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On Feedback Passivity of Discrete-Time Nonlinear Networked Control Systems with Packet Drops

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Abstract

We analyze the feedback passivity of a networked control system in which the control packets may be dropped by the communication channel. Specifically, we consider a discrete-time switched nonlinear system with relative degree zero that switches between two modes. At the instants when the communication link transmits the packet successfully, the system evolves in closed-loop and the increase in storage function is bounded below the energy supplied by the control input. At the instants when a packet drop occurs, the system evolves in open loop according to the free dynamics of the closed-loop mode and the increase in storage function is not necessarily bounded by the supplied energy. The literature on passivity of switched systems seems to consider only the case when all the modes are passive, which is not the case here. We prove that if the ratio of time steps for which the system evolves in closed-loop versus in open loop is lower bounded by a critical number, the system is locally feedback passive in a suitably defined sense. This generalized definition of feedback passivity is useful since it preserves two important properties of classical passivity - that feedback passivity implies asymptotic stabilizability for zero state detectable systems and that feedback passivity is preserved in parallel and negative feedback interconnections.

Index Terms

Networked Control Systems; Switched Nonlinear Systems; Passivity; Feedback Passivity; Zero Dynamics; Relative Degree Zero; Discrete-Time Systems

I. Introduction

Networked control systems (NCS) is now an established area of research [1]. In this paper, we consider a discrete-time nonlinear process being controlled across a communication channel that drops control packets in a non-deterministic fashion [2], [3]. In particular, we analyze the feedback passivity of a NCS

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whose increase in storage function may be greater than the supplied energy at some time steps due to packet drops. We assume that the process being controlled is not passive, but feedback passive, i.e., it can be made passive through a suitable designed state feedback control law. Due to the packet drops induced by the communication channel, the NCS evolves in two modes. At the instants when no packet is dropped, a state feedback control input is applied through the communication link and the system evolves in closed-loop. Because the process is feedback passive, the resulting increase in storage function is bounded by the energy supplied by the control input. At the instants when the communication channel erases the control packets, the system evolves in open loop according to the free dynamics of the original process. Because the process is non-passive, the storage function may increase even though no energy is being supplied by the control input. The problem we are interested in is to identify conditions on the packet drop frequency so that the resulting switched system remains feedback passive.

Passivity is widely used for analyzing the stability of interconnected dynamical systems [4]–[7]. Two properties that make passivity particularly useful are that (i) passivity implies asymptotic stability for zero state detectable (ZSD) systems using feedback [7], and (ii) both negative feedback and parallel interconnections of passive systems are passive. The classical notion of passivity has been extended to consider time-delayed [8], [9], event-triggered [10], switched [11], and hybrid systems [12]–[14]. A relaxation of passivity is the concept of feedback passivity [15], [16]. A feedback passive system is not necessarily passive for every possible input sequence. However, it is possible to construct a control law that is a function of both the state and an external input such that the system is passive with respect to this external input [16]–[18].

Under the above framework, because of the packet drops, the process evolution can be modeled as a switched system. While results are available for passivity of switched systems [11], the existing literature seems to consider only switched systems with all passive modes. In our problem, this framework does not hold. The main contributions of this paper are 1) to extend the concept of feedback passivity to such a discrete-time nonlinear switched system, 2) to show that if the frequency of the time steps at which the system evolves in open loop is bounded, the NCS is locally feedback passive, and 3) to prove that the stabilizability and compositional properties of passivity are preserved under this generalized definition. The closest work to our presentation is [11] from which we borrow the concept of allowing the storage function of switched systems to increase when a particular mode is inactive. Unlike [11], we do not assume every mode of the system to be individually passive. Also related are [19], [20] that consider the generalized asymptotic stability of nonlinear dynamical systems where the Lyapunov function is non-increasing only on certain unbounded discrete time sets. Unlike the stability analysis in these works, passivity analysis is complicated by the fact that it is an input-output property and both the inputs and the outputs are time varying. Due to this difficulty, we analyze the passivity of the switched system based on zero dynamics ([6], [15], [16], [18]) which is the internal dynamics of the system that is consistent with constraining the system output to zero. We show that more restrictive conditions than similar conditions in the literature on stability of systems with packet drops are required to guarantee local feedback passivity. To relax these conditions, we also study the concept of passivity only for any finite time, which is also compositional and implies stability.

The organization of the paper is as follows. Section II defines the problem framework. Section III provides the main results. Section III-A analyzes the passivity of the zero dynamics of the process. Section III-B studies the (finite-time) feedback passivity of the switched system based on the results from zero dynamics. Section III-C discusses the stabilizability and interconnections of feedback passive systems. We conclude the paper in Section IV.

II. Problem Formulation

Consider a discrete-time nonlinear system described by the equation

$$\begin{cases} \mathbf{x}(k+1) = f(\mathbf{x}(k), \mathbf{u}(k)) \\ \mathbf{y}(k) = h(\mathbf{x}(k), \mathbf{u}(k)) \end{cases}, \tag{1}$$

where $k \in \mathbb{Z}^+$ is the time index, $\mathbf{x}(k) \in \mathbb{R}^n$ is the state, $\mathbf{y}(k) \in \mathbb{R}^m$ is the output, and $\mathbf{u}(k) \in \mathbb{R}^m$ is the control input. Both $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ are in C^∞. All considerations are restricted to an open set $\mathbf{X} \times \mathbf{U} : \mathbf{X} \subset \mathbb{R}^n, \mathbf{U} \subset \mathbb{R}^m$ which is a neighbourhood of the origin $\mathbf{x}^* = \mathbf{0}, \mathbf{u}^* = \mathbf{0}$. Let the origin be an isolated equilibrium of (1) such that $f(\mathbf{0}, \mathbf{0}) = \mathbf{0}$ and $h(\mathbf{0}, \mathbf{0}) = \mathbf{0}$. System (1) is assumed to be locally zero state detectable (ZSD) [21] and have local relative degree zero for all the outputs at $(\mathbf{x}^*, \mathbf{u}^*)$, i.e., $\left.\frac{\partial h(\mathbf{x},\mathbf{u})}{\partial \mathbf{u}}\right|_{(\mathbf{x}^*,\mathbf{u}^*)}$ is non-singular [18]. This is a reasonable assumption because as shown in [16], a discrete-time nonlinear system can be rendered passive if and only if it has relative degree zero and passive zero dynamics[1].

Definition 2.1: ([16], [17]) A system of the form (1) is *locally passive* if there exists a positive definite

[1]Recent work [22] relaxes this assumption by using the coupled differential/difference representation (DDR) of the system. However, it requires the existence of a control $\mathbf{u}$ such that $f(\mathbf{x}, \mathbf{u})$ is invertible. Extensions of our results to such a scenario is left as future work.

function $\bar{V}:\mathbf{x}\to\mathbb{R}^+$, called the *storage function*, such that

$$\bar{V}(f(\mathbf{x}(k),\mathbf{u}(k)))-\bar{V}(\mathbf{x}(k))\leq \mathbf{u}^{\mathrm{T}}(k)\mathbf{y}(k),\ \forall\mathbf{x}(k)\in\mathbf{X},\mathbf{u}(k)\in\mathbf{U},k\in\mathbb{Z}^+.$$

We assume that process (1) is not passive and hence is open loop unstable. We also assume that $\mathbf{x}$ does not go out of $\mathbf{X}$ with packet drops; however, if the control $\mathbf{u}(k)$ is generated by a suitable state feedback control, it can be turned passive. In other words, we assume that (1) is locally feedback passive.

Definition 2.2: ([15]–[18], [23]) A system of the form (1) is *locally feedback passive* if there exist a positive definite storage function $\tilde{V}:\mathbf{x}\to\mathbb{R}^+$ and a function $\eta(\mathbf{x},\mathbf{v}):\mathbf{X}\times\mathbf{U}\to\mathbf{U}$ which is in $\mathcal{C}^\infty$ and locally regular[2], such that for any sequence $\{\mathbf{v}(0),\mathbf{v}(1),\cdots\}$ (with all $\mathbf{v}(j)\in\mathbf{U}$), the system evolving with the control input $\mathbf{u}(k)=\eta(\mathbf{x}(k),\mathbf{v}(k))$ satisfies the inequality

$$\tilde{V}(f(\mathbf{x}(k),\eta(\mathbf{x}(k),\mathbf{v}(k))))-\tilde{V}(\mathbf{x}(k))\leq \mathbf{v}^{\mathrm{T}}(k)\mathbf{y}(k),\forall\mathbf{x}(k)\in\mathbf{X},\mathbf{v}(k)\in\mathbf{U},k\in\mathbb{Z}^+. \tag{2}$$

Now assume that process (1) is controlled across a communication network that erases some of the control packets transmitted across it. At the instants when the packets are successfully received, the system evolves as described in (1). We denote the system as evolving in Mode 1 at these time steps. At the instants when the channel erases the packets, we assume for concreteness that the actuator applies zero control input, so that the system evolves as

$$\begin{cases}\mathbf{x}(k+1)=f(\mathbf{x}(k),\mathbf{0})\\ \mathbf{y}(k)=h(\mathbf{x}(k),\mathbf{0})\end{cases}. \tag{3}$$

