Passivity and Dissipativity of a System and its Approximation

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Abstract

We consider the following problem: given two mathematical system models, one representing accurately a physical system and the other representing its approximation, what passivity properties of the system can be inferred from studying only the approximate model. Our results show that an excess of passivity (whether in the form of input strictly passive, output strictly passive or very strictly passive) in the approximate model guarantees a certain passivity index for the system, at least if the norm of the error between the two models is sufficiently small. We also consider QSR dissipative systems and show that QSR dissipativity has a similar robustness property, even though the supply rate for the system and its approximation may be different.

I. Introduction

Energy dissipation is a fundamental concept in dynamical systems. Passivity and dissipativity characterize the "energy" consumption of a dynamical system and form a powerful tool in many real applications. Passivity is closely related to stability and exhibits a compositional property for parallel and feedback interconnections [1], [2], [3]. Passivity-based control is especially useful in the analysis of complex coupled systems.

It is *impossible* to precisely describe the behavior of any physical system through mathematical models. In modeling physical systems, a classical dilemma is the tradeoff between model accuracy and tractability [4]. A variety of *approximation* methods are used, for analysis, simulation or control design of the 'real' systems [5]. It is critical that the approximate model preserves properties and features of interests of the original system, such as stability, Hamiltonian structure or passivity. One example of approximation is the use of linearization methods. Nonlinear behaviors abound in the real world, including saturation, backlash and dead zone [6]. Linearized models are often used, because methods for analysis and control designs are readily available for linear systems [7]. Another example is model reduction [5]. The need for modeling accuracy may result in large-scale, higher-order and complex mathematical models. Model reduction methods lead to a lower-order, simpler system, that can be used to facilitate control designs or speed up simulations [5], [8].

We are particularly interested in the passivity of a system as inferred from studying an approximate model of its dynamical behavior. It is known that under some conditions, linearization [1], [9] and model reduction [8], [10] preserve passivity. The main contribution of this paper is the establishment of relationships between *passivity levels* of two mathematical system models, one of which could represent accurately a physical system and the other representing its approximation; Of course, the two mathematical models can represent two different approximation of the same physical system as well. The approximate model is assumed to have an *excess* of passivity, defined as passivity levels (similar to passivity indices [3]) that characterize how passive it is (how much of the energy introduced into the system is dissipated). If the error between the system and its approximation is "small" in some sense, we show that passivity levels for the system can be guaranteed. Since passivity levels (or indices) are used to design controls for the system [3], [11], these results imply that it is possible to use the simpler approximate model for control design. Also, we derive conditions under which the system remains passive if the approximate model is passive. These results may be interpreted as *robustness properties* with respect to model uncertainties [12], [13]. If the approximate model does not have an excess of passivity, we consider the case when the approximation is QSR dissipative. In this case, it is shown that if the error between the system and its approximation is "small", the system will be QSR dissipative as well but for a different supply rate.

As a particular case, we consider linear systems and their positive-real truncations [10] and derive variations in the passivity levels for the full-order and reduced-order systems. There exist various procedures for model reduction preserving passivity [8]. The works such as [5], [8], [10] focus on how to preserve passivity instead of studying the variations in the passivity levels caused by model reduction. However, our results show how passivity levels vary as a function of the order.

The rest of the paper is organized as follows. Section II provides background material on passivity and model reduction preserving passivity. Section III presents the problem statement. The main results are given in Section IV. Discussions of results in the discrete-time domain are presented in Section V. Numerical examples are provided in Section VI. Concluding remarks are given in Section VII.

Notation. The signal space under consideration is $\mathcal{L}_2$ space or the extended $\mathcal{L}_2$ space. The Euclidean space of dimension m is denoted by $\mathbb{R}^m$. Denote the truncation of $u(t)$ up to time T ($0 \leq T < \infty$) by $u_T(t)$. The inner product of truncated signals $u_T(t), y_T(t)$ is denoted by $\langle u, y \rangle_T$, where $\langle u, y \rangle_T \triangleq \int_0^T u^T(t)y(t)\mathrm{d}t$ and $u^T(t)$ denotes the transpose of $u(t)$. The $\mathcal{L}_2$-induced norm of a signal u is denoted by $\|u\|_T$, where $\|u\|_T^2 \triangleq \int_0^T u^T(t)u(t)\mathrm{d}t$. The H_∞ norm of a transfer function $G(s)$ is denoted by $\|G\|_{H_\infty}$. For a complex matrix $A \in \mathbb{C}^{n\times n}$, the minimum eigenvalue of A is denoted by $\underline{\lambda}(A)$ and the maximum eigenvalue by $\overline{\lambda}(A)$. $\mathrm{Re}[A]$ is the real part of a complex matrix A. $A \geq 0$ denotes that A is positive semi-definite and $A > 0$ implies that A is positive definite. The identity matrix is denoted by I and the dimensions are omitted when it is clear from context. The notation $\max\{a, b\}$ denotes the larger value of $a, b \in \mathbb{R}$ and $\min\{a, b\}$ denotes the smaller value of $a, b \in \mathbb{R}$.

II. Preliminaries

A. Passivity

Definition 1 ([1], [14]): Consider a system Σ with input u and output y where $u(t), y(t) \in \mathbb{R}^m$. It is called

- *passive*, if there is a constant $\beta \leq 0$ such that
$$\langle u, y \rangle_T \geq \beta.$$
- *input strictly passive* (ISP), if there exist $\nu > 0$ and a constant $\beta \leq 0$ such that
$$\langle u, y \rangle_T \geq \beta + \nu \langle u, u \rangle_T. \tag{1}$$
- *output strictly passive* (OSP), if there exist $\rho > 0$ and a constant $\beta \leq 0$ such that
$$\langle u, y \rangle_T \geq \beta + \rho \langle y, y \rangle_T. \tag{2}$$
- *very strictly passive* (VSP), if there exist $\rho > 0$ and $\nu > 0$ and a constant $\beta \leq 0$ such that
$$\langle u, y \rangle_T \geq \beta + \rho \langle y, y \rangle_T + \nu \langle u, u \rangle_T. \tag{3}$$

In all cases, the inequality should hold for $\forall u(t)$, $\forall T \geq 0$ and the corresponding $y(t)$.

A few remarks about the definitions.

1) The constant β is related to the initial conditions and plays an important role in the stability analysis of the system [14].
2) The notation $\langle u, y \rangle_T$ denotes the externally supplied energy to the system during the interval $[0, T]$. For instance, $\langle u, y \rangle$ is the instantaneous power by viewing u as the voltage and y as the current [1], [6].
3) VSP is referred to *input-output strict passivity* in [15], [16].
4) The above definitions can be viewed as special cases of QSR-dissipative systems [2], [17], defined as systems for which there exist $Q = Q^T, R = R^T$ and S, such that $\forall u(t)$, $\forall T \geq 0$ and the corresponding $y(t)$,
$$r(u, y) \triangleq \langle y, Qy \rangle_T + 2\langle y, Su \rangle_T + \langle u, Ru \rangle_T \geq 0. \tag{4}$$
The function $r(u, y)$ is called the *supply rate* for Σ. It is clear that Σ is ISP for ρ if $Q = 0, S = 0.5I, R = -\rho I$. If $Q = -\nu I, S = 0.5I, R = 0$, Σ is OSP for ν. If $Q = -\nu I, S = 0.5I, R = -\rho I$, Σ is VSP for (ρ, ν).
5) Definition 1 is the input-output description with the benefits of abstraction [18]. The definitions based on state models can be found in [14], [17]. The relations between the two descriptions are studied in [18], [1].

