# Decentralized Event-Triggered Cooperative Control with Limited Communication

**Abstract**

**This note studies event-triggered control of Multi-Agent Systems (MAS) with first order integrator dynamics. It extends previous work on event-triggered consensus by considering limited communication capabilities through strict peer-to-peer non-continuous information exchange. The approach provides both a decentralized control law and a decentralized communication policy. Communication events require no global information and are based only on local state errors; agents do not require a global sampling period or synchronous broadcasting as in sampled-data approaches. The proposed decentralized event-triggered control technique guarantees that the inter-event times for each agent are strictly positive. Finally, the ideas in this note are used to consider the practical scenario where agents are able to exchange only quantized measurements of their states.**

Keywords: event-triggered control; consensus; quantization; multi-agent systems.

## 1. Introduction

An increasing interest in controlling large scale dynamical systems composed of several to many autonomous mobile agents exists in different academic, commercial, and military areas. This thrust is related to the large number of applications in which a group of coordinated agents is potentially able to outperform a single or a number of systems operating independently (Ren, Beard, and Atkins 2007). An important problem in Multi-Agent Systems (MAS) is to design and implement decentralized algorithms for control and communication of agents. It is well understood that each agent should be able to determine its own control laws independently and based only on local information. This has been an important research topic (Ren, Beard, and Atkins 2007; Moreau 2004; Ji and Egerstedt 2007; Tanner, Jadbabaie and Pappas 2003). These papers consider agents with continuous-time dynamics and it is assumed that agents can have continuous access to the states of their neighbors. In many applications the agents transmit their relevant variables such as position, velocity, heading, etc. to a subset of the agents not continuously but at discrete points in time. It is important to discern how frequently the agents should establish communication in order to preserve properties of similar control algorithms that assume continuous information exchange. The sample-data approach is commonly used to estimate the sampling periods (Can and Ren 2009; Can and Ren 2010; Hayakawa, Matsuzawa, and Hara 2006; Liu, Xie, and Wang 2010; Qin and Gao 2012). An important drawback of periodic transmission is that it requires synchronization between the agents, that is, all agents need to transmit their information at the same time instants and, in some cases, it requires a conservative sampling period for worst case situations.

In event-triggered broadcasting (Astrom and Bernhardson 2002; Astrom 2008; Tabuada 2007; Wang and Lemon 2008; Wang and Lemon 2011; Donkers and Heemels 2010; Garcia and Antsaklis 2013; Anta and Tabuada 2010) a subsystem sends its local state to the network only when it is necessary, that is, only when a measure of the local subsystem state error is above a specified threshold. Event-triggered control schemes offer a new point of view, with respect to conventional time-driven strategies, on how information could be sampled for control purposes. The seminal work (Astrom and Bernhardson 2002) provided an interesting comparison between conventional time driven sampling and the new event-driven sampling, emphasizing the practical advantages of the latter. Tabuada (2007) presented a triggering condition based on norms of the state and the state error $e = x(t_k) - x(t)$, that is, the last measured state minus the current state of the system, where the measurement received at the controller node is held constant until a new measurement arrives. When this happens, the error is set equal to zero and starts growing until it triggers a new measurement update.

The use of event-triggered control strategies in networked systems (Dimarogonas, Frazzoli, and Johansson 2012; Dimarogonas and Johansson 2009; Dimarogonas and Frazzoli 2009; Yu and

Antsaklis 2012; Garcia and Antsaklis 2012; Sun and El-Farra 2011; Seyboth, Dimarogonas, and Johansson 2013) provides a more robust and efficient use of network bandwidth. Its implementation in MAS also provides a highly decentralized way to schedule transmission instants which does not require synchronization compared to periodic sampled-data approaches.

The work in the present paper is similar to (Dimarogonas, Frazzoli, and Johansson 2012; Dimarogonas and Johansson 2009; Dimarogonas and Frazzoli 2009) where the consensus problem with single integrator dynamics, event-based communication, and connected and undirected graphs was considered. The main advantage of our approach compared to these papers is that we consider both the reduction of actuation and communication updates while they only focus on reduction of update instants, i.e. they still assume that continuous communication exists among agents in order to calculate the error thresholds. Since continuous access to the states of neighbors is typically not possible we extend the work in (Dimarogonas, Frazzoli, and Johansson 2012) to consider the exchange of information among agents at discrete time instants which are, in general, non-periodic and based on local events. The present paper also provides an important extension to consider the case where the agents are able to transmit only a quantized version of its measured state. Similar work (Seyboth, Dimarogonas, and Johansson 2013) uses a different threshold that does not require continuous access to the states of neighbors. The approach in this note preserves the decentralized nature of the event computations compared to (Seyboth, Dimarogonas, and Johansson 2013) where an estimate of the second eigenvalue of the Laplacian matrix ($L$) is used to trigger communication events. The communication policy described in the present paper is decentralized in the sense that each agent computes its transmission instants based on local information. We provide asymptotic convergence to the initial average using the new threshold that considers only the last received states of the neighbors. The policy ensures strictly positive inter-event times. For the case when quantized measurements are used we are able to show convergence to a bounded region around the initial average; this bound is proportional to the quantization parameter. An extended scheme is also proposed in order to guarantee strictly positive inter-event times in the presence of quantization.