We denote the system as evolving in Mode 2 at these time steps. Note that $\mathbf{x}^*=\mathbf{0}$ is an isolated equilibrium for Mode 2. Also note that (3) is exactly the free dynamics of Mode 1 with $\mathbf{u}(k)=\mathbf{0},\ \forall k$. If Mode 2 is active at time k, the storage function $\tilde{V}(\mathbf{x}(k+1))$ may be larger than $\tilde{V}(\mathbf{x}(k))$ even though no energy is being supplied through the control input. We denote the switched system evolving as in Mode 1 and Mode 2 by $\mathcal{S}$. The mode switching sequence for $\mathcal{S}$ is defined by the specification of the value $d(k)$ for every $k\in\mathbb{Z}^+$, where $d(k)\in\{1,2\}$ is the mode active at time k. Consider the evolution of system $\mathcal{S}$ over T time steps. Let $\tau(T)$ denote the total number of open loop time steps when $\mathcal{S}$ is in Mode 2 during time period $[1,T-1]$, and $T-\tau(T)$ the total number of closed-loop time steps when $\mathcal{S}$ is in Mode 1. Let the ratio between the closed-loop time steps and the open loop time steps be $r(T)=\frac{T-\tau(T)}{\tau(T)}$. When the

[2]A nonlinear state feedback control law $\eta(\mathbf{x},\mathbf{v}):\mathbf{X}\times\mathbf{U}\to\mathbf{U}$ is locally regular if $\frac{\partial\eta}{\partial\mathbf{v}}$ is invertible for all $(\mathbf{x},\mathbf{v})\in\mathbf{X}\times\mathbf{U}$.

context is clear, we will abuse the notation and suppress the dependence of $\tau(\cdot)$ and $r(\cdot)$ on T. Without loss of generality, the system is assumed to start in Mode 1 from time step $k = 1$. If this is not the case, we can shift the time axis by defining a new time variable $k' = k_0 + k$ with an appropriately defined initial condition k_0.

The introduction of Mode 2 requires a new definition of feedback passivity. To see why this is true, we consider the extreme case when $d(k) = 2$ identically. In this case, the set of allowed control inputs is $\mathbf{u}(k) = \mathbf{0}$ and no energy is supplied to the NCS. Thus, for the system to be feedback passive according to Definition 2.2 would require the existence of a positive definite storage function $\tilde{V}$ and the control input $\mathbf{u}(k) = \mathbf{v}(k) = \mathbf{0}$ such that

$$\tilde{V}(f(\mathbf{x}, \mathbf{0})) - \tilde{V}(\mathbf{x}) \leq \mathbf{0},\ \forall \mathbf{x}(k) \in \mathbf{X}, k \in \mathbb{Z}^+.$$

However, such a storage function would be a Lyapunov function for process (1) in open loop. Since Mode 2 is unstable, such a storage function does not exist. Thus, the switched system $\mathcal{S}$ is not feedback passive. Nevertheless, it is intuitive to consider the system to be feedback passive as long as Mode 2 occurs sufficiently infrequently. To capture this intuition, we propose new generalized definitions of local passivity and local feedback passivity. Before we do that, we need to consider one more aspect of the problem, which is that the set $\mathbf{U}$ of allowable controls may differ at different time steps. In particular, in our problem, $\mathbf{u}(k)$ (and hence $\mathbf{v}(k)$) can take any value in the set $\mathbf{U}$ if $d(k) = 1$, while $\mathbf{u}(k) = \mathbf{v}(k) = \mathbf{0}$ is the only value possible if $d(k) = 2$. We introduce this notion formally.

Definition 2.3: Consider a switched system $\mathcal{S}$ evolving as in Mode 1 given by Equation (1) and Mode 2 given by Equation (3) in which the control input $\mathbf{u}(k) \in \mathbf{U}(k)$ at any time k. The system is *locally passive* if there exists a positive definite storage function $\bar{V} : \mathbf{x} \to \mathbb{R}^+$ such that the following passivity inequality holds:

$$\bar{V}(\mathbf{x}(T)) - \bar{V}(\mathbf{x}(1)) \leq \sum_{k=1}^{T-1} \mathbf{u}^{\mathrm{T}}(k)\mathbf{y}(k), \forall \mathbf{x}(k) \in \mathbf{X}, \mathbf{u}(k) \in \mathbf{U}(k), T \in \mathbb{Z}^+. \tag{4}$$

Definition 2.4: Consider a switched system $\mathcal{S}$ evolving as in Mode 1 given by Equation (1) and Mode 2 given by Equation (3) in which the control input $\mathbf{u}(k) \in \mathbf{U}(k)$ at any time k. The system is *locally feedback passive* if there exists a positive definite storage function $\tilde{V} : \mathbf{x} \to \mathbb{R}^+$ and a regular state feedback control law

$$\mathbf{u}(k) = \begin{cases} \eta(\mathbf{x}(k), \mathbf{v}(k)) & \text{if } d(k) = 1 \\ \mathbf{v}(k) = \mathbf{0} & \text{if } d(k) = 2 \end{cases} \tag{5}$$

such that the following passivity inequality holds:

$$\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) \leq \sum_{k=1}^{T-1} \mathbf{v}^{\mathrm{T}}(k)\mathbf{y}(k), \forall \mathbf{x}(k) \in \mathbf{X}, \mathbf{v}(k) \in \mathbf{U}(k), T \in \mathbb{Z}^{+}, \tag{6}$$

where $\mathbf{U}(k) = \mathbf{U}$ when $d(k) = 1$ and $\mathbf{U}(k) = \mathbf{0}$ when $d(k) = 2$.

Note that a system that is locally passive (respectively locally feedback passive) according to Definition 2.1 (resp. Definition 2.2) remains locally passive (resp. locally feedback passive) according to Definition 2.3 (resp. Definition 2.4). However, the converse is not necessarily true. It is this freedom that will allow us to define the switched system $\mathcal{S}$ as locally feedback passive.

In the case when the inequality (4) (respectively the inequality (6)) only satisfies when $0 < T < \infty$, the system is *finite-time locally passive* (reps. *finite-time locally feedback passive*).

With these definitions, we answer two questions in this paper. First, we prove the intuitive result that if the system is in Mode 2 only infrequently, the switched system $\mathcal{S}$ should be expected to remain locally feedback passive. More precisely, we prove that there is a critical ratio $r^{\star}$, such that if for every T, $r(T) > r^{\star}$, then the system is locally feedback passive. Secondly, we show that this definition preserves the following two properties of classical passivity:

- A feedback passive system is asymptotic stabilizable if it is ZSD.
- Parallel or negative feedback interconnections of feedback passive systems are feedback passive.

III. Main Results

In this section, we provide the main results and brief proofs.

A. *Passivity Analysis for Zero Dynamics*

Note that there is considerable freedom in choosing the function $\eta(\mathbf{x}(k), \mathbf{v}(k))$ in Definition 2.2 for Mode 1 as defined by Equation (1). We restrict the class of allowed functions to further satisfy $\mathbf{v}(k) = h(\mathbf{x}(k), \eta(\mathbf{x}(k), \mathbf{v}(k)))$. By the implicit function theorem [18], [24], such an η always exists since Mode 1 (Equation (1)) is assumed to have relative degree zero and η is regular. Denote the control inputs so obtained by $\bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k))$. For any given bounded vector sequence $\mathbf{v}(k) \in \mathbf{U}$, the corresponding control inputs $\bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k)) \in \mathbf{U}$ are bounded. Under these inputs, system $\mathcal{S}$ in Mode 1 evolves as

$$\begin{cases} \mathbf{x}(k+1) = f(\mathbf{x}(k), \bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k))) \triangleq \bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k)) \\ \mathbf{y}(k) = h(\mathbf{x}(k), \bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k))) = \mathbf{v}(k) \end{cases} . \tag{7}$$

This is referred as the feedback transformed system. Because $h(\mathbf{x}, \mathbf{u})\Big|_{(\mathbf{x}^*, \mathbf{u}^*)=(\mathbf{0},\mathbf{0})} = \mathbf{0}$, $(\mathbf{x}^*, \mathbf{v}^*) = (\mathbf{0}, \mathbf{0})$ remains an isolated equilibrium of (7), i.e., $\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k))\Big|_{(\mathbf{x}^*, \mathbf{v}^*)=(\mathbf{0},\mathbf{0})} = \mathbf{0}$. Note that the evolution in Mode 2 is still given by Equation (3). Denote the switched system defined by Equations (7) and (3) by $\mathcal{S}_1$.

In the particular case when $\mathbf{y}(k)$, and hence $\mathbf{v}(k)$, is identically zero, let the control inputs $\bar{\mathbf{u}}^{\mathbf{v}(k)}(\mathbf{x}(k))$ be denoted by $\tilde{\mathbf{u}}(k)$. Under $\tilde{\mathbf{u}}(k)$, Mode 1 evolves as the zero dynamics of the closed-loop system (1)

$$\begin{cases} \mathbf{x}(k+1) = f(\mathbf{x}(k), \tilde{\mathbf{u}}(k)) \triangleq \tilde{f}(\mathbf{x}(k)) \\ \mathbf{y}(k) = \mathbf{0} \end{cases} \tag{8}$$

Denote the switched system defined by (8) and (3) as $\mathcal{S}_2$. Since system $\mathcal{S}$ in Mode 1 as given by Equation (1) is locally feedback passive, the zero dynamics (8) of the closed-loop mode are also locally passive and hence stable (see [16, Theorem 7.3] and [15, Remark 2.5]). Furthermore, since for system $\mathcal{S}_2$, either the input (in Mode 2 which evolves as (3)) or the output (in Mode 1 which evolves as (8)) is identically zero at every time step, Definition 2.3 implies that system $\mathcal{S}_2$ is locally passive if there exists a positive definite storage function $V(\mathbf{x}(\cdot))$ such that the following inequality holds:

$$V(\mathbf{x}(T)) - V(\mathbf{x}(1)) \leq \sum_{k=1}^{T-1} \mathbf{u}^{\mathrm{T}}(k)\mathbf{y}(k) = 0, \forall \mathbf{x} \in \mathbf{X}, T \in \mathbb{Z}^+. \tag{9}$$

From now on, we also assume that the determinant of the Hessian matrix of $V(\mathbf{x})$ in (9) at $\mathbf{x} = \mathbf{0}$ is non-zero.

