the number of switchings in $\sigma_I$).

Problem 5 and Problem 6 are formulated as general Bolza problems with terminal cost $\psi$, running cost $\int_{t_0}^{t_f} L dt$, and switching cost $\delta$. The two problems are different due to the different exogenous input. In general, EFS problem is more difficult since we need to optimize both continuous control input $u$ and switching signal $\sigma_E$, which are strongly coupled in the optimal control problem. To address the role of EFS in optimal control problem, [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40] consider autonomous switched systems (i.e. without continuous control input $u$) and derive the optimal switching conditions in the form of switching times or switching surfaces. On the other hand, the difficulty in IFS problem is that we have no direct control of switchings assuming the internal switching conditions are given.

## III. Optimal Control of Switched Systems with Control Input

### A. *Optimal control of discrete-time switched systems*

One of the modeling frameworks used for discrete-time switched systems is *piecewise affine (PWA) systems*, defined by partitioning the state space into polyhedral regions, and associating with each region a different linear state-update equation

$$x\left(t+1\right) = A_i x(t) + B_i u(t) + f_i \qquad (3)$$

$$\text{if} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{X}_i \triangleq \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : H_i x + J_i u \leq K_i \right\}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_l}$, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_l}$, $\{\mathcal{X}_i\}_{i=0}^{s-1}$ is a polyhedral partition of the sets of state+input space $\mathbb{R}^{n+m}$, $n \triangleq n_c + n_l$, $m \triangleq m_c + m_l$. PWA systems can model a large number of physical processes, such as systems with static nonlinearities, and can approximate nonlinear dynamics via multiple linearizations at different operating points.

Consider the PWA system (3) subject to hard input and state constraints

$$Ex(t) + Lu(t) \leq M \qquad (4)$$

for $t \geq 0$, and denote by constrained PWA system (CPWA) the restriction of the PWA system (3) over the set of states and inputs defined by (4),

$$x\left(t+1\right) = A_i x(t) + B_i u(t) + f_i \qquad (5)$$

$$\text{if} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \tilde{\mathcal{X}}_i \triangleq \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : \tilde{H}_i x + \tilde{J}_i u \leq \tilde{K}_i \right\}$$

where $\left\{ \tilde{\mathcal{X}}_i \right\}_{i=0}^{s-1}$ is the new polyhedral partition of the sets of state+input space $\mathbb{R}^{n+m}$ by intersecting the polyhedrons $\mathcal{X}_i$ in (3) with the polyhedron described by (4).

Define the following cost function

$$J\left(U_0^{T-1}, x(0)\right) \triangleq \left\|Px(T)\right\|_p + \sum_{k=0}^{T-1} \left(\left\|Qx(k)\right\|_p + \left\|Ru(k)\right\|_p\right) \qquad (6)$$

and consider the finite-time optimal control problem (FTCOC)

$$J^*(x(0)) \triangleq \min_{\{U_0^{T-1}\}} J(U_0^{T-1}, x(0)) \qquad (7)$$

$$s.t. \left\{ \begin{array}{c} x(t+1) = A_i x(t) + B_i u(t) + f_i \\ \text{if} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \tilde{\mathcal{X}}_i \end{array} \right. \tag{8}$$

where the column vector $U_0^{T-1} \triangleq [u'(0), \ldots, u(T-1)']' \in R^{mT}$, is the optimization vector and $T$ is the time horizon. In (6), $\|Qx\|_p = x'Qx$ for $p = 2$ and $\|Qx\|_p = \|Qx\|_{1,\infty}$ for $p = 1, \infty$, where $R = R' \succ 0$, $Q = Q'$, $P = P' \succ 0$ if $p = 2$ and $Q$, $R$, $P$ non-singular if $p = \infty$or $p = 1$.

