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CONTROL SYSTEMS TECHNICAL NOTE

#55

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ON DYNAMIC LINEAR STATE FEEDBACK

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INTRODUCTION

The static or constant linear state feedback (clsf) control law is well understood in both state-space and differential operator (polynomial) system representations. The dynamic linear state feedback (dlsf) control law is not as well studied. Its representation and properties in the differential operator framework are discussed here apparently for the first time. Given a controllable system, it is shown that the denominator of the dlsf transfer matrix F can be almost arbitrarily chosen to guarantee controllability of the compensated system, while the numerator can be used to arbitrarily assign the closed loop eigenvalues. The question of characterizing right coprime proper and stable factorizations of the plant via dlsf is also studied, thus extending previous work involving clsf.

DYNAMIC STATE FEEDBACK

The dynamic linear state feedback (dlsf) control law is defined in the state space framework as follows:

Consider

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$. The control input u(t) is generated by:

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + B_c x(t) \\ u(t) &= C_c x_c(t) + E_c x(t) + Lr(t) \end{aligned} \tag{2}$$

where $A_c \in R^{q\times q}$, r(t) is an external input, $|L| \neq 0$ and the rest of the matrices are of appropriate dimensions. In the transform domain the dlsf is

$$u(s) = F_s(s)x(s) + Lr(s) \tag{3}$$

with

$$F_s(s){:} = C_c(sI-A_c)^{-1}B_c + E_c$$

a proper rational matrix.

Any dlsf control law is equivalent to a constant linear state feedback (clsf) applied to an extended system. To see this, combine (1) and (2) to obtain

$$\dot{x}_e(t) = A_e x_e(t) + B_e u(t) \tag{4a}$$

$$u(t) = F_{es} x_e(t) + Lr(t) \tag{4b}$$

where

$$x_e(t){:} = \begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}, \quad A_e{:} = \begin{bmatrix} A & 0 \\ B_c & A_c \end{bmatrix}, \quad B_e{:} = \begin{bmatrix} B \\ 0 \end{bmatrix}$$

and $F_{es}{:} = [E_c, C_c]$. It is clear that (4b) is a clsf control law applied to the extended system (4a).

We shall now define the dlsf in a differential operator system description framework. This can be done in terms of general polynomial matrix descriptions. It is desirable however to be able to conveniently shift, whenever necessary, to an equivalent state space representation as in the case of clsf (see [1], Structure Theorem). In this way, further insight is gained. For this, we shall assume controllability of the system and the controller.

Consider the system description ($\lambda{:} = d/dt$)

$$D(\lambda)z(t) = u(t) \tag{5}$$

(5) is controllable. Assume that (A,B) in (1) is also controllable and that

$$(\lambda I-A)S(\lambda) = BD(\lambda) \text{ with } x(t) = S(\lambda)z(t).$$

Note that this is satisfied for (A,B) in controllable companion form, $D(\lambda)$ column proper with column degrees d_i; and $S(\lambda){:} = \text{diag}[1, s \ldots, s^{d_i}]^T$ [1].

Define the dlsf law by

$$\begin{aligned} D_c(\lambda)z_c(t) &= S(\lambda)z(t) \\ u(t) &= N_c(\lambda)z_c(t) + Lr(t). \end{aligned} \tag{6}$$

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Assuming that (2) is controllable from $x(t)$, let its relation to (6) be:

$(\lambda I-A_c)S_c(\lambda) = B_cD_c(\lambda)$ with $x_c(t) = S_c(\lambda)z_c(t)$ and $N_c(\lambda) = E_cD_c(\lambda) + C_cS_c(s)$.

In the transform domain, the dlsf is

$$u(s) = F(s)z(t) + Lr(t) \tag{7}$$

where

$$F(s) = N_c(s)D_c(s)^{-1}S(s) = F_s(s)S(s)$$

thus establishing the relation with (3).

Combining (5) and (6):

$$D_e(\lambda)z_e(t) = \begin{bmatrix} I \\ 0 \end{bmatrix} u(t) \tag{8a}$$

$$u(t) = F_e(\lambda)z_e(t) + Lr(t) \tag{8b}$$

where

$$z_e: = \begin{bmatrix} z \\ z_c \end{bmatrix}, \qquad D_e: = \begin{bmatrix} D & 0 \\ -S & D_c \end{bmatrix}, \qquad F_e: = [\ 0 \quad N_c\ ].$$

(8b) indeed describes a well defined clsf control law. To see this, notice that

$$F_ez_e = N_cz_c = [\ E_c, C_c\ ] \begin{bmatrix} D_c \\ S_c \end{bmatrix} z_c = [\ E_c, C_c\ ] \begin{bmatrix} S & 0 \\ 0 & S_c \end{bmatrix} \begin{bmatrix} z \\ z_c \end{bmatrix}$$

and that $F_e(\lambda) = [\ E_cS(\lambda), C_cS_c(\lambda)\ ]$ has column degrees strictly less than the corresponding column degrees of $D_e(\lambda)$ [1]. This concludes the definition of the dlsf control law in both state-space and differential operator framework. Notice that the given system is described by (1) or (5), the dlsf control law by (2) or (6), also by (3) or (7), and the equivalent clsf is given by (4) or (8).