Our first result shows that there is a lower bound on the frequency of the steps at which system $\mathcal{S}_2$ evolves in closed-loop as defined by Equation (8) that guarantees $\mathcal{S}_2$ to be locally passive.

Lemma 3.1: Consider the switched system $\mathcal{S}_2$ defined by Equations (8) and (3). Assume there exist a positive definite storage function $V(\mathbf{x}(\cdot))$ and constants $\zeta > 1$ and $0 < \sigma \leq 1$ such that

$$\begin{aligned} V(f(\mathbf{x}(k), \mathbf{0})) &\leq \zeta V(\mathbf{x}(k)) \\ V(\tilde{f}(\mathbf{x}(k))) &\leq \sigma V(\mathbf{x}(k)). \end{aligned} \tag{10}$$

If for any time $T \in \mathbb{Z}^+$, the ratio $r(T)$ satisfies

$$r(T) \geq \frac{T \ln \zeta - \ln \sigma}{(1-T)\ln \sigma} \tag{11}$$

and $\mathbf{x}(T) \in \mathbf{X}$ irrespective of $d(1), \cdots, d(T-1)$, then system $\mathcal{S}_2$ is locally passive according to Definition 2.3.

Proof: The inequalities (10) and (11) imply $V(\mathbf{x}(T)) \leq \sigma^{T-\tau-1}\zeta^{\tau}V(\mathbf{x}(1)) \leq V(\mathbf{x}(1))$. ■

Remark 3.1: The choice of ζ and σ determines how conservative the condition (11) is. The minimum ζ and σ that satisfy the inequality (10) will result in the least conservative bound.

Remark 3.2: Note that condition (11) does not require a constant ratio $r(T)$ and the right hand side is an increasing function of T. Thus, the condition on the frequency of Mode 2 becomes progressively less stringent.

We now prove an intuitive result on the effect of increasing $r(T)$.

Corollary 3.1: Consider system $\mathcal{S}_2$ defined by Equations (8) and (3) with the conditions (10) being satisfied. If the system is locally passive with a ratio $r(T)$, it is locally passive with a ratio $r'(T) > r(T)$. Thus, decreasing the frequency of open loop time steps preserves passivity.

Proof: Let the number of open loop time steps with the ratio $r(T)$ ($r'(T)$) be $\tau(r,T)$ ($\tau(r',T)$), we have $V(\mathbf{x}(T)) \leq \sigma^{T-\tau(r',T)-1}\zeta^{\tau(r',T)}V(\mathbf{x}(1)) \leq \sigma^{T-\tau(r,T)-1}\zeta^{\tau(r,T)}V(\mathbf{x}(1)) \leq V(\mathbf{x}(1))$. ■

Remark 3.3: Define the sequence of time steps $\{k_i\}$ such that $k_0 = 1$ and $k_i =$ the least time $> k_{i-1}$ such that $d(k_i - 1) = 2$ and $d(k_i) = 1$. Assume system $\mathcal{S}_2$ is locally passive. Let τ be the total number of time steps at which the system evolves in open loop and $t_c = T - \tau$ be the number of time steps at which the system evolves in closed loop during the time period $[k_0, k_i]$. According to Remark 3.2, $r(T)$ increases with T in (11). Therefore, in the time period $[k_i, k_{i+1}]$, let τ' be the number of open loop time steps and t'_c be the number of closed-loop time steps, we must have $\frac{t'_c}{\tau'} \geq \frac{t_c}{\tau}$. Following similar derivation of Corollary 3.1, since $\mathcal{S}_2$ is locally passive in the time period $[k_0, k_i]$, it is also locally passive in the time period $[k_i, k_{i+1}]$.

B. *Feedback Passivity Analysis for the Original System*

Lemma 3.2: Consider a positive definite storage function $V(\mathbf{x}(\cdot))$ as a storage function for the zero dynamics (8) of the closed-loop system (1). Assume that $V(\tilde{f}(\mathbf{x}(k))) \leq \sigma V(\mathbf{x}(k))$ for some $0 < \sigma \leq 1$ according to (10) and that $\det\{\text{Hess}(\sigma V(\mathbf{x}) - V(f(\mathbf{x},0)))|_{\mathbf{x}=0}\} \neq 0$. Then, there exists a constant $a > 0$

such that $\tilde{V} = aV,\ a \in (0, \hat{a})$ is a storage function for the feedback transformed system (7) and

$$\tilde{V}(\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k))) - \sigma\tilde{V}(\mathbf{x}(k)) - \sigma\mathbf{v}^T(k)\mathbf{v}(k) \leq 0 \tag{12}$$

with the equality holding at $(\mathbf{x}^*, \mathbf{v}^*) = (\mathbf{0}, \mathbf{0})$.

Remark 3.4: The original proof is presented in [18] for the case when $\sigma = 1$. The detailed proof for this generalized result can be found in the Appendix for any $0 < \sigma \leq 1$.

Theorem 3.1: Let the switched system $\mathcal{S}_2$ defined by Equations (8) and (3) satisfy the inequalities (10) and (11) such that $\mathcal{S}_2$ is locally passive. Furthermore, let the switched system $\mathcal{S}$ defined by Equations (1) and (3) evolve from the same initial condition and with the same mode switching signal as $\mathcal{S}_2$. Denote by $M > 0$ the number of maximum consecutive open loop operations of system $\mathcal{S}_2$ that occur according to dynamics (3). Finally let $\det\{\text{Hess}(\sigma V(\mathbf{x}) - V(f(\mathbf{x}, 0)))|_{\mathbf{x}=0}\} \neq 0$ with some $0 < \sigma \leq 1$ and V be the storage function for $\mathcal{S}_2$. If $\zeta^M\sigma \leq 1$, then system $\mathcal{S}$ is locally feedback passive.

Proof: Here we summarize the proof and a detailed explanation can be found in the Appendix. The proof is in two steps. We use mathematical induction to prove that

$$\tilde{V}(\mathbf{x}(T)) \leq \sigma^{T-\tau(T)-1}\zeta^{\tau(T)}\tilde{V}(\mathbf{x}(1)) + \zeta^M\sigma \sum_{\substack{k:d(k)=1 \\ k \leq T-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k), \tag{13}$$

holds for all $T \geq 2$. From condition (11), we have $\sigma^{T-\tau(T)-1}\zeta^{\tau(T)} \leq 1$. By assumption, we have $\zeta^M\sigma \leq 1$. Therefore, the inequality (13) becomes

$$\tilde{V}(\mathbf{x}(T)) \leq \tilde{V}(\mathbf{x}(1)) + \sum_{\substack{k:d(k)=1 \\ k \leq T-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k).$$

This indicates that system $\mathcal{S}$ is locally feedback passive. ■

Remark 3.5: Notice that the condition $\zeta^M\sigma \leq 1$ is more restrictive than similar conditions in the literature on stability of systems with packet drops [25], [26]. We conjecture that condition (11) is enough for passivity. However, the analysis is hampered by the fact that passivity is an input-output property and conditions (10) impose restrictions only on the state. As a partial proof of the conjecture, we present the following result that shows that if we restrict Definitions 2.3 and 2.4 to hold only for all finite T, then (11) is a sufficient condition. Note that this concept of passivity is also compositional (as shown in Theorem 3.4) and implies Lyapnov stability (as shown in Theorem 3.3).

Theorem 3.2: Let the switched system $\mathcal{S}_2$ defined by Equations (8) and (3) satisfy the inequalities (10) and (11) such that $\mathcal{S}_2$ is locally passive. Furthermore, let the switched system $\mathcal{S}$ defined by Equations (1) and (3) evolve from the same initial condition and with the same mode switching signal as $\mathcal{S}_2$. Then system $\mathcal{S}$ is *finite-time* locally feedback passive in the sense that (6) is satisfied for all $\mathbf{x}(k) \in \mathbf{X}, \mathbf{v}(k) \in \mathbf{U}(k)$ and any finite T.

Proof: To prove that system $\mathcal{S}$ is *finite-time* locally feedback passive, i.e. the inequality (6) holds for any $0 < T < \infty$ in a neighborhood of $(\mathbf{x}^*, \mathbf{v}^*)$, we proceed as follows. First, if $d(k) = 1$ for all $1 \leq k \leq T-1$, then the system $\mathcal{S}$ operates in Mode 1 at every time step and thus (6) holds trivially. Second, if $\mathbf{v}(k) = 0$ at every time when $d(k) = 1$, then the system $\mathcal{S}$ is equivalent to the system $\mathcal{S}_2$ and by assumption, (6) is satisfied. Other than these two cases, we will show that $\tilde{V}(\mathbf{x}(\cdot)) = aV(\mathbf{x}(\cdot))$ (with $a > 0$) is a storage function for system $\mathcal{S}$ (Recall that $V(\mathbf{x}(\cdot))$ is the storage function for system $\mathcal{S}_2$). Since we impose $\mathbf{y}(k) = \mathbf{v}(k)$ at every time when $d(k) = 1$, proving (6) is equivalent to prove that in a neighborhood of $(\mathbf{x}^*, \mathbf{v}^*)$, the inequality

$$\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) \leq \sum_{\substack{k:d(k)=1 \\ k \leq T-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k), \ \forall \mathbf{x} \in \mathbf{X}, \mathbf{v} \in \mathbf{U}, T \in \mathbb{Z}^+. \tag{14}$$

holds for all finite T.

