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A further remark on the problem of reliable stabilization using rectangular dilated LMIs

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In this brief paper, an extension of the result of Fujisaki & Befekadu (2009, Reliable decentralized stabilization of multi-channel systems: a design method via dilated Linear Matrix Inequalities (LIMs) and unknown disturbance observers. *Int. J. Contr.*, **82**, 2040–2050.) concerning the problem of reliable stabilization for generalized multi-channel systems is given. Specifically, we use a rectangular dilated LMIs framework to provide a relaxed sufficient condition for the reliable stabilization of a multi-channel system both when all of the controllers work together and when one of the controllers ceases to function due to a failure.

Keywords: decentralized control; dilated LMIs; reliable control; stabilization.

1. Introduction

Recently, the problem of reliable stabilization for generalized multi-channel systems with a single failure in any of the control channels has been addressed by Fujisaki & Befekadu (2009) via dilated LMIs and unknown disturbance observers. In this brief paper, we extend this result for a multi-channel system using a rectangular dilated LMIs framework. The new extension can be looked as a sufficiently decoupling framework (i.e. separating the design variables from the system data) that provides a tractable (and also less-conservative) design technique for reliable stabilization of the multi-channel system.[1]

This brief paper is organized as follows. In Section 2, we present the main result where the problem of reliable stabilization for a generalized multi-channel system is formally restated. Specifically, a relaxed and verifiable sufficient condition is given in terms of a set of rectangular dilated LMIs for the reliable stabilization of the multi-channel system. In the Supplementary appendix section, for the sake of completeness, we also present a supplementary result on the rectangular dilated LMIs framework.

Notation. For a matrix $A \in \mathbb{R}^{n\times n}$, He (A) denotes a hermitian matrix defined by He $(A) \triangleq (A + A^{\mathrm{T}})$, where A^{T} is the transpose of A. For a matrix $B \in \mathbb{R}^{n\times p}$ with $r = \operatorname{rank} B$, $B^{\perp} \in \mathbb{R}^{(n-r)\times n}$ denotes an orthogonal complement of B, which is a matrix that satisfies $B^{\perp}B = 0$ and $B^{\perp}B^{\perp T} \succ 0$. $\mathcal{S}_n^+$ denotes the set of strictly positive definite $n \times n$ real matrices and $\mathbb{C}_-$ denotes the set of complex numbers with negative real parts, that is, $\mathbb{C}_- \triangleq \{s \in \mathbb{C} \mid \operatorname{Re}\{s\} < 0\}$. $\operatorname{Sp}(A)$ denotes the spectrum of a matrix $A \in \mathbb{R}^{n\times n}$, that is, $\operatorname{Sp}(A) \triangleq \{\lambda \in \mathbb{C} \mid \operatorname{rank}(A - \lambda I) < n\}$ and $\mathrm{GL}_n(\mathbb{R})$ denotes the general linear group consisting of all $n \times n$ real nonsingular matrices.

[1] Note that the problem of reliable stabilization is essentially equivalent to a strong stabilization problem that involves an intractable problem (e.g. see Vidyasagar & Viswanadham, 1985; Nemirovsk, 1993; Blondel & Tsitsiklis, 1997).

2. Main result

Consider the following finite-dimensional generalized the multi-channel system

$$\dot{x}(t) = Ax(t) + \sum_{j\in\mathcal{N}} B_j u_j(t), \quad x(0) = x_0, \tag{2.1}$$

where $x(t) \in \mathscr{X} \subseteq \mathbb{R}^n$ is the state of the system, $u_j(t) \in \mathscr{U}_j \subseteq \mathbb{R}^{r_j}$ is the control input to the jth-channel, $A \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times r_j}$ and $\mathcal{N} = \{1, 2, \ldots, N\}$ represents the set of controllers in the system.