6) Clearly, if a system Σ is ISP for $\nu > 0$, it is also ISP for $\nu - \epsilon$, where $0 \leq \epsilon < \nu$. Analogously, if Σ is OSP for $\rho > 0$, it is also OSP for $\rho - \epsilon$, where $0 \leq \epsilon < \rho$ [3]. Finally, if Σ is VSP for (ρ, ν), it is also VSP for $(\rho - \epsilon, \nu - \epsilon)$, where $0 \leq \epsilon < \min\{\rho, \nu\}$ (see Lemma 2 in the Appendix). A positive value of ρ or ν can thus be interpreted as an *excess* of passivity and these two values (called *passivity levels*) characterize 'how passive' Σ is. If ρ or ν is negative, we say Σ has a *shortage* of passivity. This intuition is captured by the concept of passivity indices [3].

Definition 2: For a system Σ with input u and output y where $u(t), y(t) \in \mathbb{R}^m$,

- the *input feedforward passivity index* (IFP) is the largest $\nu > 0$ such that (1) holds for $\forall u$ and $\forall T \geq 0$,
- the *output feedback passivity index* (OFP) is the largest $\rho > 0$ such that (2) holds for $\forall u$ and $\forall T \geq 0$.

The two indices are denoted by IFP(ν) and OFP(ρ), respectively.

Note the fact that a system has IFP(ν) and OFP(ρ) does *not* necessarily imply that the system is VSP for (ρ, ν). In other words, the system may not have IFP(ν) and OFP(ρ) *simultaneously*. A necessary condition for ρ and ν to be VSP is given by $\rho\nu \leq \frac{1}{4}, \rho > 0, \nu > 0$ (see Lemma 3 in the Appendix). As a result, for VSP, it may not make sense to define the largest $\rho > 0$ and the largest $\nu > 0$ (simultaneously) such that (3) holds for $\forall u$ and $\forall T \geq 0$, since a large ρ corresponds to a small ν from the constraint $\rho\nu \leq \frac{1}{4}$. To get around this difficulty, we define the notion of passivity levels in the following consistent manner. Consider a system Σ,

- any $\tilde{\nu} \in (0, \nu]$ is a *passivity level* of Σ if Σ has IFP(ν);
- any $\tilde{\rho} \in (0, \rho]$ is a *passivity level* of Σ if Σ has OFP(ρ);
- any $(\tilde{\rho}, \tilde{\nu})$ are *passivity levels* of Σ if Σ is VSP for (ρ, ν) such that $0 < \tilde{\rho} \leq \rho, 0 < \tilde{\nu} \leq \nu$.

B. Model Reduction Preserving Passivity

Model reduction preserving passivity is an effective *approximation* technique when dealing with large-scale systems, such as power grid and circuit interconnect [10], [19]. We are mostly interested in truncated balancing realization (TBR) for model reduction that preserves passivity, so-called positive-real TBR (PR-TBR for short) [8], [10].

For linear time invariant system with transfer function $G(s)$, a state space realization is given as

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx + Du. \end{aligned} \tag{5}$$

We assume $\{A, B\}$ is controllable and $\{A, C\}$ is observable. The following result, namely the positive real lemma, is useful to test whether (5) is passive.

Lemma 1 ([6]): (5) is passive if and only if there exist matrices $P = P^T > 0, L, W$, such that

$$\begin{aligned} PA + A^T P &= -L^T L, \\ PB &= C^T - L^T W, \\ W^T W &= D + D^T. \end{aligned} \tag{6}$$

The dual equations of (6), obtained by setting $A \to A^T, B \to C^T, C \to B^T$, are given as

$$\begin{aligned} AX + XA^T &= -KK^T, \\ XC^T &= B - KJ^T, \\ JJ^T &= D + D^T, \end{aligned} \tag{7}$$

where $X = X^T \geq 0, K, J$ are the dual set of P, L, W.

The non-negative matrices P and X are used as the basis for the PR-TBR procedure (see Algorithm 1 in the Appendix). P and X are analogous to the observability grammian W_o and the controllability grammian W_c, where

$$AW_c + W_cA^T = -BB^T,$$
$$A^TW_o + W_oA = -CC^T.$$

W_c and W_o are the basis for TBR procedure but do not guarantee passivity of the reduced model in general [8], [10] except for the following special case. The eigenvalues of the product W_cW_o are called Hankel singular values and are used to establish upper bounds on the error between the transfer functions of the full-order system (denoted by G) and its reduced-order approximation (denoted by G_a). If we denote σ_i as the ith Hankel singular values ($\sigma_1 \geq \sigma_2 \geq ...\sigma_n \geq 0$, and n is the order of G), then we obtain

$$\|G - G_a\|_{H_\infty} \leq 2\sum_{j=r+1}^{n} \sigma_i,$$

where $0 \leq r < n$ is the order of the reduced-order approximation G_a. It is obvious that the larger the order r is, the smaller the error is.

A special case of (5) is of the relaxation type, i.e.

$$A = A^T, A \leq 0, B^T = C, D \geq 0. \tag{8}$$

Relaxation systems [10], [17] play an important role in applications; examples include integrated circuits and mechanical systems in which inertial effects may be neglected. It can be verified that $P = I$ is a solution to (6), i.e. $V(x) = \frac{1}{2}x^Tx$ is a storage function for (8), where

$$\begin{aligned}\dot{V}(x) - u^Ty &= x^T(Ax + Bu) - u^T(Cx + Du)\\ &= x^TAx - u^TDu\\ &\leq -u^TDu \leq 0.\end{aligned}$$

Therefore, the system (8) is *passive*. If $D > 0$, the above inequality actually shows that the system is ISP for

$$\rho \geq \underline{\lambda}(D) > 0.$$

Furthermore, the reduced model of (8) obtained through Algorithm 1 will also be ISP for $\tilde{\rho} \geq \underline{\lambda}(D) > 0$.

Remark 1: Positive real balancing for *nonlinear* systems has been studied in [20]. Besides, there exist various approaches to reduce model order, but we do not concentrate on that problem. Model reduction of linear systems are used merely as 'examples' to illustrate our main results in Section IV.

III. Problem Statement

Consider two system models Σ_1 and Σ_2 as shown in Fig. 1. One can view Σ_i as the system we are interested in as it describes some behavior of interest and Σ_j as an *approximation* of Σ_i, where $i, j \in \{1, 2\}$ and $j \neq i$. A commonly used measure for judging how well Σ_j approximates Σ_i is to compare the outputs for the same excitation function u [5]. We denote the difference in the outputs by Δy. The error may be due to modeling, linearization or model reduction, etc. For a 'good' approximation, we require that the "worst" case Δy over all control inputs u be small. Thus, Σ_j is a *good* approximation of Σ_i if there exists a positive constant $\gamma > 0$ such that

$$\|\Delta y\|_T \leq \gamma\|u\|_T, \quad \forall u \text{ and } \forall T \geq 0. \tag{9}$$

The value of γ obviously constrains how good the approximation is. In the following analysis, without loss of generality, we view Σ_2 as an approximation of Σ_1.

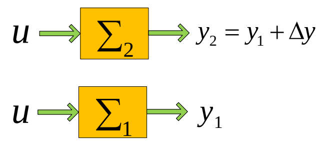

Fig. 1. Illustration of two systems: Σ_1 with input u and output y_1 and Σ_2 with input u and output $y_2 = y_1 + \Delta y$.