The proof is in two steps.

1) First, we show that it is sufficient to prove that for an appropriate $a > 0$, the inequality

$$a(\zeta - 1) \sum_{\substack{k:d(k)=2 \\ k \leq T-1}} V(\mathbf{x}(k)) \leq \sum_{\substack{k:d(k)=1 \\ k \leq T-1}} \phi(\mathbf{x}(k), \mathbf{v}(k)). \tag{15}$$

holds for all finite T. Then, assuming (15), we show that we can always choose a constant $a > 0$ such that $\tilde{V}(\mathbf{x}(\cdot))$ is a storage function for system $\mathcal{S}$. To prove that (15) holds, we proceed as follows. Two cases are possible.

   a) If $\mathbf{x}(k) = \mathbf{0}$ for all k such that $d(k) = 2$, (15) holds for any $a > 0$.

   b) For all other cases, there exists at least one time step $K < T$ such that $d(K) = 2$ and $\mathbf{x}(K) \neq 0$. Since T is finite, there exists $\overline{M} > 0$, such that

$$0 < V(\mathbf{x}(K)) \leq \sum_{\substack{k:d(k)=2 \\ k \leq T-1}} V(\mathbf{x}(k)) < \overline{M} < \infty. \tag{16}$$

Define the function $\phi(\mathbf{x}(k), \mathbf{v}(k))$ as

$$\begin{aligned}\phi(\mathbf{x}(k), \mathbf{v}(k)) &\triangleq \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k) + \tilde{V}(\mathbf{x}(k)) - \tilde{V}(\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k))) \\ &= \sum_{i=1}^{m} v_i^2(k) + \tilde{V}(\mathbf{x}(k)) - \tilde{V}(\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k))). \end{aligned} \tag{17}$$

We use the following property of $\phi(\mathbf{x}(k), \mathbf{v}(k))$ that is proved in Lemma 3.2: if $k : d(k) = 1$, $\phi(\mathbf{x}(k), \mathbf{v}(k))$ has a local minimum at $(\mathbf{x}^*, \mathbf{v}^*) = (\mathbf{0}, \mathbf{0})$ with value 0.

Since we exclude the special case that $\mathbf{v}(k) = \mathbf{0}$ at every time when $d(k) = 1$, there must be some $K' < T$ such that $\mathbf{v}(K') \neq 0$ and $d(K') = 1$. Thus, we obtain

$$\sum_{\substack{k:d(k)=1 \\ k \le T-1}} \phi(\mathbf{x}(k), \mathbf{v}(k)) \ge \phi(\mathbf{x}(K'), \mathbf{v}(K')) > 0. \tag{18}$$

Define $\tilde{a} \triangleq \frac{\phi(\mathbf{x}(K'),\mathbf{v}(K'))}{(\zeta-1)\overline{M}} > 0$. Now we choose $a \in (0, \min\{\hat{a}, \tilde{a}\})$ where $\hat{a}$ is defined in Lemma 3.2, so that the inequality (15) is satisfied.

2) Now, if (15) holds, we can show that $\tilde{V}$ is a storage function for $\mathcal{S}$ as follows. For any k such that $d(k) = 2$, systems $\mathcal{S}$ and $\mathcal{S}_2$ evolve in an identical manner as given by Equation (3). From the inequality (10), we obtain at these time steps

$$\tilde{V}(f(\mathbf{x}(k), \mathbf{0})) - \tilde{V}(\mathbf{x}(k)) = a\left(V(f(\mathbf{x}(k), \mathbf{0})) - V(\mathbf{x}(k))\right) \le a(\zeta - 1)V(\mathbf{x}(k)) \tag{19}$$

so that

$$\sum_{\substack{k:d(k)=2 \\ k \le T-1}} \left[\tilde{V}(f(\mathbf{x}(k), \mathbf{0})) - \tilde{V}(\mathbf{x}(k))\right] \le a(\zeta - 1) \sum_{\substack{k:d(k)=2 \\ k \le T-1}} V(\mathbf{x}(k)). \tag{20}$$

Now note that

$$\sum_{\substack{k:d(k)=2 \\ k \le T-1}} \left[\tilde{V}(f(\mathbf{x}(k), \mathbf{0})) - \tilde{V}(\mathbf{x}(k))\right] + \sum_{\substack{k:d(k)=1 \\ k \le T-1}} \left[\tilde{V}(\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k))) - \tilde{V}(\mathbf{x}(k))\right] = \tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)),$$

so that according to the inequalities (20) and (15), we have

$$\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) \le \sum_{\substack{k:d(k)=1 \\ k \le T-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k)$$

if a is chosen in the interval $(0, \min(\hat{a}, \tilde{a}))$. At this point, we have shown that the inequality (6) holds for any $0 < T < \infty$ in a neighborhood of $(\mathbf{x}^*, \mathbf{v}^*)$ and therefore system $\mathcal{S}$ is *finite-time* locally feedback passive.

■

C. Stability and Interconnections of Feedback Passive Systems

We now prove that the definition of feedback passivity we have introduced in Definition 2.4 for both finite and infinite T preserves some of the important properties of classical feedback passivity.

Theorem 3.3: If the switched system $\mathcal{S}$ defined by Equations (1) and (3) is locally feedback passive according to Definition 2.4 and locally ZSD, then the system is locally asymptotically stabilizable with a suitable state feedback control law. If Definition 2.4 holds merely for all finite T, then the system is locally stabilizable with a suitable state feedback control law.

Proof: Because system $\mathcal{S}$ is locally passive, choose $\mathbf{v}(k) = \mathbf{0}$, $\forall k$ when $d(k) = 1$, the inequality (14) reduces to $\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) \leq 0$, $\forall \mathbf{x}(\cdot) \in \mathbf{X}$. Following Remark 3.3, we obtain $\tilde{V}(\mathbf{x}(k_{i+1})) - \tilde{V}(\mathbf{x}(k_i)) \leq 0$, $\forall i = 0, 1, \cdots, \forall \mathbf{x} \in \mathbf{X}$. Then $\tilde{V}(\mathbf{x}(\cdot))$ is a Lyapunov function for system $\mathcal{S}$. The asymptotic stability for the case when T is infinite follows from ZSD. ■

Theorem 3.4: If two switched nonlinear systems $\mathcal{S}^1$ and $\mathcal{S}^2$ are both locally feedback passive according to the inequality (6) in Definition 2.4, then their parallel and negative feedback interconnections (as defined in Figure 1) are both locally feedback passive.

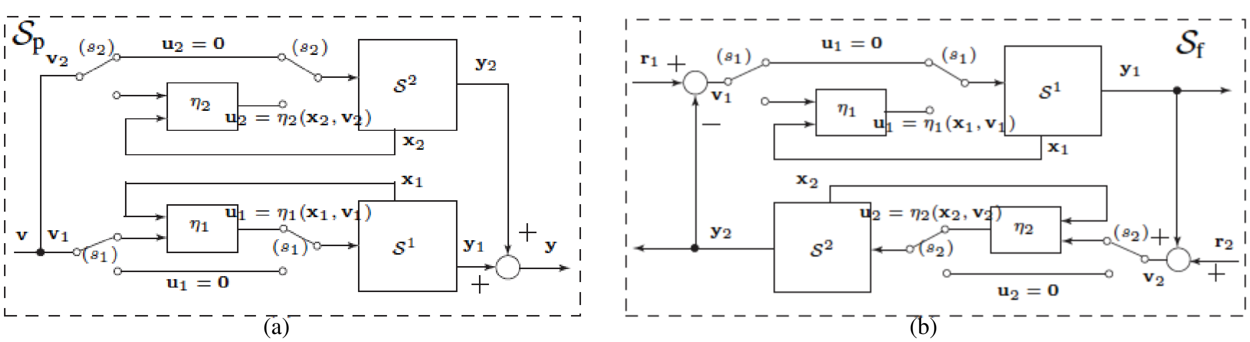

Fig. 1. (a) Parallel, and (b) negative feedback interconnections for two locally feedback passive switched nonlinear systems $\mathcal{S}^1$ and $\mathcal{S}^2$. Note that the switches marked with a same notation (s_i), $i = 1$ or 2 switch simultaneously.

Proof: For the parallel interconnection, the extrinsic control sequence $\mathbf{v}(k)$ is the same for both systems ($\mathcal{S}^1$ and $\mathcal{S}^2$) and the output $\mathbf{y}(k) = \mathbf{y}_1(k) + \mathbf{y}_2(k)$. Consider the storage function $\tilde{V}(\mathbf{x}(k)) =$

$\tilde{V}_1(\mathbf{x}_1(k)) + \tilde{V}_2(\mathbf{x}_2(k))$, we have

$$\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) \leq \sum_{k=1}^{T-1} \mathbf{v}^{\mathrm{T}}(k)\mathbf{y}(k). \tag{21}$$

Similarly, for the negative feedback interconnection, the control inputs and outputs are as $\mathbf{r}_1(k) = \mathbf{v}_1(k) + \mathbf{y}_2(k)$ and $\mathbf{r}_2(k) = \mathbf{v}_2(k) - \mathbf{y}_1(k)$. We have

$$\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) \leq \sum_{k=1}^{T-1} (\mathbf{r}_1^{\mathrm{T}}(k)\mathbf{y}_1(k) + \mathbf{r}_2^{\mathrm{T}}(k)\mathbf{y}_2(k)). \tag{22}$$

Note that in Theorem 3.4, if (6) holds for $\mathcal{S}^1$ and $\mathcal{S}^2$ only for $0 < T < \infty$, then the inequalities (21) and (22) for the interconnected system holds for $0 < T < \infty$ as well. ■

IV. Conclusions

We analyze feedback passivity for a NCS with packet drops. We model it as a discrete-time switched nonlinear system that switch between two modes - an uncontrolled mode in which the system evolves open loop, and a controlled mode in which a control is applied to the system. A new generalized definition of feedback passivity is given for such a system and it is shown that if the ratio of the time steps for which the system evolves closed-loop versus the time steps for which the system evolves open loop is bounded above a critical ratio, then the system is locally feedback passive in this sense. We show that this generalized definition is useful since it preserves the stabilizability and compositional properties of classical passivity.