For the above system, we restrict the set $\mathcal{K}$ to be the set of all linear, time-invariant (reliable) stabilizing state-feedback gains that satisfies

$$\mathcal{K} \subseteq \left\{ (K_1, K_2, \ldots, K_N) \in \prod_{i\in\mathcal{N}} \mathcal{K}_i \subseteq \prod_{i\in\mathcal{N}} \mathbb{R}^{r_i\times n} \,\middle|\, \mathrm{Sp}(A + B_{\neg j} \circ K_{\neg j}) \subset \mathbb{C}_-,\ \forall j \in \mathcal{N} \cup \{0\} \right\}, \tag{2.2}$$

where the sets $\mathcal{N}_{\neg 0} \triangleq \mathcal{N}$ and $\mathcal{N}_{\neg j}$ are defined by $\mathcal{N}_{\neg j} \triangleq \mathcal{N}\backslash\{j\}$ for $j = 1, 2, \ldots, N$ with cardinality of $|\mathcal{N}_{\neg 0}| = N$ and $|\mathcal{N}_{\neg j}| = N - 1$, respectively. Moreover, $B_{\neg 0} = (B_i)_{i\in\mathcal{N}}$, $B_{\neg j} = (B_i)_{i\in\mathcal{N}_{\neg j}}$ and $B_{\neg j} \circ K_{\neg j} \triangleq \sum_{i\in\mathcal{N}_{\neg j}} B_i K_i$ for $j \in \mathcal{N} \cup \{0\}$.

REMARK 2.1 In this brief paper, we consider the stability of the closed-loop system $(A + B_{\neg j} \circ K_{\neg j})$ under nominal operation condition (i.e. when $j = 0$) as well as under any single-channel controller failure (i.e. when $j \in \mathcal{N}$).

Let us define the following matrices that will be later used in Theorem 2.1.

DEFINITION 2.1 For $j = 0$

$$E_{\neg 0} = [\underbrace{I_{n\times n} \quad I_{n\times n} \quad \cdots \quad I_{n\times n}}_{(|\mathcal{N}_{\neg 0}|+1)\ \text{times}}], \quad \langle X_0, X_{\neg 0}\rangle = \text{block diag}\{\underbrace{X_0, \overbrace{X_0, X_0, \ldots, X_0}^{=X_{\neg 0}}}_{(|\mathcal{N}_{\neg 0}|+1)\ \text{times}}\},$$

$$[A, B]_{U_0, L_{\neg 0}} = [\underbrace{AU_0 \quad B_1L_1 \quad B_2L_2 \quad \cdots \quad B_NL_N}_{(|\mathcal{N}_{\neg 0}|+1)\ \text{times}}], \quad \langle U_0, W_{\neg 0}\rangle = \text{block diag}\{\underbrace{U_0, \overbrace{W_1, W_2, \ldots, W_N}^{=W_{\neg 0}}}_{(|\mathcal{N}_{\neg 0}|+1)\ \text{times}}\},$$

and for $j \in \mathcal{N}$

$$E_{\neg j} = [\underbrace{I_{n\times n} \quad \cdots \quad I_{n\times n} \quad I_{n\times n} \quad \cdots \quad I_{n\times n}}_{(|\mathcal{N}_{\neg j}|+1)\ \text{times}}], \quad \langle X_j, X_{\neg j}\rangle = \text{block diag}\{\underbrace{X_j, \overbrace{X_j, \ldots, X_j, X_j, \ldots, X_j}^{=X_{\neg j}}}_{(|\mathcal{N}_{\neg j}|+1)\ \text{times}}\},$$

$$[A, B]_{U_j, L_{\neg j}} = [\underbrace{AU_j \quad B_1L_1 \quad \cdots \quad B_{j-1}L_{j-1} \quad B_{j+1}L_{j+1} \quad \cdots \quad B_NL_N}_{(|\mathcal{N}_{\neg j}|+1)\ \text{times}}],$$