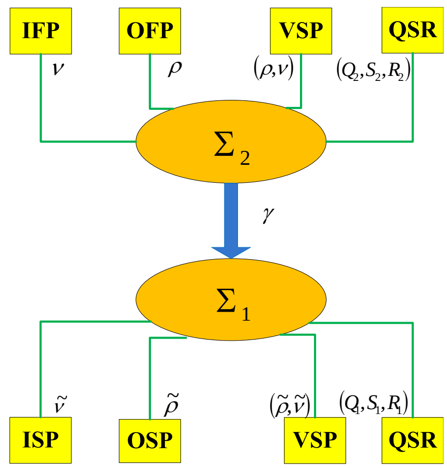

Fig. 2. Problem Statement

Remark 2: For stable linear systems, Σ_1 (resp. Σ_2) is characterized by the transfer function G_1 (resp. G_2). Defining $\Delta G = G_1 - G_2$, we obtain from (9) that

$$\|\Delta G\|_{H_\infty} \leq \gamma.$$

In this case, γ is an upper bound on the H_∞ norm of the error in the transfer functions G_1 and G_2.

We are now ready to state the problem of interest (see Fig. 2). Assume that Σ_2 has an *excess* of passivity, namely Σ_2 has IFP(ν) or OFP(ρ) or is VSP for (ρ, ν). What passivity property for Σ_1 can be *inferred* from that of Σ_2? For the case when Σ_2 does not have an *excess* of passivity, we assume it to be (Q_2, S_2, R_2)-dissipative and consider the problem of obtaining conditions under which Σ_1 is (Q_1, S_1, R_1)-dissipative as well. The problem is summarized as follows.

Problem 1: Suppose that an approximate model Σ_2

- has IFP(ν); or
- has OFP(ρ); or
- is VSP for (ρ, ν); or
- is (Q_2, S_2, R_2)-dissipative.

The aim is to derive the corresponding passivity property for the system Σ_1 based on conditions on γ in (9), such that

1) Σ_1 has ISP level $(\tilde{\nu})$; or
2) Σ_1 has OSP level $(\tilde{\rho})$; or
3) Σ_1 is VSP for $(\tilde{\rho}, \tilde{\nu})$; or
4) Σ_1 is (Q_1, S_1, R_1)-dissipative.

IV. Main Results

In this section, we present our main results. We begin by considering the case when the approximate model is ISP and then move on to the cases when the approximation is OSP, VSP or QSR-dissipative.

A. *Input Strictly Passive Systems*

We have the following result that guarantees a certain passivity level given the error constraint γ and an IFP level for the approximate model.

Theorem 1: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (9) is satisfied for some $\gamma > 0$. If Σ_2 has IFP(ν) and $\gamma < \nu$, then Σ_1 will be ISP for $\tilde{\nu} = \nu - \gamma$.

Proof: From Cauchy-Schwarz inequality and the assumption (9), we obtain

$$|\langle u, \Delta y\rangle_T| \leq \sqrt{\langle u, u\rangle_T}\sqrt{\langle \Delta y, \Delta y\rangle_T} \leq \gamma\langle u, u\rangle_T, \tag{10}$$

For the system Σ_2 with input u and output y_2, we have

$$\begin{aligned}
&\langle u, y_2\rangle_T - \nu\langle u, u\rangle_T \\
=&\langle u, y_1\rangle_T - \nu\langle u, u\rangle_T + \langle u, \Delta y\rangle_T \\
\leq&\langle u, y_1\rangle_T - \nu\langle u, u\rangle_T + |\langle u, \Delta y\rangle_T| \\
\leq&\langle u, y_1\rangle_T - (\nu - \gamma)\langle u, u\rangle_T.
\end{aligned}$$

Now, by assumption, Σ_2 is ISP for $\nu > 0$, then

$$\langle u, y_2\rangle_T - \nu\langle u, u\rangle_T \geq \beta.$$

Therefore, defining $\tilde{\nu} = \nu - \gamma > 0$, we obtain $\langle u, y_1\rangle_T - \tilde{\nu}\langle u, u\rangle_T \geq \beta$. This implies Σ_1 is ISP for $\tilde{\nu} > 0$. ■

Note $\tilde{\nu}$ does *not* represent the IFP of Σ_1 (Σ_1 may have IFP larger than $\tilde{\nu}$). By viewing Δy as model uncertainties that are not captured by the approximation Σ_2, the results can be interpreted as *robustness properties* [3]. The following result regarding robust passivity is *less* restrictive than Theorem 1.

Corollary 1: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (9) is satisfied for some $\gamma > 0$. If Σ_2 has IFP(ν) and $\gamma \leq \nu$, then, Σ_1 will be passive.

Proof: From (10) and $\gamma \leq \nu$, we obtain

$$|\langle u, \Delta y\rangle_T| \leq \gamma\langle u, u\rangle_T \leq \nu\langle u, u\rangle_T.$$

The following relation holds for Σ_1

$$\begin{aligned}
\langle u, y_1\rangle_T &= \langle u, y_2\rangle_T - \langle u, \Delta y\rangle_T \\
&\geq \langle u, y_2\rangle_T - |\langle u, \Delta y\rangle_T| \\
&\geq \langle u, y_2\rangle_T - \nu\langle u, u\rangle_T \geq \beta.
\end{aligned}$$

Therefore, $\langle u, y_1\rangle_T \geq \beta$, i.e. Σ_1 is passive. ■

B. *Output Strictly Passive Systems*

Computing OFP of a system is more difficult then IFP because of the feedback loops involved. For linear systems, we assume along the lines of [3] that Σ_2 is *minimum phase* so that the inverse of Σ_2, denoted by Σ_2^{-1}, is causal and stable (i.e. all the poles of Σ_2^{-1} are with negative real parts).

Assumption 1: Consider Σ_2 with input u and output y_2. Assume the inverse of Σ_2 is causal and stable, i.e. there exist $\eta > 0$, such that $\forall y_2$, $\forall T \geq 0$ [16]

$$\|u\|_T \leq \eta\|y_2\|_T. \tag{11}$$

Note that Assumption 1 is not necessary, however, OFP can be conveniently computed in this way. For linear systems, the OFP for $G(s)$ is shown to be equivalent to the IFP of the inverse of $G(s)$, denoted by $G^{-1}(s)$.