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APPENDIX

Proof for Lemma 3.1.

Proof: For any time $T \in \mathbb{Z}^{+}$, (10) implies that $V(\mathbf{x}(T)) \leq \sigma^{T-\tau-1}\zeta^{\tau}V(\mathbf{x}(1))$. Since (11) implies $\sigma^{T-\tau-1}\zeta^{\tau} \leq 1$, we obtain that $V(\mathbf{x}(T)) \leq V(\mathbf{x}(1))$ for any T, if the conditions (10) in the theorem are met. From Definition 2.3, system $\mathcal{S}_2$ is locally passive. ■

Proof for Corollary 3.1.

Proof: At time $T \in \mathbb{Z}^{+}$, denote the number of time steps for which the system evolves open loop with the ratio $r(T)$ by $\tau(r,T)$ and with the ratio $r'(T)$ by $\tau(r',T)$. Conditions (10) yield $V(\mathbf{x}(T)) \leq \sigma^{T-\tau(r,T)-1}\zeta^{\tau(r,T)}V(\mathbf{x}(1))$ and $V(\mathbf{x}(T)) \leq \sigma^{T-\tau(r',T)-1}\zeta^{\tau(r',T)}V(\mathbf{x}(1))$. Since the system is locally passive with ratio $r(T)$, $\sigma^{T-\tau(r,T)-1}\zeta^{\tau(r,T)} \leq 1$. The proof follows by noting that $\tau(r',T) < \tau(r,T)$ and thus, $\sigma^{T-\tau(r',T)-1}\zeta^{\tau(r',T)} < \sigma^{T-\tau(r,T)-1}\zeta^{\tau(r,T)} \leq 1$. ■

Proof for Lemma 3.2.

Proof: Define the function

$$\phi(\mathbf{x}(k),\mathbf{v}(k)) \triangleq \sigma\mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k) + \sigma\tilde{V}(\mathbf{x}(k)) - \tilde{V}(\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k))) = \sigma\sum_{i=1}^{m} v_i^2(k) + \sigma\tilde{V}(\mathbf{x}(k)) - \tilde{V}(\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k))). \quad (23)$$

For notational convenience, we suppress the dependence on k and denote the pair $(\mathbf{x}^*,\mathbf{v}^*)$ by $(\mathbf{0},\mathbf{0})$. Thus, consider the first order derivatives of $\phi(\mathbf{x},\mathbf{v})$ at $(\mathbf{0},\mathbf{0})$. We have for $i = 1,\cdots,n$, $r = 1,\cdots,m$,

$$\begin{aligned}
\left.\frac{\partial\phi(\mathbf{x},\mathbf{v})}{\partial x_i}\right|_{\mathbf{x}^*=\mathbf{0},\mathbf{v}^*=\mathbf{0}} &= \left[\sigma\frac{\partial\tilde{V}}{\partial x_i} - \sum_{h=1}^{n}\frac{\partial\tilde{V}}{\partial\bar{f}_h^{\mathbf{v}}}\frac{\partial\bar{f}_h^{\mathbf{v}}(\mathbf{x})}{\partial x_i}\right]_{(\mathbf{0},\mathbf{0})}\\
\left.\frac{\partial\phi(\mathbf{x},\mathbf{v})}{\partial v_r}\right|_{\mathbf{x}^*=\mathbf{0},\mathbf{v}^*=\mathbf{0}} &= \left[2\sigma v_r - \sum_{h=1}^{n}\frac{\partial\tilde{V}}{\partial\bar{f}_h^{\mathbf{v}}}\frac{\partial\bar{f}_h^{\mathbf{v}}(\mathbf{x})}{\partial v_r}\right]_{(\mathbf{0},\mathbf{0})}.
\end{aligned}$$

The storage function $V(\mathbf{x}(k))$, and hence the function $\tilde{V}(\mathbf{x}(k)) = aV(\mathbf{x}(k))$, has a local minimum at $\mathbf{x}^* = \mathbf{0}$ because V is positive definite with $V(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$. Moreover, the origin is an isoalated local equilibrium of the system; thus, at $\mathbf{x}^* = \mathbf{v}^* = \mathbf{0}$, $\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k)) = \mathbf{0}$. Combining these facts, we see that

$$\left.\frac{\partial \phi(\mathbf{x},\mathbf{v})}{\partial x_i}\right|_{\mathbf{x}^*=\mathbf{0},\mathbf{v}^*=\mathbf{0}} = 0,\ i = 1,\cdots,n,$$
$$\left.\frac{\partial \phi(\mathbf{x},\mathbf{v})}{\partial v_r}\right|_{\mathbf{x}^*=\mathbf{0},\mathbf{v}^*=\mathbf{0}} = 0,\ r = 1,\cdots,m.$$

Next, we check the elements of the Hessian matrix of $\phi(\mathbf{x},\mathbf{v})$ at $(\mathbf{0},\mathbf{0})$. We have for $i,j = 1,\cdots,n$ and $r,s = 1,\cdots,m$,

$$\left.\frac{\partial^2 \phi(\mathbf{x},\mathbf{v})}{\partial x_j \partial x_i}\right|_{\mathbf{x}^*=\mathbf{0},\mathbf{v}^*=\mathbf{0}} = a\left[\sigma\frac{\partial^2 V}{\partial x_j \partial x_i} - \sum_{h,l=1}^{n}\frac{\partial^2 V}{\partial \bar{f}_h^{\mathbf{v}}\partial \bar{f}_l^{\mathbf{v}}}\frac{\partial \bar{f}_h^{\mathbf{v}}}{\partial x_i}\frac{\partial \bar{f}_l^{\mathbf{v}}}{\partial x_j}\right]_{\mathbf{x}^*=\mathbf{0},\mathbf{v}=^*\mathbf{0}}$$
$$\left.\frac{\partial^2 \phi(\mathbf{x},\mathbf{v})}{\partial v_r \partial x_i}\right|_{\mathbf{x}^*=\mathbf{0},\mathbf{v}^*=\mathbf{0}} = -a\left[\sum_{h,l=1}^{n}\frac{\partial^2 V}{\partial \bar{f}_h^{\mathbf{v}}\partial \bar{f}_l^{\mathbf{v}}}\frac{\partial \bar{f}_h^{\mathbf{v}}}{\partial x_i}\frac{\partial \bar{f}_l^{\mathbf{v}}}{\partial v_r}\right]_{\mathbf{x}^*=\mathbf{0},\mathbf{v}^*=\mathbf{0}}$$
$$\left.\frac{\partial^2 \phi(\mathbf{x},\mathbf{v})}{\partial v_s \partial v_r}\right|_{\mathbf{x}^*=\mathbf{0},\mathbf{v}^*=\mathbf{0}} = 2\sigma\delta_{rs} - a\left[\sum_{h,l=1}^{n}\frac{\partial^2 V}{\partial \bar{f}_h^{\mathbf{v}}\partial \bar{f}_l^{\mathbf{v}}}\frac{\partial \bar{f}_h^{\mathbf{v}}}{\partial v_r}\frac{\partial \bar{f}_l^{\mathbf{v}}}{\partial v_s}\right]_{\mathbf{x}^*=\mathbf{0},\mathbf{v}^*=\mathbf{0}}.$$

Denote $\tilde{\phi}(\mathbf{x}(k)) = \phi(\mathbf{x}(k),\mathbf{0}) = a\left(\sigma V(\mathbf{x}(k)) - V(\bar{f}^{\mathbf{0}}(\mathbf{x}(k)))\right)$, so that

$$\left.\frac{\partial^2 \phi(\mathbf{x},\mathbf{v})}{\partial x_j \partial x_i}\right|_{\mathbf{x}^*=\mathbf{0},\mathbf{v}^*=\mathbf{0}} = \left.\frac{\partial^2 \tilde{\phi}(\mathbf{x})}{\partial x_j \partial x_i}\right|_{\mathbf{x}^*=\mathbf{0}}. \tag{24}$$

The zero dynamics (8) are locally passive and hence satisfy the passivity inequality (9). Because $\bar{f}^{\mathbf{0}}(\mathbf{x}(k)) = \tilde{f}(\mathbf{x}(k))$, the term $\tilde{\phi}(\mathbf{x}(k))$ has a local minimum at $\mathbf{x}^* = \mathbf{0}$. By the assumption that $\det\{\text{Hess}(\sigma V(\mathbf{x}) - V(f(\mathbf{x},0)))|_{\mathbf{x}^*=0}\} \neq 0$ for some $0 < \sigma \leq 1$, we obtain that the eigenvalues of the Hessian matrix of $\tilde{\phi}(\mathbf{x})$ at $\mathbf{x}^* = \mathbf{0}$ are all positive. Denote these eigenvalues by λ_i, $\forall i = 1,2,\cdots,n$. Furthermore, the Hessian matrix of $\tilde{\phi}(\mathbf{x})$ at $\mathbf{x}^* = \mathbf{0}$ is symmetric and can be diagonalized. Thus, with an appropriate choice of coordinates, the Hessian matrix of $\phi(\mathbf{x},\mathbf{v})$ at $(\mathbf{0},\mathbf{0})$ can be evaluated to be of the form

$$\begin{bmatrix} a\lambda_1 & \cdots & 0 & ab_{11} & \cdots & ab_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a\lambda_n & ab_{n1} & \cdots & ab_{nm} \\ ab_{11} & \cdots & ab_{n1} & 2\sigma + ac_{11} & \cdots & ac_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ ab_{1m} & \cdots & ab_{nm} & ac_{m1} & \cdots & 2\sigma + ac_{mm} \end{bmatrix} \tag{25}$$