$$\langle U_j, W_{\neg j}\rangle = \text{block diag}\{\underbrace{U_j, \overbrace{W_1, \ldots, W_{j-1}, W_{j+1}, \ldots, W_N}^{=W_{\neg j}}}_{(|\mathcal{N}_{\neg j}|+1)\ \text{times}}\},$$

where $X_j \in \mathcal{S}_n^+$, $U_j \in \mathrm{GL}_n(\mathbb{R})$ for $j = 0, 1, \ldots, N$, $W_i \in \mathrm{GL}_n(\mathbb{R})$ and $L_i \in \mathbb{R}^{r_i\times n}$ for $i = 1, 2, \ldots, N$.

REMARK 2.2 Note that the above set of matrices allows us to introduce a common set of matrix variables $\{L_i, W_i\}_{i\in\mathcal{N}}$ that will be useful for the main result of this section.

Next we can characterize the set $\mathcal{K}$ using a new-class of dilated LMIs (i.e. rectangular dilated LMIs) as follow.

THEOREM 2.1 Suppose there exist $X_j \in \mathcal{S}_n^+$, $\epsilon_j > 0$, $U_j \in \mathrm{GL}_n(\mathbb{R})$, $j = 0, 1, \ldots, N$, $W_i \in \mathrm{GL}_n(\mathbb{R})$ and $L_i \in \mathbb{R}^{r_i\times n}$, $i = 1, 2, \ldots, N$ such that

$$\begin{aligned}&\begin{bmatrix} 0_{n\times n} & E_{\neg j}\langle X_j, X_{\neg j}\rangle \\ \langle X_j, X_{\neg j}\rangle E_{\neg j}^{\mathrm{T}} & 0_{(|\mathcal{N}_{\neg j}|+1)n\times(|\mathcal{N}_{\neg j}|+1)n}\end{bmatrix}\\ &\quad + \mathrm{He}\left(\begin{bmatrix}[A,B]_{U_j,L_{\neg j}} \\ -\langle U_j, W_{\neg j}\rangle\end{bmatrix} \times [E_{\neg j}^{\mathrm{T}} \quad \epsilon_j I_{(|\mathcal{N}_{\neg j}|+1)n\times(|\mathcal{N}_{\neg j}|+1)n}]\right) \prec 0 \quad \forall j \in \mathcal{N}\cup\{0\}. \end{aligned} \tag{2.3}$$

For any family of $|\mathcal{N}_{\neg 0}|$-tuples $(L_1, L_2, \ldots, L_N)$ and $(W_1, W_2, \ldots, W_N)$ as above, if we set $K_i = L_i W_i^{-1}$ for each $i = 1, 2, \ldots, N$, then the matrices $(A + B_{\neg j} \circ K_{\neg j})$ are Hurwitz for all $j \in \mathcal{N}\cup\{0\}$, that is, $(L_i W_i^{-1})_{i\in\mathcal{N}} \in \mathcal{K}$.[2]

Here, we give a short proof which is based on the result of Lemma 2.1 (see Supplementary appendix for details).

Proof. The above result can be verified by using Finsler's lemma (e.g. Skelton *et al.*, 1998), which is a specialized version of the elimination lemma, with

$$\begin{bmatrix}[A,B]_{U_j,L_{\neg j}} \\ -\langle U_j, W_{\neg j}\rangle\end{bmatrix}^{\perp} = [I_{n\times n} [A,B]_{U_j,L_{\neg j}}\langle U_j, W_{\neg j}\rangle^{-1}], \tag{2.4}$$

$$\begin{bmatrix} E_{\neg j} \\ \epsilon_j I_{(|\mathcal{N}_{\neg j}|+1)n\times(|\mathcal{N}_{\neg j}|+1)n}\end{bmatrix}^{\perp} = [\epsilon_j I_{n\times n} \quad -E_{\neg j}], \tag{2.5}$$

for $j = 0, 1, \ldots, N$.