Theorem 2: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (9) holds for some $\gamma > 0$ and (11) holds for some $\eta > 0$. If Σ_2 has OFP(ρ) and $\gamma < \rho$, then Σ_1 will be OSP for $\tilde{\rho} = \rho - \gamma$ if

$$\frac{1}{\eta^2} - \left(1 + 2(\rho - \gamma)\frac{1}{\rho} + (\rho - \gamma)\gamma\right) \geq 0. \tag{12}$$

Proof: We use the relation from [6] that

$$u^T y_2 - \rho y_2^T y_2 \leq \frac{1}{2\rho} u^T u - \frac{\rho}{2} y_2^T y_2.$$

Σ_2 is assumed to be OSP for $\rho > 0$, thus

$$\frac{1}{2\rho}\langle u, u\rangle_T - \frac{\rho}{2}\langle y_2, y_2\rangle_T \geq \langle u, y_2\rangle_T - \rho\langle y_2, y_2\rangle_T \geq \beta,$$

and therefore $\langle y_2, y_2\rangle_T \leq \frac{1}{\rho^2}\langle u, u\rangle_T - \frac{2\beta}{\rho}$. From Cauchy-Schwarz inequality, (9) and the fact $\beta \leq 0$, we obtain

$$\begin{aligned} |\langle y_2, \Delta y\rangle_T| &\leq \sqrt{\langle \Delta y, \Delta y\rangle_T}\sqrt{\langle y_2, y_2\rangle_T} \\ &\leq \frac{\gamma}{\rho}\sqrt{\langle u, u\rangle_T}\sqrt{\langle u, u\rangle_T - 2\beta\rho} \\ &\leq \frac{\gamma}{\rho}\left(\langle u, u\rangle_T - 2\beta\rho\right) = \frac{\gamma}{\rho}\langle u, u\rangle_T - 2\beta\gamma. \end{aligned} \tag{13}$$

Together with (10), if we define $a \triangleq \rho - \gamma > 0$, we obtain

$$\begin{aligned} \Phi \triangleq & \gamma\langle y_2, y_2\rangle_T - \langle u, \Delta y\rangle_T + 2a\langle \Delta y, y\rangle_T - a\langle \Delta y, \Delta y\rangle_T \\ \geq & \gamma\langle y_2, y_2\rangle_T - |\langle u, \Delta y\rangle_T| - 2a|\langle \Delta y, y_2\rangle_T| - a\gamma^2\langle u, u\rangle_T \\ \geq & \gamma\langle y_2, y_2\rangle_T - \left(\gamma + 2a\frac{\gamma}{\rho} + a\gamma^2\right)\langle u, u\rangle_T + 4a\beta\gamma. \end{aligned}$$

If (12) is satisfied, from assumption (11), we obtain

$$\begin{aligned} &\gamma\langle y_2, y_2\rangle_T - \left(\gamma + 2a\frac{\gamma}{\rho} + a\gamma^2\right)\langle u, u\rangle_T \\ &\geq \left[\frac{1}{\eta^2} - \left(1 + 2a\frac{1}{\rho} + a\gamma\right)\right]\gamma\eta^2\langle y_2, y_2\rangle_T \geq 0. \end{aligned}$$

Thus, $\Phi \geq 4a\beta\gamma$. For Σ_1 with $y_1 = y_2 - \Delta y$,

$$\begin{aligned} &\langle u, y_1\rangle_T - (\rho - \gamma)\langle y_1, y_1\rangle_T \\ =&\langle u, y_2\rangle_T - \rho\langle y_2, y_2\rangle_T + \Phi \geq \beta + 4a\beta\gamma \triangleq \bar{\beta}, \end{aligned}$$

for all functions u, all $T \geq 0$ and $\bar{\beta} \leq 0$. Therefore, for $\gamma < \rho$, Σ is OSP for $\tilde{\rho} = \rho - \gamma$. ■

Note that Σ may have OFP larger than $\tilde{\rho}$. The following result is immediate regarding *robust passivity*.

Corollary 2: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (9) holds for some $\gamma > 0$ and (11) holds for some $\eta > 0$. If Σ_2 has OFP(ρ) and $\gamma\eta^2 \leq \rho$, then, Σ_1 will be passive.

Proof: From (10) and the assumption (11), we obtain

$$|\langle u, \Delta y\rangle_T| \leq \gamma\langle u, u\rangle_T \leq \gamma\eta^2\langle y_2, y_2\rangle_T.$$

Thus, the following relation holds if $\gamma\eta^2 \leq \rho$,

$$\begin{aligned} \langle u, y_1\rangle_T &= \langle u, y_2\rangle_T - \langle u, \Delta y\rangle_T \\ &\geq \langle u, y_2\rangle_T - \rho\langle y_2, y_2\rangle_T - |\langle u, \Delta y\rangle_T| + \rho\langle y_2, y_2\rangle_T \\ &\geq \beta + (\rho - \gamma\eta^2)\langle y_2, y_2\rangle_T \geq \beta. \end{aligned}$$

Therefore, $\langle u, y_1\rangle_T \geq \beta$, i.e. Σ_1 is passive. ■

C. Very Strictly Passive Systems

We have the following result.

Theorem 3: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (9) holds for some $\gamma > 0$. Suppose Σ_2 is VSP for (ρ, ν), where $\rho > \gamma, \nu > \gamma$. Then, Σ_1 is VSP for $(\rho - \gamma, \nu - \gamma)$ if

$$\nu^2 - \frac{2(\rho-\gamma)}{\rho} - (\rho-\gamma)\gamma \geq 0. \tag{14}$$

Proof: We use the relation $u^T y_2 - \nu u^T u \leq \frac{1}{2\nu} y_2^T y_2 - \frac{\nu}{2} u^T u$. Σ_2 is assumed to be ISP for $\nu > 0$, thus

$$\frac{1}{2\nu}\langle y_2, y_2\rangle_T - \frac{\nu}{2}\langle u, u\rangle_T \geq \langle u, y_2\rangle_T - \nu\langle u, u\rangle_T \geq \beta,$$

and therefore $\langle y_2, y_2\rangle_T \geq \nu^2\langle u, u\rangle_T + 2\beta\nu$. Also, Σ_2 is OSP for $\rho > 0$, thus (13) is satisfied. Together with (9) and (10), if we define $a = \rho - \gamma > 0, \psi = 2a\langle y, \Delta y\rangle_T - \langle u, \Delta y\rangle_T - a\langle \Delta y, \Delta y\rangle_T$, we obtain

$$\begin{aligned}|\psi| &= |\langle u, \Delta y\rangle_T| + 2a|\langle y, \Delta y\rangle_T| + a\langle \Delta y, \Delta y\rangle_T \\ &\leq \left(\gamma + 2a\frac{\gamma}{\rho} + a\gamma^2\right)\langle u, u\rangle_T - 4a\beta\gamma.\end{aligned}$$

Thus, the following relation holds

$$\begin{aligned}&\gamma\langle u, u\rangle_T + \gamma\langle y_2, y_2\rangle_T + \psi \\ \geq&\gamma(1+\nu^2)\langle u, u\rangle_T + 2\beta\nu\gamma - |\psi| \\ \geq&\left[\gamma(1+\nu^2) - (\gamma + 2a\frac{\gamma}{\rho} + a\gamma^2)\right]\langle u, u\rangle_T + 2\beta\nu\gamma + 4a\beta\gamma \\ =&\gamma\left(\nu^2 - \frac{2a}{\rho} - a\gamma\right)\langle u, u\rangle_T + 2\beta\nu\gamma + 4a\beta\gamma.\end{aligned}$$

We assume that $\nu^2 - \frac{2a}{\rho} - a\gamma \geq 0$ from (14), thus

$$\gamma\langle u, u\rangle_T + \gamma\langle y_2, y_2\rangle_T + \psi \geq 2\beta\nu\gamma + 4a\beta\gamma.$$

For Σ_1 with input u and output $y_1 = y_2 - \Delta y$, we have

$$\begin{aligned}&\langle u, y_1\rangle_T - (\nu-\gamma)\langle u, u\rangle_T - (\rho-\gamma)\langle y_1, y_1\rangle_T \\ =&\langle u, y_2\rangle_T - \nu\langle u, u\rangle_T - \rho\langle y_2, y_2\rangle_T \\ &+ \gamma\langle u, u\rangle_T + \gamma\langle y_2, y_2\rangle_T + \psi \\ \geq&\langle u, y_2\rangle_T - \nu\langle u, u\rangle_T - \rho\langle y_2, y_2\rangle_T + 2\beta\nu\gamma + 4a\beta\gamma.\end{aligned}$$

Σ_2 is assumed to be VSP for (ρ, ν) and therefore

$$\langle u, y_2\rangle_T - \nu\langle u, u\rangle_T - \rho\langle y_2, y_2\rangle_T \geq \beta.$$

Defining $\bar{\beta} = \beta + 2\beta\nu\gamma + 4a\beta\gamma \leq 0$, we have

$$\langle u, y_1\rangle_T - (\nu-\gamma)\langle u, u\rangle_T - (\rho-\gamma)\langle y_1, y_1\rangle_T \geq \bar{\beta}.$$

Thus, for $\gamma < \rho, \gamma < \nu$, Σ_1 is VSP for $(\rho-\gamma, \nu-\gamma)$. ■

Σ_1 is VSP for (ρ, ν) implies that ρ is a passivity level for OSP and ν is a passivity level for ISP. The OFP for Σ_1 is larger than ρ and the IFP is larger than ν in general.