Now, we apply [18, Lemma 12] which states that for $\lambda_i > 0$ and $\forall a = (0, \hat{a}),\ \hat{a} = \min_j a_j^u$ where

$$a_j^u = \min\left\{1, \frac{2^j\sigma^j\lambda_1\cdots\lambda_n - \epsilon}{|\alpha_1| + \cdots + |\alpha_j|}\right\},\ j = 1, \cdots, m \tag{26}$$

with $0 < \epsilon \ll 1$ and $\alpha_l,\ l = 1, \cdots, j$ being some constants related to λ_i, b_{il} and $c_{rl},\ i = 1, \cdots, n$, $r = 1, \cdots, j$, $l = 1, \cdots, j$, the determinant of matrix (25) is greater than zero. Sylester's criterion now readily yields that the Hessian matrix of $\phi(\mathbf{x},\mathbf{v})$ at $(\mathbf{0},\mathbf{0})$ as evaluated in (25) is positive definite. Therefore, $\phi(\mathbf{x},\mathbf{v})$ has a local minimum at $(\mathbf{0},\mathbf{0})$. Because the storage function V is positive definite and $\bar{f}^{\mathbf{v}(k)}(\mathbf{x}(k))\Big|_{\mathbf{x}^*=\mathbf{0},\mathbf{v}^*=\mathbf{0}} = 0$, we obtain $\phi(\mathbf{x},\mathbf{v}) = 0$ at $(\mathbf{0},\mathbf{0})$. ∎

Proof for Theorem 3.1.

Proof: The proof is in two steps. First, we prove that (13) holds for all $T \geq 2$. We use mathematical induction. Recall that $\tau(T)$ denotes the total number of open loop time steps when $\mathcal{S}$ is in Mode 2 during the time period $[1, T-1]$. For $T = 2$, by assumption we have $d(1) = 1$. Thus, $\tau(T) = 0$ and from (12), it follows that $\tilde{V}(\mathbf{x}(2)) \leq \sigma\tilde{V}(\mathbf{x}(1)) + \sigma\mathbf{v}^{\mathrm{T}}(1)\mathbf{v}(1)$. Since $\zeta^M \geq 1$, the inequality (13) holds in this case. Next, assume that (13) holds for T. Two cases are possible at time T:

- If $d(T) = 1$, then $\tau(T+1) = \tau(T)$. From (12), we have $\tilde{V}(\mathbf{x}(T+1)) \leq \sigma\tilde{V}(\mathbf{x}(T)) + \sigma\mathbf{v}^{\mathrm{T}}(T)\mathbf{v}(T)$. From (13), we obtain that

$$\tilde{V}(\mathbf{x}(T+1)) \leq \sigma^{T+1-\tau(T)-1}\zeta^{\tau(T)}\tilde{V}(\mathbf{x}(1)) + \zeta^M\sigma^2 \sum_{\substack{k:d(k)=1\\k\leq T-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k) + \sigma\mathbf{v}^{\mathrm{T}}(T)\mathbf{v}(T).$$

Since $\zeta^M\sigma^2 \leq \zeta^M\sigma, \sigma \leq \zeta^M\sigma$ and $\tau(T+1) = \tau(T)$, we have

$$\tilde{V}(\mathbf{x}(T+1)) \leq \sigma^{T+1-\tau(T+1)-1}\zeta^{\tau(T+1)}\tilde{V}(\mathbf{x}(1)) + \zeta^M\sigma \sum_{\substack{k:d(k)=1\\ k\leq(T+1)-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k),$$

i.e., the inequality (13) holds for $T+1$ as well if $d(T)=1$.

- If $d(T)=2$, then $\tilde{V}(\mathbf{x}(T+1)) \leq \zeta\tilde{V}(\mathbf{x}(T))$ according to condition (10). By assumption that the number of consecutive open loop steps is bounded by M, hence there exists a time step t such that $d(t)=1$ and $t \in [T-M, T-1]$. Thus, according to (12), $\tilde{V}(\mathbf{x}(t+1)) \leq \sigma\tilde{V}(\mathbf{x}(t)) + \sigma\mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t)$. Let $b(m)$ denote the total number of open-loop time steps when $\mathcal{S}$ is in mode 2 during the time interval $[t, t+m]$, where $T = t+m$. We prove that for any $m \in [1, M]$

$$\tilde{V}(\mathbf{x}(T+1)) = \tilde{V}(\mathbf{x}(t+m+1)) \leq \zeta^{b(m)}\sigma^{m+1-b(m)}\tilde{V}(\mathbf{x}(t)) + \zeta^{b(m)}\sigma \sum_{\substack{k=t\\ d(k)=1}}^{t+m} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k). \tag{27}$$

We also prove the inequality (27) by mathematical induction.

1) When $m=1$, if $d(t+1)=1$, according to (12), we have

$$\begin{aligned}
\tilde{V}(\mathbf{x}(t+2)) &\leq \sigma\tilde{V}(\mathbf{x}(t+1)) + \sigma\mathbf{v}^{\mathrm{T}}(t+1)\mathbf{v}(t+1) \\
&\leq \sigma^2\tilde{V}(\mathbf{x}(t)) + \sigma^2\mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t) + \sigma\mathbf{v}^{\mathrm{T}}(t+1)\mathbf{v}(t+1) \\
&\leq \sigma^2\tilde{V}(\mathbf{x}(t)) + \sigma\left(\mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t) + \mathbf{v}^{\mathrm{T}}(t+1)\mathbf{v}(t+1)\right).
\end{aligned}$$

Therefore, the inequality (27) is satisfied with $b(m)=0$.

If $d(t+1)=2$, according to condition (10) and the inequality (12), we have

$$\tilde{V}(\mathbf{x}(t+2)) \leq \zeta\tilde{V}(\mathbf{x}(t+1)) \leq \zeta\left(\sigma\tilde{V}(\mathbf{x}(t)) + \sigma\mathbf{v}^{\mathrm{T}}(t)\mathbf{v}(t)\right).$$

Therefore, the inequality (27) is satisfied with $b(m)=1$. Hence, the inequality (27) is satisfied when $m=1$.

2) Assume that (27) holds for $m-1$, i.e.,

$$\tilde{V}(\mathbf{x}(t+m)) \le \zeta^{b(m-1)}\sigma^{m-b(m-1)}\tilde{V}(\mathbf{x}(t)) + \zeta^{b(m-1)}\sigma \sum_{\substack{k=1 \\ d(k)=1}}^{t+m-1} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k). \tag{28}$$

If $d(t+m)=1$, we have $b(m-1)=b(m)$. From (12) and (28), it follows that

$$\begin{aligned}
\tilde{V}(\mathbf{x}(t+m+1)) &\le \sigma\tilde{V}(\mathbf{x}(t+m)) + \sigma\mathbf{v}^{\mathrm{T}}(t+m)\mathbf{v}(t+m) \\
&\le \zeta^{b(m-1)}\sigma^{m+1-b(m-1)}\tilde{V}(\mathbf{x}(t)) + \zeta^{b(m-1)}\sigma^2 \sum_{\substack{k=1 \\ d(k)=1}}^{t+m-1} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k) \\
&+ \sigma\mathbf{v}^{\mathrm{T}}(t+m)\mathbf{v}(t+m).
\end{aligned} \tag{29}$$

Since $\zeta^{b(m-1)}\sigma^2 = \zeta^{b(m)}\sigma^2 \le \zeta^{b(m)}\sigma$ and $\sigma \le \zeta^{b(m)}\sigma$, from (29), we obtain

$$\tilde{V}(\mathbf{x}(t+m+1)) \le \zeta^{b(m)}\sigma^{m+1-b(m)}\tilde{V}(\mathbf{x}(t)) + \zeta^{b(m)}\sigma \sum_{\substack{k=1 \\ d(k)=1}}^{t+m} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k),$$

i.e., the inequality (27) holds for m if $d(t+m)=1$.

If $d(t+m)=2$, we have $b(m)=b(m-1)+1$. According to condition (10) and the inequality (28), we have

$$\begin{aligned}
\tilde{V}(\mathbf{x}(t+m+1)) &\le \zeta\tilde{V}(\mathbf{x}(t+m)) \\
&\le \zeta^{b(m-1)+1}\sigma^{m-b(m-1)}\tilde{V}(\mathbf{x}(t)) + \zeta^{b(m-1)+1}\sigma \sum_{\substack{k=1 \\ d(k)=1}}^{t+m-1} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k), \\
&= \zeta^{b(m)}\sigma^{m+1-b(m)}\tilde{V}(\mathbf{x}(t)) + \zeta^{b(m)}\sigma \sum_{\substack{k=1 \\ d(k)=1}}^{t+m} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k),
\end{aligned}$$

i.e., the inequality (27) holds for m if $d(t+m)=2$. Hence, the inequality (27) holds for both $d(t+m)=1$ and $d(t+m)=2$ where $1<m\le M$.

3) Therefore, by mathematical induction, the inequality (27) is satisfied for all $m \in [1, M]$.