Note that if we eliminate $\langle U_j, W_{\neg j}\rangle$ from (2.3) by using (2.4) and (2.5). Then, we have the following matrix inequalities

$$\begin{aligned}&[I_{n\times n} \quad [A,B]_{U_j,L_{\neg j}}\langle U_j, W_{\neg j}\rangle^{-1}]\begin{bmatrix} 0_{n\times n} & E_{\neg j}\langle X_j, X_{\neg j}\rangle \\ \langle X_j, X_{\neg j}\rangle E_{\neg j}^{\mathrm{T}} & 0_{(|\mathcal{N}_{\neg j}|+1)n\times(|\mathcal{N}_{\neg j}|+1)n}\end{bmatrix}\begin{bmatrix} I_{n\times n} \\ (\langle U_j, W_{\neg j}\rangle^{-1})^{\mathrm{T}}[A,B]_{U_j,L_{\neg j}}^{\mathrm{T}}\end{bmatrix}\\ &\quad = \mathrm{He}((A + B_{\neg j}\circ K_{\neg j})X_j) \prec 0, \end{aligned} \tag{2.6}$$

$$[\epsilon_j I_{n\times n} \quad -E_{\neg j}]\begin{bmatrix} 0_{n\times n} & E_{\neg j}\langle X_j, X_{\neg j}\rangle \\ \langle X_j, X_{\neg j}\rangle E_{\neg j}^{\mathrm{T}} & 0_{(|\mathcal{N}_{\neg j}|+1)n\times(|\mathcal{N}_{\neg j}|+1)n}\end{bmatrix}\begin{bmatrix}\epsilon_j I_{n\times n} \\ -E_{\neg j}^{\mathrm{T}}\end{bmatrix} = -2\epsilon_j(|\mathcal{N}_{\neg j}| + 1)X_j \prec 0. \tag{2.7}$$

Hence, we see that Equations (2.6) and (2.7) state the Liapunov stability condition with $X_j \in \mathcal{S}_n^+$ and state-feedback gains $(L_i W_i^{-1})_{i\in\mathcal{N}} \in \mathcal{K}$. □

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[2] Note that the theorem is solvable only if all of the pairs $(A, B_{\neg j})$ for $j \in \mathcal{N}$ are stabilizable. Moreover, the stabilizability of one of the pairs implies stabilizability of $(A, B_{\neg 0})$, thus we do not have to assume this explicitly.

REMARK 2.3 We remark that the above dilated LMI framework stated in Theorem 2.1 is useful in the context of reliable control for a system with generalized multi-channel configurations, since the framework effectively separates design variables such as X_j from the system data $(A, B_{\neg j})$ for all $j \in \mathcal{N} \cup \{0\}$.

Note that Theorem 2.1 is a generalization of the square dilated LMIs technique that has been considered by Fujisaki & Befekadu (2009) in the context of reliable stabilization for multi-channel systems (e.g. see Geromel *et al.*, 1998; Ebihara & Hagiwara, 2005; Fujisaki & Befekadu, 2007; Pipeleers *et al.*, 2009 and references therein for a review of square dilated LMI technique). In fact, if we multiply Equation (2.3) from the left side by

$$\Gamma_{\neg j} = \begin{bmatrix} (|\mathcal{N}_{\neg j}| + 1)I_{n\times n} & 0_{n\times(|\mathcal{N}_{\neg j}|+1)n} \\ 0_{n\times n} & E_{\neg j} \end{bmatrix}, \tag{2.8}$$

and from the right side by the transpose matrix $\Gamma_{\neg j}^{\mathrm{T}}$. Finally, making use of the relation $E_{\neg j}E_{\neg j}^{\mathrm{T}} = (|\mathcal{N}_{\neg j}| + 1)I_{n\times n}$ and setting $W_i \to W$ for $i = 1, 2, \cdots, N$ and $U_j \to W$ for $j = 0, 1, \cdots, N$ (which also gives us the condition $\langle W, W_{\neg j}\rangle E_{\neg j}^{\mathrm{T}} = E_{\neg j}^{\mathrm{T}} W$), then (2.3) reduces to

