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SENSITIVITY CONSIDERATIONS IN THE CONTROL OF GENERALIZED PLANTS

by

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Abstract

A new comparison sensitivity matrix is derived for a generalized plant where the controlled and measured variables are not necessarily the same. It contains a constant term, which imposes a constraint on sensitivity reduction that can be overcome when the controlled and measured variables are related appropriately. Its relation to exogenous signal attenuation is also shown.

Introduction

The role of feedback in reducing the sensitivity of a linear control system to plant parameter variations is well known [1-7]. In multivariable systems, comparison sensitivity is related to the inverse of the return difference matrix (when the loop is broken at the "output") and it can be used to introduce a measure of feedback performance [3-7]. Most of these results consider the case where all the controlled variables of the plant are measured. In [7], a plant-sensor configuration is considered. When the controlled and measured variables are not necessarily the same, the relationship between sensitivity and the return difference matrix needs to be reevaluated. In fact, in [8] it is stated without proof that feedback can be used to reduce the sensitivity of a control system to plant parameter variations only if all the controlled variables are measured; it is assumed that the measured variables include the controlled variables. Here, we derive a new comparison sensitivity matrix when the plant's controlled and measured variables are not necessarily the same. It contains a constant term unless the controlled and measured variables are related; for example, the controlled and measured variables coincide or are related by a sensor. The constant term imposes a constraint on sensitivity reduction that can be overcome when the controlled and measured variables are related appropriately.

Problem Formulation

In [3], the scalar concept of a sensitivity function was generalized to the multivariable case. A linear relation was derived between the errors, due to plant parameter variations, in the response of the controlled variables to the command input in a feedback and an "equivalent" open-loop system. In [3], the controlled (y_c) and measured (y_m) variables coincide. Here, similar results are derived when the controlled and measured variables are not necessarily the same. The same approach as in [3] is used to compare the response of the system $\Sigma(S_P,S_C)$ in Figure 1 and of the open loop system in Figure 2, leading to a *comparison sensitivity* matrix, when $y_c \neq y_m$.

Consider the linear, time-invariant, finite dimensional multivariable system $\Sigma(S_P,S_C)$,

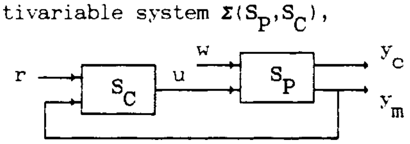

FIGURE 1. The compensated system $\Sigma(S_P,S_C)$.

where S_P and S_C denote the plant and controller, respectively. Assume that the plant and controller are controllable and observable. Let an input-output description of the plant be

$$\begin{bmatrix} y_m \\ y_c \end{bmatrix} = P \begin{bmatrix} u \\ w \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \tag{1}$$

where the vector w contains all the variables that affect the plant, but are not manipulated by the controller (for example, nonmeasurable disturbances and initial conditions); u is the vector of control inputs; and P_{ij} $(i,j=1,2)$ are proper transfer matrices. The control u is given by $u = C[y_m^t,\ r^t]^t$, where ^{t} denotes transpose, $C=[-C_y \quad C_r]$ is the transfer matrix of the controller, and r is the vector of command inputs.

This general plant model is useful because it unifies the study of plants where $y_c \neq y_m$, and where exogenous signals are present. It has been used recently in the formulation and analysis of multi-objective control problems in [9-13].

In the following it is assumed that an internally stabilizing controller C exists and it has been found. So that the compensated system in Figure 1 is internally stable.

Main Results

Let T_{cr} and T_{mr} represent the transfer matrices from r to y_c and y_m, respectively. For the feedback system in Figure 1, we have

$$T_{mr_1} = P_{11}(I+C_yP_{11})^{-1}C_r \quad \text{and} \tag{2}$$

$$T_{cr_1} = P_{21}(I+C_yP_{11})^{-1}C_r. \tag{3}$$

Note that the subscript $(_1)$ refers to quantities of the feedback system in Figure 1; $(_2)$ refers to quantities of the open-loop system in Figure 2.

If no uncertainty is present (this also implies no exogenous signals), it is known that the transfer matrices in (2) and (3) can be attained using the open-loop configuration:

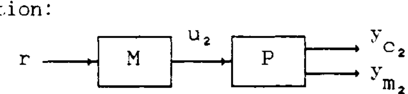

FIGURE 2. Open-loop compensation with $w \equiv 0$.

where it is assumed that y_{c_2} and y_{m_2} are equal to y_{c_1} and y_{m_1} respectively, the outputs of the feedback system in Figure 1. It is easily verified that an appropriate value for the open-loop controller M is $M=(I+C_yP_{11})^{-1}C_r$.