By induction, the inequality (13) holds for all time steps before T. Hence, we have

$$\tilde{V}(\mathbf{x}(t)) \leq \sigma^{t-\tau(t)-1}\zeta^{\tau(t)}\tilde{V}(\mathbf{x}(1)) + \zeta^{M}\sigma \sum_{\substack{k:d(k)=1\\ k\leq t-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k). \tag{30}$$

Substitute (30) into (27), we obtain

$$\begin{aligned}\tilde{V}(\mathbf{x}(T+1)) &\leq \zeta^{b(m)}\sigma^{m+1-b(m)}\sigma^{t-\tau(t)-1}\zeta^{\tau(t)}\tilde{V}(\mathbf{x}(1)) + \zeta^{b(m)}\sigma^{m+1-b(m)}\zeta^{M}\sigma \sum_{\substack{k:d(k)=1\\ k\leq t-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k)\\ &+ \zeta^{b(m)}\sigma \sum_{\substack{k=t\\ d(k)=1}}^{t+m} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k). \end{aligned} \tag{31}$$

Since $\tau(t)+b(m)=\tau(T+1),\ b(m)\leq m\leq M$, it follows that

$$\begin{cases} \zeta^{b(m)+\tau(t)}\sigma^{m+1-b(m)+t-\tau(t)-1} = \zeta^{\tau(T+1)}\sigma^{T+1-\tau(T+1)-1} \\ \zeta^{b(m)+M}\sigma^{m+1-b(m)}\sigma \leq \zeta^{2M}\sigma^{m+2-b(m)} \leq \zeta^{2M}\sigma^{2} \end{cases}.$$

Hence, the inequality (31) becomes

$$\tilde{V}(\mathbf{x}(T+1)) \leq \zeta^{\tau(T+1)}\sigma^{T+1-\tau(T+1)-1}\tilde{V}(\mathbf{x}(1)) + \zeta^{2M}\sigma^{2} \sum_{\substack{k:d(k)=1\\ k\leq t-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k) + \zeta^{M}\sigma \sum_{\substack{k=t;\\ d(k)=1}}^{t+m} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k).$$

By assumption that $\zeta^{M}\sigma\leq 1$, it follows that $\zeta^{2M}\sigma^{2}=(\zeta^{M}\sigma)^{2}\leq\zeta^{M}\sigma$. Therefore, we have

$$\tilde{V}(\mathbf{x}(T+1)) \leq \zeta^{\tau(T+1)}\sigma^{T+1-\tau(T+1)-1}\tilde{V}(\mathbf{x}(1)) + \zeta^{M}\sigma \sum_{\substack{k:d(k)=1\\ k\leq (T+1)-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k),$$

i.e., the inequality (13) holds for $T+1$ as well if $d(T)=2$.

Hence, for both cases (either $d(T)=1$ or $d(T)=2$) we prove that (13) holds with $T+1$. From condition (11), we have $\sigma^{T-\tau(T)-1}\zeta^{\tau(T)}\leq 1$, and by assumption $\zeta^{M}\sigma\leq 1$. Therefore, the inequality (13) becomes

$$\tilde{V}(\mathbf{x}(T)) \leq \tilde{V}(\mathbf{x}(1)) + \sum_{\substack{k:d(k)=1\\ k\leq T-1}} \mathbf{v}^{\mathrm{T}}(k)\mathbf{v}(k).$$

This completes the proof. ■

Proof for Theorem 3.3.

Proof: Since System $\mathcal{S}$ is locally passive, we can follow the proof of Theorem 3.1 and construct a control law $\mathbf{u}(k)$ as defined by Equation (5) that guarantees that for any $\mathbf{v}(k) \in \mathbf{U}$, $\mathbf{y}(k) = \mathbf{v}(k)$ if $d(k) = 1$ and the inequality (14) holds.

Now, we choose $\mathbf{v}(k) = \mathbf{0},\ \forall k$. Thus, the control law is given by

$$\mathbf{u}(k) = \begin{cases} \eta(\mathbf{x}(k), \mathbf{0}) & \text{if } d(k) = 1 \\ \mathbf{0} & \text{if } d(k) = 2 \end{cases},$$

so that $\mathbf{y}(k) = \mathbf{0}$ if $d(k) = 1$. In this case, the inequality (14) reduces to

$$\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) \leq 0,\ \forall \mathbf{x}(\cdot) \in \mathbf{X}, \forall T.$$

In other words, there exists a function η and a positive definite storage function $\tilde{V}(\mathbf{x}(\cdot)) = aV(\mathbf{x}(\cdot))$ such that the inequality (14) holds.

Recall the sequence of time steps $\{k_i\}$ such that $k_0 = 1$ and $k_i =$ the least time $> k_{i-1}$ such that $d(k_i - 1) = 2$ and $d(k_i) = 1$. Choosing $T = k_1$ yields in particular $\tilde{V}(\mathbf{x}(k_1)) - \tilde{V}(\mathbf{x}(1)) \leq \mathbf{0}, \forall \mathbf{x}(\cdot) \in \mathbf{X}$. Following Remark 3.3, we can repeat the same argument starting from time k_i with $\mathbf{x}(k_i)$ as the initial condition. Thus we obtain the series of inequalities

$$\tilde{V}(\mathbf{x}(k_{i+1})) - \tilde{V}(\mathbf{x}(k_i)) \leq 0,\ \forall i = 0, 1, \cdots, \forall \mathbf{x} \in \mathbf{X}.$$

When T is infinite, since Mode 1 is active infinitely often, $\{k_i\}$ is an infinite sequence. Then $\tilde{V}(\mathbf{x}(\cdot))$ is a Lyapunov function for system $\mathcal{S}$ which implies that the system is Lyapunov stable with the given control law. The asymptotic stability then follows from ZSD. Observe that all the trajectories of the closed-loop system eventually approach the invariant set $I = \{\mathbf{x} \in \mathbb{R}^n : V(\mathbf{x}(k+1)) = V(\mathbf{x}(k))\}$. Since $\mathbf{y}(k) = 0$ and by ZSD $\lim_{k\to\infty} \mathbf{x}(k) = \mathbf{0}$. The system is thus locally asymptotically stable with the given control law.

When T is finite, the sequence $\{k_i\}$ is also finite and depends on T. However, since the number of time steps in Mode 2 is bounded and the ratio between Mode 1 and 2 is given by (11) , the state will always be bounded in a neighborhood of the origin and also satisfies the inequality $\tilde{V}(\mathbf{x}(k_{i+1})) - \tilde{V}(\mathbf{x}(k_i)) \leq 0,\ \forall i = 0, 1, \cdots, \forall \mathbf{x} \in \mathbf{X}$. This implies that the system is stable with the given control law. ■

Proof for Theorem 3.4.

Proof: If System $\mathcal{S}^1$ (respectively $\mathcal{S}^2$) is locally feedback passive, then there exist a control law $\mathbf{u}_1(k) = \eta_1(\mathbf{x}_1(k), \mathbf{v}_1(k))$ when $d_1(k) = 1$ and $\mathbf{u}_1(k) = \mathbf{0}$ when $d_1(k) = 2$ (resp. $\mathbf{u}_2(k) = \eta_2(\mathbf{x}_2(k), \mathbf{v}_2(k))$ when $d_2(k) = 1$ and $\mathbf{u}_2(k) = \mathbf{0}$ when $d_2(k) = 2$) and a positive definite storage function $\tilde{V}_1(\mathbf{x}_1(\cdot))$ (resp. $\tilde{V}_2(\mathbf{x}_2(\cdot))$) such that the inequality (6) is satisfied for any sequence $\mathbf{u}(k) \in \mathbf{U}(k)$. For the parallel interconnection, the extrinsic control sequence $\mathbf{v}(k)$ is the same for both systems and the output $\mathbf{y}(k) = \mathbf{y}_1(k) + \mathbf{y}_2(k)$. Consider the control law $\mathbf{u}(k) = [\mathbf{u}_1^{\mathrm{T}}(k) \quad \mathbf{u}_2^{\mathrm{T}}(k)]^{\mathrm{T}}$ and the storage function $\tilde{V}(\mathbf{x}(k)) = \tilde{V}(\mathbf{x}_1(k), \mathbf{x}_2(k)) = \tilde{V}_1(\mathbf{x}_1(k)) + \tilde{V}_2(\mathbf{x}_2(k))$. For any time $T \in \mathbb{Z}^+$, we have $\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) = (\tilde{V}_1(\mathbf{x}_1(T)) - \tilde{V}_1(\mathbf{x}_1(1))) + (\tilde{V}_2(\mathbf{x}_2(T)) - \tilde{V}_2(\mathbf{x}_2(1))) \leq \sum_{k=1}^{T-1} \mathbf{v}^{\mathrm{T}}(k)\mathbf{y}_1(k) + \sum_{k=1}^{T-1} \mathbf{v}^{\mathrm{T}}(k)\mathbf{y}_2(k) \leq \sum_{k=1}^{T-1} \mathbf{v}^{\mathrm{T}}(k)\mathbf{y}(k)$.