The FTCOC can be views as IFS problem since the switchings are implicitly determined by the partition of the state space. The main results on FTCOC can be found in [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53] by Bemporad and Morari *et al*. The work can be viewed as consisting of two kinds of results: theoretical and computational. For the theoretical results, it has been proved that the closed form of the state-feedback solution to finite time optimal control based on quadratic or linear norms performance criteria is a time-varying piecewise affine feedback control law. For the computational results, two computational methods are provided to numerically find the optimal solution.

One way is by describing the PWA system by a set of inequalities with integer variables as the system switches between the different dynamics. An appropriate modeling framework for such class of systems is mixed logical dynamic (MLD) framework where the switching behavior as well as the constraints of the system are modeled with inequality conditions. So consider the equivalent MLD system of the PWA system. Problem (6)-(8) can be rewritten as:

$$\min_{\{U_0^{T-1}\}} J\left(U_0^{T-1}, x(0)\right) \triangleq \|Px(T)\|_p + \sum_{k=0}^{T-1} (\|Qx(k)\|_p + \|Ru(k)\|_p) \tag{9}$$

$$subj.\ to \left\{ \begin{array}{c} x(k+1) = Ax(k) + B_u u(k) + B_\delta \delta(k) + B_z z(t), \\ E_\delta \delta(k) + E_z z(k) \leq E_u u(t) + E_x x(t) + E \end{array} \right. \tag{10}$$

The optimal control problem in (9)-(10) can be formulated as a Mixed Integer Quadratic Program (MIQP) when the squared Euclidean norm $p = 2$ is used [54], or as a Mixed Integer Linear Program (MILP), when $p = \infty$ or $p = 1$ [48]. In addition, multiparametric programming can be used to efficiently compute the explicit form of the optimal state-feedback control law. Then, for performance indices based on the $\infty$-norm or 1-norm, the optimization problem can be treated as a multi-parametric MILP (mp-MILP) [48], [52], while for performance indices based on the 2-norm, the optimization problem can be treated as a multi-parametric MIQP (mp-MIQP).

In addition to the algorithms based on the Mixed Integer Program (MIP), a more efficient way that combines a dynamic programming strategy with a multi-parametric program solver is proposed. The equivalent dynamic program is of the following form

$$J_j^*(x(j)) \triangleq \min_{u(j)} \|Qx(j)\|_p + \|Ru(j)\|_p + J_{j+1}^*(f_{PWA}(x(j), u(j))), \tag{11}$$

$$subj.\ to\ f_{PWA}(x(j), u(j)) \in \mathcal{X}^{j+1} \tag{12}$$

for $j = T-1, \ldots, 0$, with boundary conditions

$$X^T = X^f, \text{ and} \tag{13}$$

$$J_T^*(x(T)) = \|Px(T)\|_p \tag{14}$$

where

$$X^j = \left\{x \in \mathbb{R}^n | \exists u,\ f_{PWA}(x,u) \in \mathcal{X}^{j+1}\right\} \tag{15}$$

is the set of all initial states for which the problem (11)-(12) is feasible.

When the problem is a PWA system with a quadratic performance criterion, i.e. $p = 2$, the algorithm is based on a dynamic programming recursion and a multiparametric quadratic solver [47], [49]. Similarly, when the problem is a PWA system with a linear performance index, i.e. $p = 1$, or $p = \infty$, the algorithm is based on dynamic programming recursion and a multiparametric linear program solver [41], [49], [53]. Compared with the former algorithm based on MIP, the dynamic programming algorithm is more efficient and less complex due to fewer underlying inequality constraints. Also, the dynamic programming algorithm can be used to approximate infinite time horizon solutions through finite time horizon solutions. Recent work [50], [42] shows how to exploit the underlying geometric structure of the optimization problem with a linear performance index in order to significantly improve the efficiency of the off-line computations. By using algebraic geometry methods, [45], [46] study the constrained finite-time optimal control problem of discrete-time nonlinear systems.