Corollary 3: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (9) holds for some $\gamma > 0$. If Σ_2 is VSP for (ρ, ν) and $\rho\nu^2 + \nu - \gamma \geq 0$, then, Σ_1 will be passive.

Proof: Σ_2 is ISP for ν, it has been shown that $\langle y_2, y_2\rangle_T \geq \nu^2\langle u, u\rangle_T + 2\beta\nu$. From (10), we obtain

$$\begin{aligned}\chi \triangleq & -|\langle u, \Delta y\rangle_T| + \rho\langle y_2, y_2\rangle_T + \nu\langle u, u\rangle_T \\ & \geq (\rho\nu^2 + \nu - \gamma)\langle u, u\rangle_T + 2\beta\rho\nu.\end{aligned}$$

Thus, if $\rho\nu^2+\nu-\gamma \geq 0$, we obtain $\chi \geq 2\beta\rho\nu$. Σ_2 is VSP for (ρ,ν), thus $\langle u, y_2\rangle_T - \rho\langle y_2, y_2\rangle_T - \nu\langle u, u\rangle_T \geq \beta$. For Σ_1 with input u and output y_1, we have

$$\begin{aligned}\langle u, y_1\rangle_T &= \langle u, y_2\rangle_T - \langle u, \Delta y\rangle_T \\ &\geq \langle u, y_2\rangle_T - \rho\langle y_2, y_2\rangle_T - \nu\langle u, u\rangle_T + \chi \\ &\geq \beta + 2\beta\rho\nu \triangleq \bar{\beta}.\end{aligned}$$

Thus, $\langle u, y_1\rangle_T \geq \bar{\beta}$ and $\bar{\beta} \leq 0$, i.e. Σ_1 is passive. ■

Remark 3: It can be verified that the above results hold when Σ_1 and Σ_2 exchange places. In other words, it does not really matter whether we view Σ_1 as an approximation of Σ_2 or Σ_2 as an approximation of Σ_1. In practice, however, a simple model is usually used as an approximation of a complex system, e.g. linearized model vs. nonlinear model and lower-order model vs. higher-order model.

Remark 4: Theorem 1-3 relate passivity levels between Σ_1 and Σ_2 for ISP, OSP and VSP systems. It is worth stressing that these results are applicable to *any* approximation methods and *any* system structure in general. In particular, if we consider *linear* systems and *PR-TBR* as a particular approximation approach, then the error γ in (9) is characterized by the Hankel singular values, and the results in Theorem 1-3 provide a tool to trade off the error as a function of *variations in the passivity levels* for the full-order system Σ_1 (or Σ_2) and the reduced-order system Σ_2 (or Σ_1).

D. Extension to QSR-dissipative Systems

In this section, we extend the results to QSR-dissipative systems, for which the system may be not passive or have a *shortage* of passivity.

Theorem 4: Consider Σ_1 and Σ_2 in Fig. 1. Suppose (9) holds for some $\gamma > 0$. Let Σ_2 be (Q_2, S_2, R_2)-dissipative and assume $S_1 - S_2 = 0, Q_1 - Q_2 > 0, R_1 - R_2 > 0$. If there exists a $\xi > 0$ such that

$$\begin{aligned}\underline{\lambda}(R_1 - R_2) - \frac{\gamma^2}{\xi} - 2\lambda_1\gamma - b \geq 0,\\ \underline{\lambda}(Q_1 - Q_2) - \xi\lambda_2 \geq 0,\end{aligned} \tag{15}$$

where $b = 2\max\{0, \overline{\lambda}(-Q_1)\gamma^2\}$, and

$$\lambda_1 \triangleq \sqrt{\overline{\lambda}(S_1^T S_1)} \geq 0, \lambda_2 \triangleq \sqrt{\overline{\lambda}(Q_1^T Q_1)} \geq 0,$$

then Σ_1 is (Q_1, S_1, R_1)-dissipative.

Proof: From Cauchy-Schwarz inequality, we obtain

$$|\langle S_1 u, \Delta y\rangle_T| \leq \sqrt{\overline{\lambda}(S_1^T S_1)}\gamma\langle u, u\rangle_T = \lambda_1\gamma\langle u, u\rangle_T.$$

Also, for some $\xi > 0$, the following relation holds

$$2\langle Q_1 y_2, \Delta y\rangle_T \leq \frac{\gamma^2}{\xi}\langle u, u\rangle_T + \xi\lambda_2\langle y_2, y_2\rangle_T.$$

Define the supply rate for Σ_i as $r_i(u, y_i) \triangleq \langle y_i, Q_i y_i\rangle_T + 2\langle y_i, S_i u\rangle_T + \langle u, R_i u\rangle_T$, then

$$\begin{aligned}r_1 =& r_2 + \langle y_2, (Q_1 - Q_2)y_2\rangle_T + \langle u, (R_1 - R_2)u\rangle_T \\ & - 2\langle y_2, Q_1\Delta y\rangle_T + \langle \Delta y, Q_1\Delta y\rangle_T - 2\langle \Delta y, S_1 u\rangle_T \\ \geq & r_2 + (\underline{\lambda}(Q_1 - Q_2) - \xi\lambda_2)\,\|y_2\|_T^2 \\ & + \left(\underline{\lambda}(R_1 - R_2) - \frac{\gamma^2}{\xi} - 2\lambda_1\gamma\right)\|u\|_T^2 + \langle \Delta y, Q_1\Delta y\rangle_T.\end{aligned}$$

Since Σ_2 is (Q_2, S_2, R_2)-dissipative, $r_2 \geq 0$. Two cases are possible. If $Q_1 > 0$, we have $b = 0, \langle \Delta y, Q_1 \Delta y\rangle_T \geq 0$. Thus, from (15), we obtain $r_1 \geq r_2 \geq 0$. If $Q_1 \leq 0$, we have $b = \overline{\lambda}(-Q_1)\gamma^2$ and from (9),

$$\langle \Delta y, Q_1 \Delta y\rangle_T \geq -\overline{\lambda}(-Q_1)\gamma^2 \langle u, u\rangle_T.$$

If (15) holds, we obtain $r_1 \geq r_2 \geq 0$. In summary, $r_1 \geq 0$ if (15) is satisfied and thus Σ_1 is (Q_1, S_1, R_1)-dissipative. ■

Remark 5: (1). Similar arguments can be developed when $S_1 - S_2 = 0, Q_1 - Q_2 > 0, R_1 - R_2 > 0$ does not hold. However, the analysis is more involved. (2). When $S_i = 1/2I$, $Q_i > 0$ or $R_i > 0$ $(i = 1, 2)$ indicates the system has a shortage of passivity.