Similarly, for the negative feedback interconnection, the control inputs and outputs are as $\mathbf{r}_1(k) = \mathbf{v}_1(k) + \mathbf{y}_2(k)$ and $\mathbf{r}_2(k) = \mathbf{v}_2(k) - \mathbf{y}_1(k)$. Consider the control law $\mathbf{u}(k) = [\mathbf{u}_1^{\mathrm{T}}(k) \ \ \mathbf{u}_2^{\mathrm{T}}(k)]^{\mathrm{T}}$ and the storage function $\tilde{V}(\mathbf{x}_1(k), \mathbf{x}_2(k)) = \tilde{V}_1(\mathbf{x}_1(k)) + \tilde{V}_2(\mathbf{x}_2(k))$, we have $\tilde{V}(\mathbf{x}(T)) - \tilde{V}(\mathbf{x}(1)) \leq \sum_{k=1}^{T-1}(\mathbf{r}_1^{\mathrm{T}}(k)\mathbf{y}_1(k) + \mathbf{r}_2^{\mathrm{T}}(k)\mathbf{y}_2(k))$. ■

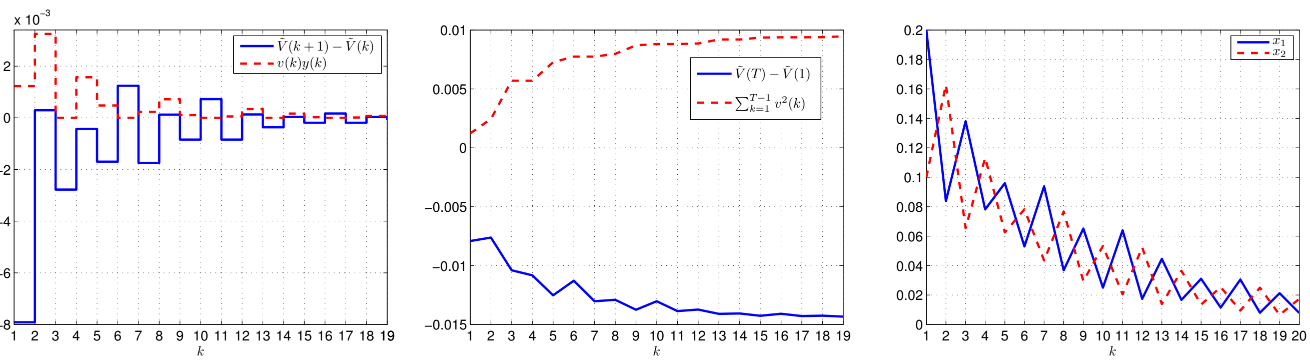
EXAMPLES

A. Example 1

In this example, we passify a nonlinear switched system by applying a regular state feedback control law across a network with packet drops. Consider a system of the form

$$\begin{aligned} x_1(k+1) &= -0.3x_1^2(k)x_2(k) + 1.2x_2(k) + u(k) \\ x_2(k+1) &= 0.82x_1(k) - u^2(k) \\ y(k) &= 0.7x_2(k) + u(k), \end{aligned} \tag{32}$$

with initial states $x_1(1) = 0.2, x_2(1) = 0.1$. Note that system (32) is locally ZSD and has relative degree zero. As discussed earlier, we construct $\eta(\mathbf{x}(k), v(k))$ by imposing $v(k) = y(k)$. This leads to $u(k) = \eta(\mathbf{x}(k), v(k)) = v(k) - 0.7x_2(k)$. The resulting feedback transformed system has a passive zero dynamics with $v(k) = 0$, and hence the system is feedback passive for any possible $v(k)$. For the purpose of numerical illustration, we choose the external input as $v(k) = 0.35x_2(k)$, which leads to the controller $u(k) = -0.35x_2(k)$. The evolution of the system in Mode 2 is given by Equation (3) with $u(k) = 0$. In Mode 1, the transformed dynamics and the zero dynamics of system (32) can be obtained as in Equations (7) and (8). Given the zero dynamics, we choose a quadratic storage function $V(\mathbf{x}(k)) = \mathbf{x}(k)^{\mathrm{T}} P \mathbf{x}(k) = x_1^2(k) + 0.5x_2^2(k)$. We can verify that the determinant of the Hessian matrix of $V(\mathbf{x}(k))$ at $\mathbf{x}(k) = [0\ 0]^{\mathrm{T}}$ is not zero.



(a) (b) (c)

Fig. 2. (a) Passivity check for the switched system in the time interval $[1, 20]$ according to classical passivity Definition 2.2, (b) Passivity check for the switched system according to the generalized feedback passivity Definition 2.4, and (c) State dynamics of the switched system.

More insight can be obtained if we consider the system to operate over a finite horizon T. Consider the system operation from $k = 1$ to 20. The parameters in the condition (10) are $\zeta = 2.88$ and $\sigma = 0.5516$. The condition $\sigma\zeta^M < 1$ required for infinite T will be too restrictive in this case. According to (11) then, choosing the ratio $r(T)$ to satisfy

$$r(T) \geq \frac{1.0578(T-1)}{1.0578 + 0.5949T} \tag{33}$$

would guarantee system passivity. This condition is satisfied, e.g., by a periodic system in which at every third time step (i.e., at $k = 3, 6, 9, \cdots$) the system is in Mode 2. However, the system need not be periodic to satisfy (33). If the system starts in Mode 1, then any communication protocol that guarantees that out of every 3 consecutive control packets, at most one packet is not delivered would guarantee passivity. We consider the system to be in Mode 2 at time steps $k = 3, 6, 10, 13, 16, 19$ as shown in Figure 2(a). Thus, the classical feedback passivity inequality (2) does not necessarily hold at these time steps. The storage function $\tilde{V}(\mathbf{x}(k))$ for the transformed system is chosen as $0.32V(\mathbf{x}(k))$ with $\hat{a} = 0.49$ and $\tilde{a} = 1.9996$. Figure 2(b) shows the corresponding generalized feedback passivity inequality (6) for the system. We can see that unlike the classical case, the storage function is now allowed to be greater than the supplied energy instantaneously; however, the general passivity inequality is satisfied at every time till T. Figure 2(c) shows the evolution of the state dynamics of the switched system. If we choose the control to be $\mathbf{u}(k) = -0.7x_2(k)$, the system can achieve local stability.

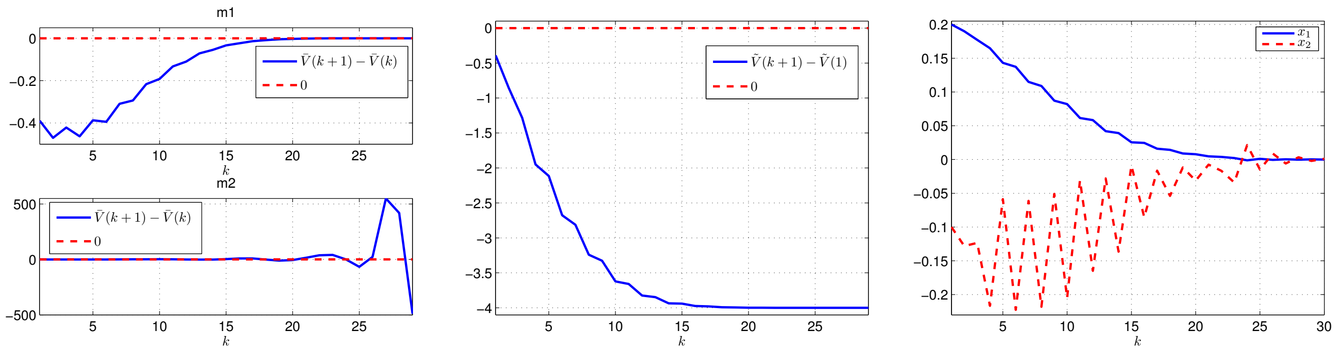
B. Example 2

Consider the following nonlinear mass-damper-spring system which is controlled through a network with packet drops. A negative damper is used so that the system is non-passive and open loop unstable. We use the proposed method to passify and stabilize the system.

$$\begin{aligned}
x_1(k+1) &= x_1(k) + Tx_2(k) \\
x_2(k+1) &= -\frac{K}{m}Tx_1(k) + \left(1 - \frac{c}{m}T\sin(x_1(k))\right)x_2(k) + \frac{T}{m}u(k) \\
y(k) &= 18x_2(k) + u(k),
\end{aligned}$$

where x_1 and x_2 are the displacement and velocity and $u(k)$ is the force. We set the sampling period $T = 0.1$s, mass $m = 0.5$kg, stiffness $K = 1$N/m, viscous damping coefficient $c = 3\text{N}\cdot\text{s/m}$ and initial conditions $x_1 = 0.2$m, $x_2 = -0.1$m/s. We choose the controller by imposing $f_2(\mathbf{x}, u) = \bar{f}^{\mathbf{0}}(\mathbf{x})$. The resulting controller is $u(k) = -5x_1(k) - 10x_2(k)$ with $v(k) = 0$ which renders the system locally passive and stable. The evolution of the system in Mode 2 is given in Figure 3 when $u(k) = \mathbf{0}$. We choose a storage function $V(\mathbf{x}(k)) = 100x_1^2 + 0.01x_2^2$. We can also verify that the determinant of the Hessian matrix of $V(\mathbf{x}(k))$ at $\mathbf{x}(k) = [0\ 0]^{\mathrm{T}}$ is not zero. We consider the system to operate from $k = 1$ to 30 and with

$d(k) = 2$ at time steps $k = 2, 11, 20, 29$. The parameters in condition (10) are $\zeta = 1.23$ and $\sigma = 0.92$. Figure 3(a) shows the corresponding passivity inequality for Mode 1 and 2, respectively. Figure 3(b) shows the generalized passivity inequality according to 6. Figure 3(c) shows the evolution of the state dynamics of the switched system. Both states are locally stable.



(a) (b) (c)

Fig. 3. (a) Passivity check for Mode 1 and 2 according to classical feedback passivity definition 2.2, (b) Passivity check for the switched system according to the generalized feedback passivity definition 2.4, and (c) State dynamics of the switched system.