Gorges *et al* [55] study the optimal control and scheduling problem of *discrete-time switched linear systems* by assuming the switching is EFS. The model considered is

$$x\,(k+1) = A_{j(k)}x(k) + B_{j(k)}u(k). \tag{16}$$

Switching between subsystems is described by the switching index $j(k)$ which is subject to control. Further, the cost function is in the quadratic form

$$J_N(k) = x^T(k+N)Q_0x(k+N) + \sum_{i=0}^{N-1} l(k+i) \tag{17}$$

with step cost $l(i) = l(x(i), u(i), j(i)) \geq 0$ defined by $l(i) = x^T(i)Q_{1j(i)}x(i) + u^T(i)Q_{2j(i)}u(i)$ where $Q_0$ and $Q_{1j(i)}$ are symmetric positive definite; $Q_{2j(i)}$ is symmetric positive semi-definite. $N$ is the prediction horizon. $k$ denotes the current time instant and $i$ denotes the time instant with the prediction horizon. The optimal control problem is then formulated using receding horizon control and scheduling strategy, as in Problem 7.

**Problem 7.** [55] For the switched system (16) and the current state $x(k)$ find a control sequence

$$\mathcal{U}^*(k, k+N-1) = (u^*(k), \ldots, u^*(k+N-1))$$

and a switching sequence

$$\mathcal{J}^*(k, k+N-1) = (j^*(k), \ldots, j^*(k+N-1))$$

such that the cost function (17) is minimized over the prediction horizon $N$, i.e.

$$\min_{\mathcal{U}(k,k+N-1),\mathcal{J}(k,k+N-1)} J_N(k)$$

subject to $x\,(k+1+i) = A_{j(k+i)}x(k+i) + B_{j(k+i)}u(k+i)$ with $i = 0, \ldots, N-1$.

By solving a set of difference Riccati Equations (DRE), the resulting optimal control input $u(k)$ is in piecewise linear state feedback form and the switching sequence and time are obtained through dynamic programming.

Wei Zhang [56], [57], [58], [59] considers the same discrete-time switched linear system model as in [55]. Here the optimal control problem studied is a discrete-time switched LQR problem with the cost function defined as

$$J_N(u,j) = x^T(k)Q_f x(k) + \sum_{i=0}^{N-1} \left(x^T(i)Q_{j(i)}x(i) + u^T(i)R_{j(i)}u(i)\right) \tag{18}$$

where $Q_f$ and $Q_j$ are symmetric positive semi-definite and $R_j$ is symmetric positive definite. The explicit feedback form of the optimal control input is obtained through dynamic programming, which is similar to the result in [55]. It is worth pointing out that both [56] and [55] face the problem that the size of the positive semi-definite matrix set, obtained by solving a set of DREs, will grow exponentially as the time evolves. To reduce the size of the positive semi-definite matrix set, [55] considers a sub-optimal cost function using which the optimal control problem can be approached by relaxed dynamic programming [60], [61]. The different approach in [57], [58] resorts to finding the minimum equivalent subset of the positive semi-definite matrix set by removing the redundant matrices. It should be noted that the optimality of the problem is not jeopardized in [57], [58]. [57] also shows that a similar algorithm can be extended to the case when sub-optimal performance is acceptable.

A more recent result [62], considers the optimal control problem of *discrete-time nonlinear switched systems* and the goal is to find the optimal continuous control input and switching signal. It shows that the problem can be numerically solved by dividing the original control problem into two sub-problems. The optimal continuous control input can be obtained from the first sub-problem for a given switching sequence. The second sub-problem will search for the optimal switching sequence through the discrete filled function method. The global optimal solutions (for most cases) can be found by iteratively solving the two sub-problems.