V. Discussions in the Discrete-time Setting

In this section, we consider the same problem (i.e. Problem 1) in the discrete-time domain. In this case, the signal space under consideration is ℓ_2 space or the extended ℓ_2 space. The set of time instants is $\mathbb{Z} = \{0, 1, 2, ...\}$. The inner product of truncated signals $u_T(k), y_T(k)$ is defined as $\langle u, y\rangle_T \triangleq \sum_0^T u^T(k)y(k)$ where $0 \leq T < \infty$. The ℓ_2-induced norm of a signal u is denoted by $\|u\|_T$, where $\|u\|_T^2 \triangleq \sum_0^T u^T(k)u(k)$.

The definitions of passivity in the discrete-time domain can be found in e.g. [21], [22], [23]. In fact, we can apply Definition 1 as well if we use *the time instant* k and *the inner product introduced above*. Analogously, we can define *passivity indices* and *passivity levels* of a discrete-time system Σ as in the continuous-time domain.

To study Problem 1 in the discrete-time setting, similar arguments in the continuous-time domain can be developed. In fact, the results derived in this paper (Lemma 2-3, Theorem 1-4 and Corollary 1-3), apply to discrete-time domain. The only difference is that in discrete-time setting, the time instants are integers and the inner product is defined as summations.

VI. Numerical Examples

In this section, we consider numerical examples to illustrate our results. In the following examples, Σ_1 is considered to be a linear system of relaxation type (denoted by G) and Σ_2 is an approximation of Σ_1 (denoted by G_a) obtained from the PR-TBR procedure (e.g. in [10]).

Example 1 (ISP): Consider the following relaxation system

$$\dot{x} = \begin{pmatrix} -1.62 & -1.522 \\ -1.522 & -4.18 \end{pmatrix} x + \begin{pmatrix} -3.876 \\ -2.01 \end{pmatrix} u,$$
$$y = \begin{pmatrix} -3.876 & -2.01 \end{pmatrix} x + 0.5u,$$

which is a minimal realization of

$$G_a(s) = \frac{0.5s^2 + 21.96s + 47.85}{s^2 + 5.8s + 4.456}.$$

This second-order system is obtained from the PR-TBR procedure (see Algorithm 1). We have shown that $G_a(s)$ is ISP for $\rho \geq D = 0.5$. In fact, the IFP(ρ) for G_a (defined in [3]) can be computed as

$$\rho = \min_{w \in \mathbb{R}} \text{Re}[G_a(jw)] = 0.5.$$

The original system $G(s)$ given by (16) is of order 8. The Hankel singular values, i.e. the eigenvalues of the product W_cW_o, are given by Λ in (17) and ordered as $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_8$. Therefore, we have [10]

$$\|G_r - G_a\|_{H_\infty} \leq 2 \sum_{k=3}^{8} \sigma_k = 0.0803.$$

$$G^1 = \frac{0.5s^8 + 28.6s^7 + 352.2s^6 + 1887s^5 + 5299s^4 + 8295s^3 + 7190s^2 + 3173s + 542.9}{s^8 + 18.5s^7 + 133.5s^6 + 496.1s^5 + 1047s^4 + 1290s^3 + 911.1s^2 + 337.5s + 50.18} \tag{16}$$

$$\Lambda = \text{diag}\{4.6357, 0.4834, 0.0375, 0.0023, 3.5 \times 10^{-4}, 1.9 \times 10^{-5}, 0, 0\}. \tag{17}$$

$$A = \begin{pmatrix} -5 & 0.1 & 1.2 & 0 & 0 & 1 \\ 0.1 & -3 & 0 & -0.3 & 0 & -1 \\ 1.2 & 0 & -6 & -2 & 0.5 & -2 \\ 0 & -0.3 & -2 & -4 & 0.4 & 0.5 \\ 0 & 0 & 0.5 & 0.4 & -4 & -0.8 \\ 1 & -1 & -2 & 0.5 & -0.8 & -8 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \\ 2 \\ 0.8 \end{pmatrix}, C = B^T, D = 2. \tag{18}$$

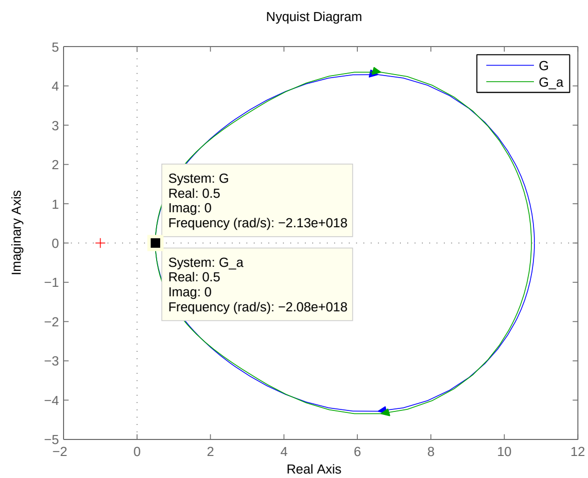

Fig. 3. The Nyquist Plots of G and G_a in Example 1.

Thus, γ in (9) is given by $\gamma = 0.0803 < 0.5$. According to Theorem 1, Σ_1 (G) is input strictly passive for

$$\tilde{\nu} = \nu - \gamma = 0.5 - 0.0803 = 0.4197.$$

This is true because the passivity index for $G_r(s)$ is actually 0.5, which is greater than $\tilde{\nu} = 0.4197$.

The Nyquist plots of G and G_a are given in Figure 3. Figure 3 demonstrates the second-order system $G_a(s)$ approximates the real system $G(s)$ very well and the IFP for the two systems are both 0.5. If we use a forth-order approximate model, $\gamma = 8 \times 10^{-4}, \nu = 0.5$. Thus, the error in the transfer function is upper bounded by 8×10^{-4}. Besides, the passivity level for G is then given by $\nu - \gamma = 0.5 - 8 \times 10^{-4}$, very close to its passivity index 0.5.

Example 2 (OSP): Consider the following system

$$G_a(s) = \frac{1.8s + 19.37}{s + 4.132},$$

which is obtained from the PR-TBR algorithm. It is obvious that $G_a^{-1}(s)$ exits and stable. Also, we have

$$\eta = \|G_a^{-1}(s)\|_{H_\infty} = 0.5556,$$
$$\nu = \min_{w \in \mathbb{R}} \text{Re}[G_a^{-1}(jw)] = 0.213.$$

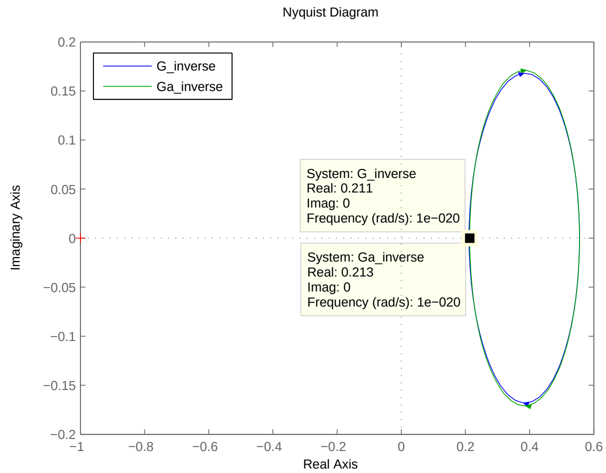

Fig. 4. The Nyquist Plots of G^{-1} and of G_a^{-1} in Example 2.