To derive the comparison sensitivity matrix one compares the responses of the feedback system in Figure 1 and of the open-loop system in Figure 2. For this, let the errors between the nominal $(^\circ)$ and true transfer matrices for the feedback $(_1)$ and open-loop $(_2)$ configurations be:

$$\begin{bmatrix} E_{m_1} \\ E_{c_1} \end{bmatrix} = \begin{bmatrix} T_{mr_1} - T^\circ_{mr_1} \\ T_{cr_1} - T^\circ_{cr_1} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_{m_2} \\ E_{c_2} \end{bmatrix} = \begin{bmatrix} T_{mr_2} - T^\circ_{mr_2} \\ T_{cr_2} - T^\circ_{cr_2} \end{bmatrix}. \tag{4}$$

A convenient way to derive the desired result is to consider an augmented output signal $[y_c^t,\ y_m^t]^t$. Using then an approach similar to the one in [3], the following relation can be derived [11]:

$$\begin{bmatrix} E_{m_1} \\ E_{c_1} \end{bmatrix} = S_a^\circ \begin{bmatrix} E_{m_2} \\ E_{c_2} \end{bmatrix}, \quad S_a^\circ = \begin{bmatrix} (I+P_{11}^\circ C_y)^{-1} & 0 \\ -P_{21}^\circ C_y(I+P_{11}^\circ C_y)^{-1} & I \end{bmatrix}. \tag{5}$$

This transfer matrix is the comparison sensitivity matrix when $y_c \neq y_m$.

Observe that the relation between the feedback and open-loop errors for T_{mr} has the form of the usual sensitivity matrix $S_1^\circ=(I+P_{11}^\circ C_y)^{-1}$, as of course should be expected. However, the control designer is really interested in T_{cr} and the relation between the errors in this case. From (5), the difference $E_{c_1}-E_{c_2}$ is linearly related to E_{m_2}, the open-loop error in T_{mr}. For special cases of S_p, there is a relation between E_{c_2} and E_{m_2} (see remarks below), but, in general, E_{c_1} depends on both open-loop errors: E_{m_2} and E_{c_2}.

Remark 1: When $y_c=y_m$, then $P_{11}=P_{21}$ and $E_{m_2}=E_{c_2}$, giving $E_{c_1}=(I-P_{11}^\circ C_y(I+P_{11}^\circ C_y)^{-1}E_{c_2}=(I+P_{11}^\circ C_y)^{-1}E_{c_2}$. So the comparison sensitivity matrix in (5) reduces to the usual one when $y_c=y_m$, as expected.

Remark 2: Consider $y_m=H_s y_c$, where H_s corresponds to the transfer matrix of a sensor, as in, for example, [7]. Then $P_{11}=H_s P_{21}$ and assuming $P_{11}^\circ=H_s P_{21}^\circ$, gives $E_{c_1} \equiv (I+P_{21}^\circ C_y H_s)^{-1}E_{c_2}$.

Remark 3: When $y_c=Q_m y_m$ (Q_m proper and stable), then $P_{21}=Q_m P_{11}$, and assuming $P_{21}^\circ=Q_m P_{11}^\circ$, gives $E_{c_1} \equiv Q_m(I+P_{11}^\circ C_y)^{-1}E_{m_2}$, where $E_{c_2}=Q_m E_{m_2}$. This is the case considered in [8] with $Q_m=[I,\ 0]$.

The comparison sensitivity matrix derived here as well as the classical one, is in terms of true plant parameters; this makes it unsuitable for direct use in control systems design [14]. Some ways to amend this shortcoming have appeared in [4-7,14]. Specifically, in [6,7], particular representations of the plant parameter uncertainties are used. As a consequence, the comparison sensitivity matrix can be written in terms of these representations, and a new relation between T_{c_1} and $T_{c_1}^\circ$ can be derived, which can be quite simple if appropriate uncertainty representations are used [7]. The problem is then reduced to obtaining bounds on the size of the plant parameter variations. In [7], characterizations of the controlled output and the control input sensitivity to plant and some compensator parameter variations are given. A similar approach is taken in [11] to give sensitivity matrices in terms of the nominal parameters.

In control systems design, sensitivity matrices in terms of the nominal quantities are being used as explained above. Sensitivity matrices in terms of the actual quantities can also be directly useful as they provide considerable insight. Since we are interested here in the relation between the feedback error and open loop error for T_{cr} define the output comparison sensitivity matrix to be

$$S_o^\circ = [-P_{21}^\circ C_y(I+P_{11}^\circ C_y)^{-1}, \quad I]. \tag{6}$$

Notice that the constant term in (6) appears to indicate that the feedback system in Figure 1 cannot perform any better than the open loop one in Figure 2. The performance of the feedback and open-loop systems can be compared using the integral of the quadratic errors [3] or using the largest singular value of S_o° over the frequency band of interest [6]. In the case of scalar systems, it is easily shown that the largest singular value of S_o° is greater than or equal to 1 whenever (6) has a constant term. Cases when the constant term in (6) disapears, and the largest singular value depends directly on the return difference matrix, include the cases considered in the remarks, where a relation between y_c and y_m exists. In general, feedback can improve performance only when there is no constant term in (6), implying that a relation between y_c and y_m is present.

Additional insight into the role of (6) is gained by considering its relation to exogenous signal attenuation, which corresponds to minimizing T_{cw}, the transfer matrix from w to y_c. For the feedback system in Figure 1, $T_{cw}=P_{22}-P_{21}C_y(I+P_{11}C_y)^{-1}P_{12}$, so that

$$T_{cw}^\circ = S_o^\circ \begin{bmatrix} P_{12}^\circ \\ P_{22}^\circ \end{bmatrix}. \tag{7}$$

This has been done in [11]; it extends results for $y_c=y_m$.

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