### B. Optimal control of continuous-time switched systems

The model of continuous-time switched systems is adopted from Definition 1, as

$$\dot{x}(t) = f_{i(t)}(x(t), u(t)) \tag{19}$$

where $i(t)$ is the switching signal and it is of the EFS type. The set of vector fields $f_i(x,u)$ can be nonlinear or linear functions of $x$ and $u$. The form of cost function is as in (1). Due to computability of different numerical algorithms, the form of cost function is subject to change when a different optimal control problem formulation is considered.

Xu and Antsaklis consider the optimal control problem of seeking both the optimal control switching instants and continuous control input of *continuous-time switched nonlinear systems* [63], [64], [65], [14], [66], [67], [68], [69], [70], [71], [72], [73], [74]. A computational method based on a two stage optimization method is proposed and proved to solve the EFS problem assuming a prespecified sequence of active subsystems is given. The method is based on the fact that the following equation holds.

$$J\left(\sigma_E^*, u^*\right) = \min_{ad.(\sigma_E,u)'s} J\left(\sigma_E, u\right) = \min_{\sigma\in\{\sigma_E|\exists u,(\sigma_E,u)\ is\ ad.\}} \ \min_{u\in\{u|(\sigma_E,u)\ is\ ad.\}} J\left(\sigma_E, u\right). \tag{20}$$

Here '$ad.$' stands for '*admissible*'. and $J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))dt$.

The right hand side of (20) needs twice the minimization processes. This implies that the following two stage optimization can be applied.

Stage 1. (a) Fix the total number of switchings to be $K$ and the sequence of active subsystems and let the minimum value of $J$ with respect to $u$ to be a function of the $K$ switching instants, i.e., $J_1 = J_1(t_1, \cdots t_K)$ for $K \geq 0$ ($t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f$). Find $J_1$.

Stage 1. (b) Minimize $J_1$with respect to $t_{1,} \cdots, t_K$. i.e.

$$\min_{\hat{t}} J_1(\hat{t})$$

subject to $\hat{t} \in T$, where $T \triangleq \{\hat{t} = (t_1, \cdots, t_K)^T | t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f\}$.

Stage 2. (a) Vary the sequence of active subsystems to find an optimal solution under K switchings.

Stage 2. (b) Vary the value of $K$ to find an optimal solution for Problem 5.

It can be seen that the stage 1 and stage 2 are relatively decoupled problems and therefore can be solved separately. The stage 1 optimization proposed is an interesting topic in the optimal control of switched systems, while the stage 2 optimization is more suitable to be formulated as a searching problem. Consequently the research mainly focuses on the stage 1 optimization.

A computational method to solve stage 1 optimization is proposed based on nonlinear programming [64], [67]. The conceptual computational algorithm for stage 1 optimization is as follows.

1) Set the iteration index $j = 0$. Choose an initial $\hat{t}^j$.
2) By solving an optimal control problem (i.e., stage (a)), find $J_1(\hat{t}^j)$.
3) Find $(\partial J_1/\partial \hat{t})(\hat{t}^j)$ (and $(\partial^2 J_1/\partial \hat{t}^2)(\hat{t}^j)$ if second-order method is to be used).
4) Use some feasible direction method to update $\hat{t}^j$ to be $\hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$ (here $d\hat{t}^j$ is formed by using the gradient information of $J_1$, the step-size $\alpha^j$ can be chosen using some step-size rule, e.g., Armijo's rule). Set the iteration index $j = j + 1$.
5) Repeat Steps (2), (3), (4) and (5), until a prespecified termination condition is satisfied (e.g., the norm of the projection of $(\partial J_1/\partial \hat{t})(\hat{t}^j)$ on any feasible direction is smaller than a given small number $\epsilon$).