The real system $G(s)$ is of order 5 and given through

$$\frac{1.8s^5 + 53.56s^4 + 590.8s^3 + 3034s^2 + 7279s + 6543}{s^5 + 23s^4 + 203.1s^3 + 861.7s^2 + 1759s + 1382},$$

and the error in the transfer function is given by the Hankle singular values σ_k (in a decreasing order), where

$$\|G - G_a\|_{H_\infty} \leq 2\sum_{k=2}^{5} \sigma_k = 0.0461.$$

For $\gamma = 0.0461 < \nu$, (12) holds because

$$\frac{1}{\eta^2} - \left(1 + 2(\nu - \gamma)\frac{1}{\nu} + (\nu - \gamma)\gamma\right) = 0.6695 > 0.$$

From Theorem 2, we can conclude that G is OSP for

$$\tilde{\nu} = \nu - \gamma = 0.213 - 0.0461 = 0.1669.$$

This is true because the OFP for G is given by 0.211, which is larger than $\tilde{\nu} = 0.1669$.

The Nyquist plots of G^{-1} and G_a^{-1} are given in Figure 2. From this figure, we can read the OFP indices: 0.213 for G_a and 0.211 for G, respectively. If a second-order approximation is used, we can obtain a smaller error in the transfer function with $\gamma = 0.0015$, and for which the passivity level for G is given by $\tilde{\nu} = 0.2095$ from Theorem 2, which is very close to the OFP for G (0.211).

Remark 6: For linear systems, a higher-order reduced model will result in smaller error in the transfer function and the passivity level, as indicated by Example 1 and 2. Therefore, there exists a tradeoff between how simple (i.e. small order) Σ_2 is and how accurate Σ_2 is.

Example 3 (VSP): The original system G is given by (4). Its second-order approximation is given by

$$G_a = \frac{2s^2 + 42.06s + 183.8}{s^2 + 11.22s + 26.79},$$

which is VSP for (ρ, ν), where $\nu = 1.2, \rho = 0.01$. This can be verified through $\Pi \leq 0$ [23], where Π is given by

$$\begin{bmatrix} A^TP + PA + \rho C^TC & PB - (1/2C^T - \rho C^TD) \\ B^TP - (1/2C - \rho D^TC) & \nu I + \rho D^TD - D \end{bmatrix},$$

with A, B, C, D as a minimal realization of G_a and $P = I$.

The error in G_a and G is given by $\gamma = 0.0042$. For our choice of ρ, ν, we obtain

$$\nu^2 - 2(\rho - \gamma)/\rho - (\rho - \gamma)\gamma = 0.2869 > 0,$$

therefore (14) is satisfied. According to Theorem 3, the original system G is VSP for $(\tilde{\rho}, \tilde{\nu})$, where

$$\tilde{\nu} = \nu - \gamma = 1.1958, \tilde{\rho} = \rho - \gamma = 0.0058.$$

This can also be verified through $\Pi \leq 0$ by setting $P = I$ and substituting $\tilde{\rho}, \tilde{\nu}$ for ρ, ν, respectively.

Example 4 (QSR): Consider a simple example, where the original system G is given by $C = B^T, D = -1$ and

$$A = \begin{pmatrix} -2 & 0.1 & 1.2 & 0 & 0 \\ 0.1 & -1 & 0 & -0.3 & 0 \\ 1.2 & 0 & -4 & -2 & 0.5 \\ 0 & -0.3 & -2 & -3 & 0.4 \\ 0 & 0 & 0.5 & 0.4 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}.$$

The reduced-order model G_a is obtained through the standard truncated balanced realization [10], for which W_c and W_o are the basis for transformation. G_a is given as

$$G_a(s) = \frac{-s^2 + 7.402s + 21.96}{s^2 + 3.485s + 2.139}.$$

It can be verified that G_a is (Q_2, S_2, R_2)-dissipative for $Q_2 = 0.1, R_2 = 1, S_2 = 0.5$. This can be done by testing $\Pi \leq 0$ with $P = 0.5I, \rho = -0.1, \nu = -1$.

The error in the transfer functions is given as $\gamma = 0.0318$. From the assumption that $Q_1 > Q_2 = 0.1$. Choose $\xi = 0.5, Q_1 = 0.2$, we obtain $Q_1 - Q_2 - \xi Q_1 = 0$. Also, $b = 0$ for this example, we can choose $R_1 > R_2 + 2\gamma^2 + \gamma = 1.0338$ from (15), for instance, $R_1 = 1.1$. According to Theorem 4, G is (Q_1, S_1, R_1)-dissipative, where $Q_1 = 0.2, S_1 = 0.5, R_1 = 1.1$. Again, this can be verified through $\Pi \leq 0$ by setting $P = 0.5I, \rho = -0.2, \nu = -1.1$.

Example 5 (Sector Nonlinearity): Consider a feedback connection as shown in Figure 5, represented by a linear system and a feedback loop containing a memoryless nonlinearity [6], [15]. This connection is often used in absolute stability analysis. Here, we are more interested in passivity of the closed-loop system Σ_1 with input u and output y_1. We use the linear system $G_a(s)$ with input u and output y_2 as an approximation of Σ_1. The simulink model for the two system models is built in Fig. 6.

The linear system is given by

$$G_a(s) = \frac{2s^2 + 9.04s + 8.48}{s^2 + 4s + 3}.$$

The difference of the outputs for the same input function $u(t) = \cos(t) + 2$ is shown in Fig. 7. The error γ is upper bounded by 0.3. One can verify that the conditions in Corollary 1-3 are satisfied. Thus, the nonlinear system Σ_1 is passive as well. If we plot the product of $u^T y$, we can see from Fig. 8 that $u^T y \geq 0$ for all time t. Therefore, the system Σ_1 is passive from Definition 1. (One can verify the results for other choices of input as well.)

VII. Concluding Remarks

In this paper, we established conditions under which the passivity properties of a system can be obtained by analyzing its approximation. The approximate model is assumed to be input/output/very strictly passive and the results are of the form that if the error between the system and its approximation is small, the original system has a guaranteed passivity level. The analysis is extended to a general case when the approximation is QSR dissipative (not necessarily passive). The results may be interpreted as robustness properties of passivity with respect to model uncertainties. It has also been shown that our results can be used to derive variations in the passivity levels of a linear system and its reduced-order approximation.

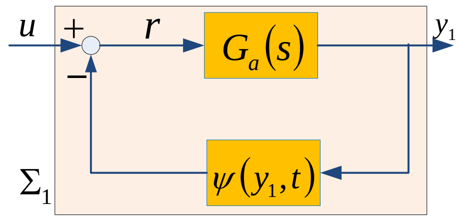

Fig. 5. Feedback connection

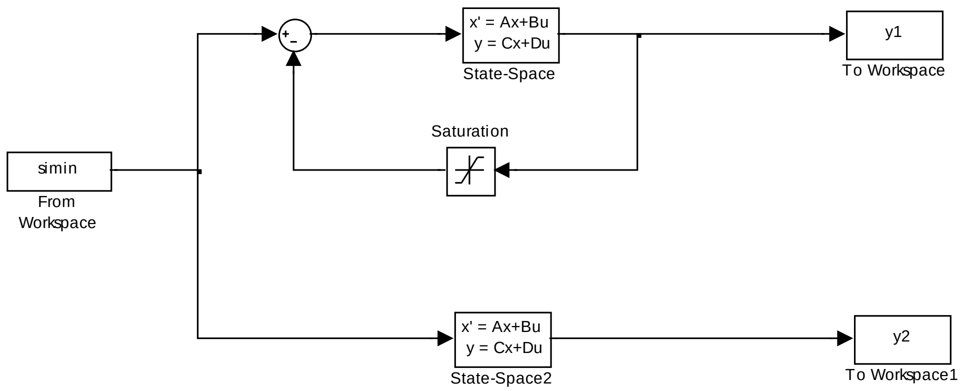

Fig. 6. Simulink Model for Example 5.