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Controllability, Observability and Eigenvalue Assignment

To arbitrarily assign all closed loop eigenvalues using dlsf, the extended system (4a) must be controllable. This is true if and only if

$$\text{rank}\begin{bmatrix} s_iI-A & 0 & B \\ -B_c & s_iI-A_c & 0 \end{bmatrix} = n+q\ , \quad \text{for all } s_i \tag{9}$$

It can be easily seen that (A,B) and (A_c,B_c) controllable are necessary conditions. For sufficiency, additional conditions dependent on the relation between A, B and A_c, B_c are needed. They can perhaps be easier seen by considering (8a) where controllability of the plant and the controller has already been assumed. Here, (8a) is controllable if and only if

$$\text{rank}\begin{bmatrix} D(s_i) & 0 & I \\ -S(s_i) & D_c(s_i) & 0 \end{bmatrix} = m+n\ , \quad \text{for all } s_i \tag{10}$$

or if and only if $S(s)$, $D_c(s)$ are left prime i.e if and only if (6) is controllable from $z(t)$.

Given (A,B) controllable, a controllable pair (A_c,B_c) can be easily found so that the extended system is fully controllable; actually almost any such pair will suffice. Then the clsf $F_{es} = [\ E_c, C_c\ ]$ in (4) can be chosen to arbitrarily assign the (q+n) closed loop eigenvalues. In the transform domain, this translates into the fact that the denominator $D_c(s)$ of $F_s(s)$ in (3) (or $F(s)$ in (7)) can be almost arbitrarily chosen (as long as $S(s)$, $D_c(s)$ are left prime); then the numerator $N_c(s)$ can always be appropriately selected for arbitrary eigenvalue assignment, for instance. In the cases when (A_c,B_c) (or $D_c(s)$) are given and the extended system is not fully controllable, only the controllable eigenvalues can be arbitrarily assigned; the remaining uncontrollable eigenvalues are fixed and they can be found from (9) and (10) above. Consider the output, to (4), equation:

$$\begin{aligned} y(t) &= Cx(t) + Eu(t) \\ &= C_e x_e(t) + Eu(t) \end{aligned} \tag{11}$$

where $C \varepsilon R^{p\times n}$, $E \varepsilon R^{p\times m}$ and $C_e := [\ C,0\ ]$. It is clear that (A_e,C_e) is unobservable; all the eigenvalues of A_c generate unobservable modes which cannot be seen from the output y. Similarly in (8), if

$$y(t) = N(\lambda)z(s) = N_e(\lambda)z_e(t) \tag{12}$$

where $N_e := [\ N,0\ ]$, $\left(D_e(\lambda),N_e(\lambda)\right)$ is not right coprime and the zeros of $|D_c(\lambda)|(=\alpha|\lambda I-A_c|)$ appear as unobservable modes from y.

The closed loop internal description is

$$\begin{bmatrix} D & -N_c \\ -S & D_c \end{bmatrix}\begin{bmatrix} z \\ z_c \end{bmatrix} = \begin{bmatrix} L \\ 0 \end{bmatrix} r, \qquad y = [\ N,0\ ]\begin{bmatrix} z \\ z_c \end{bmatrix} \tag{13}$$

using (8); and the closed loop eigenvalues are the zeros of $|D|\ |D_c-SD^{-1}N_c|$ or of $|(sI-A)D_c(s)-BN_c(s)|$. Clearly $F_s = N_cD_c^{-1}$ can be determined by solving a Diophantine equation; in this way the eigenvalue assignment problem can be solved. Alternatively, (A_c,B_c) is first chosen for full controllability of (4a) and then the clsf $F_{es} = [\ E_c,C_c\ ]$ is determined using one of the existing methods; $F_s(s)$ is then given by (3). In other words $D_c(s)$ is first determined to be l.c. with S(s); this corresponds to choosing (A_c,B_c). $F_{es} = [\ E_c,C_c\ ]$ is then chosen and this specifies $N_c(s) = E_cD_c(s) + C_cS_c(s)$. Note that the zeros of determinant of any greatest common left divisor of S and D_c will be uncontrollable modes and will appear as closed loop eigenvalues.