In fact, Step 2) poses an obstacle for the usage of the algorithm because $(\partial J_1/\partial \hat{t})(\hat{t}^j)$ and $(\partial^2 J_1/\partial \hat{t}^2)(\hat{t}^j)$ are not readily available. In [64], [67], a method based on direct differentiations of the value function is proposed to approximate the values of $(\partial J_1/\partial \hat{t})(\hat{t}^j)$ and $(\partial^2 J_1/\partial \hat{t}^2)(\hat{t}^j)$. Later in [63], [68], based on the equivalent problem formulation, a method based on the solution of a two point boundary value differential algebraic equation (DAE) is then developed for deriving accurate values of $(\partial J_1/\partial \hat{t})(\hat{t}^j)$ and $(\partial^2 J_1/\partial \hat{t}^2)(\hat{t}^j)$.

It has been shown that the proposed methods can also be applied to other problems under different switched systems models. [63], [64] show the method of applying the algorithm to general switched linear quadratic problems. [65] talks about the approach to finding the optimal switching instants for switched autonomous systems. [69] extends the method to solve IFS problem. In [70], [72], [74], optimal control problems for switched systems with state jumps are studied. [71], [73] examine the approach on time optimal control integrator switched systems with state constraints.

A similar multi-stage optimization mechanism proposed by Sastry and Tomlin can be found in [75], [76]. The innovation introduced by Sastry and Tomlin is that the proposed method aims to find the optimal continuous control input as well as the full information of switching signal (switching sequence and time instants), without assuming the switching sequence is prefixed. The switched system model used is *constrained switched nonlinear systems*. By "constrained switched nonlinear systems", it means the state in (19) is constrained to a set described by

$$x(t) \in \{x \in \mathbb{R}^n | h_j(x) \leq 0, \ j = 1, ..., N\}. \tag{21}$$

for all time and the cost function is defined as (1) but without the switching cost term. Based on this model, [75] develops a bi-level hierarchical algorithm that divides the problem into two nonlinear constrained optimization problems, as presented below.

Stage 1: Give a switching sequence $\sigma$, calculate the optimal switching time sequence $s$ and the optimal control input $u$.

Stage 2: Calculate a new sequence $\tilde{\sigma}$, which is the result of the insertion of a new switching into the original sequence $\sigma$. Repeat Stage 1 using $\tilde{\sigma}$.

At the lower level, the algorithm assumes a fixed modal sequence and determined the optimal mode duration and optimal continuous input. At the higher level, the algorithm employs a single mode insertion technique to construct a new lower cost sequence. The search for all possible switching sequences, which could grow exponentially after each iteration, can be avoided by a single mode insertion technique. An improved algorithm is presented recently [76], in which new features are implemented to overcome certain shortcomings of the original algorithm.

Essentially different from the results mentioned above, Bengea and DeCarlo [77], [78] consider solving the optimal control problem for *continuous-time switched nonlinear systems* through embedding. The switched system is first embedded into a larger family of continuous systems as

$$x(t) = \sum_{i=1}^{N} v_i(t) f_i(x(t), u_i(t)) \tag{22}$$

where $v_i \in [0, 1]$ and $\sum_{i=1}^{N} v_i(t) = 1$.

Then the embedded system can be solved using conventional optimal control techniques. By adopting such problem transformation, there is no need to make any assumptions about the number of switches nor about the mode sequence at the beginning of the optimization. The theoretical results in [77] show that the set of trajectories of the switching system is dense in the set of trajectories of the embedded system. Furthermore, the results also imply that if one solves the embedded optimal control problem and obtains a solution, either the solution is the solution of the original problem, or suboptimal solutions can be constructed. Recently, [78] further explores the possible numerical nonlinear programming techniques under this framework. It shows that sequential quadratic programming (SQP) can be utilized to reduce the computational complexity introduced by mixed integer programming (MIP). The effectiveness of the proposed approach is demonstrated through several examples.