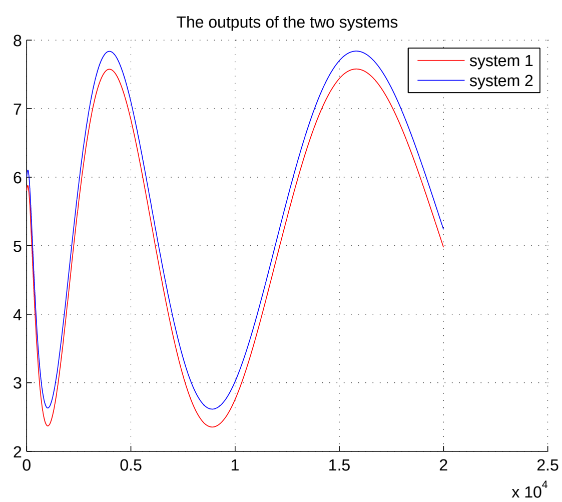

Fig. 7. The outputs of the two systems for the same control input u in Example 5.

VIII. Acknowledgment

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IX. Appendix

Lemma 2: If a system is VSP for (ρ, ν), then for any $0 \leq \epsilon < \min\{\rho, \nu\}$, it is also VSP for $(\rho - \epsilon, \nu - \epsilon)$.

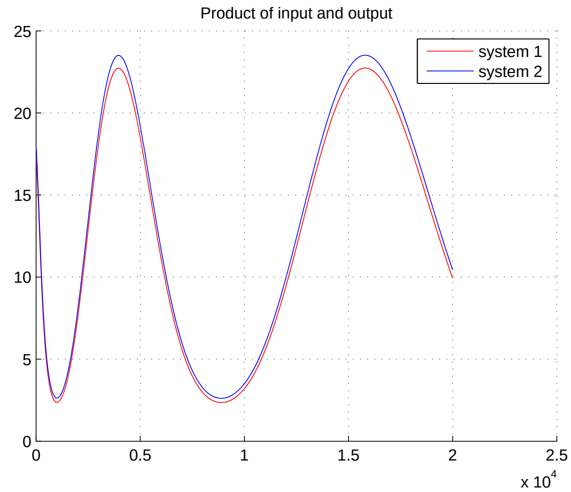

Fig. 8. The product of y_1 and control input u in Example 5.

Proof: First note that for $\epsilon \geq 0$,

$$\epsilon\langle u, u\rangle_T \geq 0, \epsilon\langle y, y\rangle_T \geq 0.$$

Therefore, we have the following relation

$$\begin{aligned}&\langle u, y\rangle_T - (\rho - \epsilon)\langle u, u\rangle_T - (\nu - \epsilon)\langle y, y\rangle_T \\ &= \langle u, y\rangle_T - \rho\langle u, u\rangle_T - \nu\langle y, y\rangle_T + \epsilon\langle u, u\rangle_T + \epsilon\langle y, y\rangle_T \\ &\geq \langle u, y\rangle_T - \rho\langle u, u\rangle_T - \nu\langle y, y\rangle_T.\end{aligned}$$

Next, from the definition for VSP systems, we obtain

$$\langle u, y\rangle_T - \rho\langle u, u\rangle_T - \nu\langle y, y\rangle_T \geq \beta.$$

Therefore, for $\epsilon < \min\{\rho, \nu\}$, the following relation holds,

$$\langle u, y\rangle_T - (\rho - \epsilon)\langle u, u\rangle_T - (\nu - \epsilon)\langle y, y\rangle_T \geq \beta,$$

thus the system is VSP for $(\rho - \epsilon, \nu - \epsilon)$. ■

The constraints on ρ and ν for Σ to be VSP are given through the following lemma. A similar problem is studied in [16] for QSR dissipative systems (4) where $Q = -\nu I, R = -\rho I, S = \delta I$. Their result is based on the eigenvalues of a dissipativity matrix, however, we use a different proof for the special case of VSP in this paper.

Lemma 3: If a system is VSP for (ρ, ν), where $\rho > 0, \nu > 0$, then ρ, ν satisfy $\rho\nu \leq \frac{1}{4}$.

Proof: It is equivalent to say, if $\rho\nu > \frac{1}{4}$, the system is not VSP for (ρ, ν). To see this, we use the following relation

$$(\sqrt{\rho}u - \sqrt{\nu}y)^T(\sqrt{\rho}u - \sqrt{\nu}y) \geq 0.$$

Therefore, for all u, all $T \geq 0$, we have

$$\rho\langle u, u\rangle_T + \nu\langle y, y\rangle_T - 2\sqrt{\rho\nu}\langle u, y\rangle_T \geq 0.$$

From the above inequality, we can derive that

$$\begin{aligned}&\langle u, y\rangle_T - \rho\langle u, u\rangle_T - \nu\langle y, y\rangle_T \\ &\leq \frac{1}{2\sqrt{\rho\nu}}\left(\rho\langle u, u\rangle_T + \nu\langle y, y\rangle_T\right) - \rho\langle u, u\rangle_T - \nu\langle y, y\rangle_T \\ &= \left(\frac{1}{2\sqrt{\rho\nu}} - 1\right)\left(\rho\langle u, u\rangle_T + \nu\langle y, y\rangle_T\right).\end{aligned}$$

If $\rho\nu > \frac{1}{4}$, then $\frac{1}{2\sqrt{\rho\nu}} - 1 < 0$, and thus $\forall u$, $\forall T \geq 0$,

$$\langle u, y\rangle_T - \rho\langle u, u\rangle_T - \nu\langle y, y\rangle_T \leq 0,$$

and the equality holds only for $u = 0, y = 0$. Therefore, the system *cannot* be VSP for (ρ, ν). ■

A PR-TBR procedure is given in [10] and shown in Algorithm 1 for completeness.

Algorithm 1 ([10]): PR-TBR

1) Solve (6) for P.
2) Solve (7) for X.
3) Compute Cholesky factors $P = L_1 L_1^T, X = L_2 L_2^T$.
4) Compute singular value decomposition of $U\Lambda V = L_1^T L_2$, where Λ is diagonal positive and U, V have orthonormal columns.
5) Compute the balancing transformations $T = L_2 V \Lambda^{-1/2}$ and $T^{-1} = \Lambda^{-1/2} U^T L_1^T$.
6) Form the balanced realization $\hat{A} = T^{-1}AT, \hat{B} = T^{-1}B, \hat{C} = CT$.
7) Select the reduced model order and partition $\hat{A}, \hat{B}, \hat{C}$ conformally.
8) Truncate $\hat{A}, \hat{B}, \hat{C}$ to form the reduced realization $\tilde{A}, \tilde{B}, \tilde{C}$.

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