$$\begin{bmatrix} 0 & (|\mathcal{N}_{\neg j}| + 1)X_j \\ (|\mathcal{N}_{\neg j}| + 1)X_j & 0 \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} (AW + B_{\neg j} \circ L_{\neg j}) \\ -W \end{bmatrix} [(|\mathcal{N}_{\neg j}| + 1)I_{n\times n} \quad \epsilon_j I_{n\times n}]\right) \prec 0, \tag{2.9}$$

where $B_{\neg j} \circ L_{\neg j} = \sum_{i\in\mathcal{N}_{\neg j}} B_i L_i$ for $j \in \mathcal{N}$. Note that the above equation is basically a square dilated LMI condition presented in Fujisaki & Befekadu (2009), that is, if we let further $\epsilon_j \to (|\mathcal{N}_{\neg j}| + 1)\epsilon_j'$ for all $j \in \mathcal{N} \cup \{0\}$, we then have

$$\begin{bmatrix} 0 & X_j \\ X_j & 0 \end{bmatrix} + \mathrm{He}\left(\begin{bmatrix} (AW + B_{\neg j} \circ L_{\neg j}) \\ -W \end{bmatrix} [I_{n\times n} \quad \epsilon_j' I_{n\times n}]\right) \prec 0. \tag{2.10}$$

Moreover, for any family of $|\mathcal{N}_{\neg 0}|$-tuple $(L_1, L_2, \ldots, L_N)$ and $W \in \mathrm{GL}_n(\mathbb{R})$ as above, if we set $K_i = L_i W^{-1}$ for each $i = 1, 2, \ldots, N$, then the matrices $(A + B_{\neg j} \circ K_{\neg j})$ are Hurwitz for all $j \in \mathcal{N} \cup \{0\}$.

We remark that Equation (2.3) describes a new class of dilated LMI conditions in terms of $W_i \in \mathrm{GL}_n(\mathbb{R})$, $L_i \in \mathbb{R}^{r_i\times n}$, $i = 1, 2, \ldots, N$, and $U_j \in \mathrm{GL}_n(\mathbb{R})$, $X_j \in \mathcal{S}_n^+$, $j = 0, 1, \ldots, N$. Note also that a common set of matrix variables $\{L_i, W_i\}_{i\in\mathcal{N}}$ is used for all failure modes, that is, for all $j \in \mathcal{N} \cup \{0\}$. This is because we need an $|\mathcal{N}_{\neg 0}|$-tuple state-feedback gain $K \triangleq (K_1\ K_2\ \ldots\ K_N)$ with $K_i \in \mathcal{K}_i$ for $i \in \mathcal{N}$ that ensures stability for all possible closed-loop systems. However, it should be noted that, since we use a new class of dilated LMI framework, we do not require either a common quadratic Liapunov stability certificate $X \in \mathcal{S}_n^+$ as in the case of quadratic Liapunov technique or a common $W \in \mathrm{GL}_n(\mathbb{R})$ and $\{L_i\}_{i\in\mathcal{N}}$ that will be needed in the case of a square dilated LMI technique for all possible failure modes (cf. Equation (2.10)). In this sense, the new extension, which is based on Theorem 2.1, is not as conservative as the quadratic Liapunov technique or the square dilated LMIs technique.

Note that although we have considered a reliable state-feedback stabilization problem, the problem of reliable stabilization via multi-controller configuration that was actually explored by Fujisaki & Befekadu (2009) can be treated in the same way. In fact, that paper also presented a tractable design method which covers a class of plants that can be stabilized reliably using dynamic output feedback controllers (with fixed order of the controllers).

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Supplementary material

Supplementary material is available at *IMA Journal of Mathematical Control and Information* online.

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