### C. *Software packages*

Software packages to compute the optimal control solutions are available based on Mixed Integer Program or multi-parametric program. The Multi-Parametric Toolbox (MPT) [79] is a free MATLAB toolbox for design, analysis and deployment of optimal controllers for constrained linear, nonlinear and hybrid systems. Efficiency of the code is guaranteed by the extensive library of algorithms from the field of computational geometry and multi-parametric optimization. YALMIP [80] features an intuitive and flexible modeling language for solving optimization problems in MATLAB. The main emphasis is on convex conic programming (linear, quadratic, second order cone and semi-definite programming), but the toolbox supports also integer programming and non-convex problems. The toolbox additionally includes modules for moment and sum of squares programming, mixed integer conic programming and global non-convex optimization. A large number of solvers (both free and commercial) are interfaced and YALMIP will automatically use the most suitable solver it can find. YALMIP can also be used together with the MPT toolbox to setup and solve multiparametric programs.

## IV. Optimal Control of Autonomous Switched Systems

### A. *Optimal control of autonomous switched linear systems*

We present the model of s*witched affine systems with state jumps* as a general mathematical representation of autonomous switched linear systems, as in (23) and (24).

$$\dot{x}(t) = A_{i(t)}x(t) + f_{i(t)},\ i(t) \in \mathcal{S}, \tag{23}$$

$$x(t^+) = J_{i,j}x(t^-)\ if\ i(t^-) = i,\ i(t^+) = j, \tag{24}$$

where $i(t)$ is a controlled switching signal and $\mathcal{S} \triangleq \{1, 2, \ldots, s\}$ is a finite set of index. Equation (24) models a jump condition. It means whenever at time $t$ a switch from mode $i(t^-) = i$ to mode $i(t^+) = j$ occurs, the state jumps from $x(t^-)$ to $x(t+) = J_{i,j}x(t^-)$, where $J_{i,j} \in \mathbb{R}^{n\times n}$ are constant matrices.

The main objective is to solve the optimal control problem

$$V_N^* \triangleq \min_{\mathcal{I},\mathcal{T}}\{F(\mathcal{I},\mathcal{T}) = \int_0^\infty x'(t)Q_{i(t)}x(t)dt + \sum_{k=1}^{N} H_{i_{k-1},i_k}\} \tag{25}$$

$$s.t. \begin{cases} \dot{x}(t) = A_{i(t)}x(t) + f_{i(t)}, \\ x(0) = x_0, \\ i(t) = i_k \in \mathcal{S}\ for\ \tau_k \leq t \leq \tau_{k+1},\ k = 0, \ldots, N, \\ 0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_N < \tau_{N+1} = +\infty, \\ x(\tau_k^+) = J_{i_{k-1},i_k} \ldots J_{i_{h-1},i_h}x(\tau_h^-),\ if\ \tau_{h-1} < \tau_h = \ldots = \tau_k < \tau_{k+1}, \end{cases} \tag{26}$$

where $Q_i$ are positive semi-definite matrices, and $x_0$ is the initial state of the system. The cost consists of two components: a quadratic cost that depends on the time evolution (the integral) and a cost that depends on the switches (the sum), where $H_{i,j} \geq 0$, $i, j \in \mathcal{S}$, is the cost for commuting from mode $i$ to mode $j$, with $H_{i,i} = 0$, $\forall i \in \mathcal{S}$. In this optimization problem there two types of decision variables: a finite sequence of switching times $\mathcal{T} \triangleq \{\tau_1, \ldots, \tau_N\}$ and a finite sequence of modes $\mathcal{I} \triangleq \{i_0, \ldots, i_N\}$.

To solve such an optimal control problem, two different computational approaches are proposed by Giua *et al* [15], [16], [17], [18], [19], [20], [21], [22], [23], [24]. The first approach is called *master-slave procedure* (MSP) [19], [24] and exploits a synergy of discrete-time and continuous-time techniques alternating between two different procedures. The "master" procedure is based on mixed-integer quadratic programming (MIQP) and finds an optimal switching sequence for a given initial state, assuming the switching instants are known. The "slave" procedure [17], [20] is based on the construction of the switching regions and finds the optimal switching instants, assuming the mode sequence is known. The results show that it is possible to numerically compute a region of the states space in which an optimal control switch should occur.