Alternative Representation. Equivalence of dlsf and clsf

Given

$$Dz = u,\ y = Nz,\ u = Fz + Lr \tag{14}$$

with $F = N_FD_F^{-1}$ dlsf, let $\bar{z} = D_F^{-1}z$ and rewrite as

$$(DD_F)\bar{z} = u,\ y = (ND_F)\bar{z},\ u = N_F\bar{z} + Lr \tag{15}$$

Note that $N_F(DD_F)^{-1} = FD^{-1} = F_sSD^{-1}$ is strictly proper. This implies that the dlsf can be seen as clsf of an extended system where all poles of F (in D_F) are unobservable. This is an alternative formulation of the result we had before, in (12). The closed loop system is, in this case,

$$(DD_F-N_F)\bar{z} = Lr \;, \qquad y = ND_F\bar{z}$$

where N_F can arbitrarily alter $DD_F - N_F$, except the highest column degree coefficients. To determine appropriate F, one could work with D_F and N_F directly. To guarantee that $N_F D_F^{-1} = F = F_s S = N_c D_c^{-1} S$ with F_s proper, as it should be for dlsf, is nontrivial in this case and the following procedure is suggested:

Choose $D_F(m\times m)$ nonsingular so that $\alpha_1 S + D_F\alpha_2 = I$ [2,3] has a solution Note that almost any D_F will suffice. Then there exist $\bar{S}$ and D_c such that $\bar{S}D_F^{-1} = D_c^{-1}S$ coprime. N_F is then determined from $N_F = N_c\bar{S}$ where N_c is found using, say, E_c and C_c in the state space.

Conversely, given an unobservable system and a clsf as in (15), it does not necessarily follow that this is equivalent to a reduced system with dlsf as in (14), where the unobservable modes are now the poles of F. In a state space framework this is true when (A_e,B_e,C_e) can be reduced as in (4) and (11); note that while (A_e,C_e) can always be appropriately structured, this is not so for B_e. In differential operator terms in (15), this can be done only when $F = N_F D_F^{-1}$ can be written as $F = F_s S$ with F_s proper. This is true when $\alpha_1 S + D_F\alpha_2 = I$ has a solution and $N_F = N_c\bar{S}$ as in the procedure described above.

These results are used in the following section where it is shown that not all r.c. proper and stable factorizations can be generated using dlsf applied to an observable realization of the plant

<u>Relation to Proper and Stable Factorizations</u>

Consider $Dz = u$, $y = Nz$ compensated by dlsf $u = Fz + Lr$ where

$$F = N_F D_F^{-1} = F_s S = N_c D_c^{-1} S = \tilde{D}_c^{-1}\tilde{N}_c S$$

with F_s proper.

Then

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} D \\ N \end{bmatrix} z \tag{16}$$

with

$$z = (D-F)^{-1}Lr = D_F(DD_F-N_F)^{-1}Lr = (D-F_sS)^{-1}Lr$$
$$= (D-N_cD_c^{-1}S)Lr = (\tilde{D}_cD-\tilde{N}_cS)^{-1}\tilde{D}_cLr$$

Let

$$\begin{bmatrix} D' \\ N' \end{bmatrix} = \begin{bmatrix} D \\ N \end{bmatrix} \Pi, \qquad \Pi = (D-F)^{-1}L \tag{17}$$

$P = N'D'^{-1}$ is r.c. proper, stable factorization of the plant transfer matrix P if and only if Π, Π^{-1} stable and $(D\Pi)$ biproper [4]. The fact that FD^{-1} is strictly proper guarantees that $D\Pi$ is biproper. Π stable requires that $u = Fz + Lr$ is a stabilizing dlsf, and Π^{-1} stable requires that F is stable. Notice that (17) are actually the r to u,y maps. Therefore: Stable, stabilizing dlsf generates r.c. proper, stable factorizations.

Can all such factorizations be generated by dlsf on minimal realizations of P?

Given Π which satisfies the conditions, write

$$(D\Pi)^{-1} = I - FD^{-1}$$

F which is stable, must be dlsf transfer matrix for the system; that is, we must be able to write F as $F = F_sS$ with F_s proper. This imposes restriction of Π; therefore, not all rc factorizations can be obtained this way. The difficulty in terms of D_F, N_F in (16) translates into being able to arbitrarily choose D_F. D_F must be such that $\alpha_1 S + D_F\alpha_2 = I$ has a solution, and this imposes restric tions. These are the restrictions encountered in the previous section, while attempting to interpret clsf of an unobservable system as dlsf, with poles of F the unobservable modes. This is not always possible. Note however that all r.c. factorizations are generated via clsf on unobservable realizations [4].

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