The second approach, called *switching table procedure* (STP) [18], [23], is based on dynamic programming ideas and allows one to avoid the explosion of the computational burden with the number of possible switching sequences. It relies on the construction of switching tables which specify when the switching should occur and what the next mode should be. Therefore it can be seen as a generalization of the slave procedure. Moreover, [15], [21], [16], [22] show that STP can be applied to other hybrid system frameworks. [15] deals with *linear hybrid automata*, namely switched linear autonomous systems whose mode of operation is determined by a controlled automaton. The quantities to be optimized are the sequence of switching times and the sequence of modes under some constraints. In [21], the results apply to *autonomous hybrid automata*, which have two types of edges: a controllable edge represents a switching which can be triggered by the controller; an autonomous edge represents a switching which is triggered by the continuous state of the system. Recently, [16] addressed an optimal control problem for *switched affine systems under safety and liveness constraints*. The solution is based on a hierarchical decomposition of the problem, where the low-level controller enforces safety and liveness constraints while the high-level controller exploits the remaining degrees of freedom for performance optimization. Similar technique can also be applied to *discrete-time hybrid automata* [22].

It is pointed out that both procedures have pros and cons in terms of computational complexity and global optimality and preferring one over another will depend on the application at hand. STP is guaranteed to find the optimal solution and provides a "global" closed-loop solution but has high computational cost. On the other hand, MSP is not guaranteed to converge to a global optimum: it only provides an open-loop solution for a given initial state. Furthermore, MSP can handle more general cost functions than STP, such as penalties associated with mode switching, with a lighter computational effort.

Similar to the embedding method in [77], [78], [81] embeds the *autonomous switched linear system* in a larger family of systems and hence the optimization problem is generalized by treating the switching sequence, number of switchings and switching instants, all as decision variable. After applying Pontryagin's Minimum Principle, the optimal solution can be determined by solving the ensuing two-point boundary value problem. It should be noted that the complexity of analysis and numerical algorithm in [81] is less than the embedding method proposed in [77], [78] since the continuous control input is absent from the model.

### *B. Optimal control of autonomous switched nonlinear systems*

Consider an *autonomous switched nonlinear system*

$$\dot{x} = f_i(x(t)), \tag{27}$$

for all $t \in [\tau_i, \tau_{i+1})$, and for every $i \in \{0, \ldots, N\}$, with the given initial condition $x(0) = x_0$. $\tau_i$ are a sequence of switching times and define $\tau_0 = 0$ and $\tau_{N+1} = T$. $T$ is fixed. Let $L : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, and consider the cost function $J$, defined by

$$J = \int_0^T L(x(t))dt. \tag{28}$$

The problem is to find a optimal switching sequence and switching times to minimize the cost function $J$.

As a starting point, Egerstedt and Wardi *et al* [30], [36] consider a simpler version of this problem by fixing the switching sequence (i.e. assuming that the switching sequence is given). Therefore, the controlled parameter to optimize the cost function is just a sequence of switching times, denoted by $\bar{\tau} := (\tau_1, \ldots, \tau_N)^T$. In [30], [36], a simple formula for the gradient $\nabla J(\bar{\tau})$ is derived, which leads itself to be directly used on gradient-descent algorithms. Later, various practical issues, such as state estimation [25], [35] and on-line computation [37], [38], [39], [40], are addressed when finding the optimal switching times. [25], [35] present the algorithms to solve the problem for autonomous switched systems where the state of the system is only partially known through the outputs. A method is presented that both guarantees that the current switch-time estimates remain optimal as the state estimates evolve, and that ensures this in a computationally feasible manner, thus rendering the method applicable to real-time applications. Inspired by this work, more results [37], [38], [39], [40] have appeared on the research of on-line optimization of switched systems. The need for real-time on-line algorithms typically arises when complete information about the system is not available apriori but the algorithm can acquire partial information about it in real time. In these situations the objective is not to optimize the cost functional defined by (28), but rather to reduce the cost to go at certain times. [38], [40] consider the case when the state variable cannot be measured and hence it has to be estimated by a suitable observer. An on-line, Newton-like optimization algorithm to optimize the cost-to-go function by recursively updating the switching times in real time is proposed. A similar algorithm is presented in [37] to optimally update the switching times when the instantaneous cost is assumed to be unavailable before run-time, but can be measured in real time. The recent paper [39] shows that a first-order convergent algorithm can be applied in on-line optimization of switching times when the dynamic response functions (state equations) associated with the modes are not known in advance.

When the state jumps in autonomous switched nonlinear systems are considered, Teo *et al* [82] shows that an approximate solution for this optimal control problem can be computed by solving a sequence of conventional dynamic optimization problem. This approach can reduce the excessive switching between subsystems by merging two or more switching times into one.

For completely solving the problem of finding the optimal switching sequence and switching times in a compact manner, [28], [29] propose an algorithmic framework regarding the variable parameters consisting of the switching times and the switching sequence. At the lower level, the algorithm considers a fixed switching sequence and minimizes the cost functional with respect to the switching times. At the upper level, it updates the switching sequence by inserting two switching points at some time with a system modal between them. Since the algorithms proposed above is cast in the form of a nonlinear programming problem defined on a sequence of nested Euclidean spaces with increasing dimensions, convergence analysis cannot be carried out by the theory of nonlinear programming. Therefore, a notion of local optimality and a suitable concept of convergence are defined in [32] to devise a provably convergent optimization algorithm. [83], however, proposes a different approach to address the problem. The original problem is first transformed into a continuous polynomial systems and then the method of moments can be applied. The necessary and sufficient condition for optimality is obtained and the example shows that existing numerical methods in convex programming can used to solve the problem.

In addition to the problem of finding the optimal switching sequence and times, the other topic in optimal control of autonomous switched systems is the optimization of the cost function (28) by constructing state-dependent switching surfaces. In this problem, we consider a switched system described by (27) with

$$i^+(t) = s(x(t), i(t)) \tag{29}$$

(29) describes the discrete event dynamics of the system: the switch from the mode $i$ to mode $i^+$ occurs at the trajectory of the system, in the continuous state space, intersects a guard set described by a function of the kind $g_i(x, a_i) = 0$, i.e.,

$$s(x,i) = \begin{cases} i, \ g_i(x,a_i) \neq 0 \\ i^+, \ g_i(x,a_i) = 0 \end{cases} \tag{30}$$

Here $g_i$ is assumed to be continuously differentiable in both arguments. $a_i \in \mathbb{R}^m$ is a controllable switching parameter. The problem considered here is how to choose such parameters in order to minimize a suitable cost function (28). [27] derives the gradient of the cost function with respect to the switching surface parameters in a costate-based formula. It then applies the formula in a gradient-descent algorithm for solving an obstacle-avoidance problem in robotics. Furthermore, [33] considers the minimization of a given cost-functional with respect to the switching parameter under the assumption that the initial state of the system is not completely known. By assuming that the initial state can be anywhere in a given set, the proposed approach is based on minimizing the worst possible cost over the given set of initial states using results from mini-max optimization. [34], [26] propose an improved algorithm, allowing to build switching surfaces which are optimal for any possible initial conditions. The algorithm is based on the sensitivity analysis of the optimal switching times with respect to the initial conditions and on the identification of the set of initial conditions maximizing the information relevant to the design of the surface.

## V. SUMMARY

This report surveys recent computational results on optimal control of switched systems. We first present the general optimal control problem formulation and then the main methodologies under different problem formulations are summarized. Among the results, we focus on the difference of their problem formulations, assumptions and computational techniques.

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