The remainder of this document is organized as follows: Section 2 addresses the event-triggered control strategy that considers limited knowledge of states of neighbors. Section 3 presents similar results using quantized measurements. Section 4 provides illustrative examples and conclusions are given in Section 5.

## 2. Decentralized Consensus

We consider a set of *n* agents that are modeled as a single integrator:

$$\dot{x}_i = u_i, \quad i = 1 \ldots n. \tag{1}$$

where $x_i \in \mathbb{R}$ is the state and $u_i \in \mathbb{R}$ is the control input associated to agent *i*. Since continuous measurements from neighbors are not available to each agent, then the control input is obtained using the last measurements received from each neighbor $j \in N_i$ as follows:

$$u_i(t) = u_i(t_{k_i}, t_{k_j}) = -\sum_{j \in N_i} (x_i(t_{k_i}) - x_j(t_{k_j})), \quad i = 1 \ldots n \tag{2}$$

where $x_i(t_{k_i})$ represents the last measurement transmitted by agent *i* at its update time $t_{k_i}$ and $N_i$ is the set of neighbors of agent *i*. Similarly, $x_j(t_{k_j})$ represents the last measurements received from neighbor *j* at the corresponding time $t_{k_j}$. In general, the update intervals are nonperiodic and the update instants for each agent are different from those of other agents, i.e. $t_{k_i}$ and $t_{k_j}$ are not necessarily equal.

The events are also computed based only on local information, that is, events are designed based on information that is available to each agent. We propose the following threshold:

$$e_i^2(t) > \frac{\sigma_i a(1-a|N_i|)}{|N_i|} z_i^2(t_{k_i}, t_{k_j}) \tag{3}$$

where $e_i(t) = x_i(t_{k_i}) - x_i(t)$ represents the novel information with respect to the last transmitted measurement, $0 < a < (1/|N_i|)$, $0 < \sigma_i < 1$, $|N_i|$ is the cardinality of $N_i$, and

$$z_i(t_{k_i}, t_{k_j}) = \sum_{j \in N_i} (x_i(t_{k_i}) - x_j(t_{k_j})). \tag{4}$$

In this paper we use the notation $(t_{k_i}, t_{k_j})$ to represents piecewise constant variables that are updated at times $t_{k_i}$, when the local agent transmits an update, and also at all times $t_{k_j}$ for $j \in N_i$, when the agent receives an update from any of its neighbors.

At each node the updates of the piecewise constant versions of the states $x_i(t_{k_i})$ and $x_j(t_{k_j})$ are as follows. When an event is triggered at time $t = t_{k_i}$ the local agent updates its local piecewise constant version of its state using the current measurement $x_i(t)$, i.e. $x_i(t_{k_i}) = x_i(t)$ and transmits this measurement to its neighbors.

On the other hand, when the local agent receives an update from any of its neighbors $j \in N_i$ at corresponding times $t = t_{k_j}$ containing a current measurement $x_j(t)$, the local agent uses this measurement to update its piecewise constant version of the state $x_j$, that is, $x_j(t_{k_j}) = x_j(t)$. Note that (2) and (4) are functions of $x_i(t_{k_i})$ and all neighbors states $x_j(t_{k_j})$ for $j \in N_i$, therefore, they are updated at all corresponding time instants $t_{k_i}$ and $t_{k_j}$.

Eq. (3) is similar to the threshold in [18]; however, the threshold in (Dimarogonas, Frazzoli, and Johansson 2012) is based on the continuous variable $z_i(t)$ which is given by:

$$z_i(t) = \sum_{j \in N_i} (x_i(t) - x_j(t)). \tag{5}$$

It is clear that $z_i(t)$ is a function of the continuous measurements of local agent $x_i(t)$, and it is also a function of the continuous measurements of all neighbors $x_j(t)$. It is evident that the local agent is not able to design this threshold since continuous measurements from neighbors are not available. In this work we try to reduce both the actuation updates and the communication updates, while the authors of (Dimarogonas, Frazzoli, and Johansson 2012) only considered the reduction of actuation updates assuming that the agents can have access to the continuous states of their neighbors. Therefore the threshold in (Dimarogonas, Frazzoli, and Johansson 2012) cannot be used in the present paper.

When an event is triggered by agent *i* we have $e_i(t_{k_i}) = x_i(t_{k_i}) - x_i(t) = x_i(t_{k_i}) - x_i(t_{k_i}) = 0$ because $t = t_{k_i}$ is an event time for agent *i*. We also have that

$$e_i^2(t) \le \frac{\sigma_i a(1-a|N_i|)}{|N_i|} z_i^2(t_{k_i}, t_{k_j}) \tag{6}$$

holds for any value of $z_i(t_{k_i}, t_{k_j})$. Note that the triggering condition (3) guarantees that (6) is satisfied. Let $x(t_{k_1} \ldots t_{k_n}) = [x_1(t_{k_1}) \ldots x_n(t_{k_n})]^T$ represent the vector containing the latest broadcasted updates by each agent in the network, that is, this vector is a function of all update times $t_{k_i}$ for *i*=1…*n*. Assume that input and communication delays are negligible. The next result shows convergence for a group of agents using the new threshold (3) under control (2).

**Theorem 1.** Consider a group of agents $\dot{x}_i = u_i$ for *i*=1…*n*, with control inputs given by (2) and with event-based updates given by (3). Assume that the communication graph is connected and undirected. Then all agents asymptotically stabilize to their initial average.

*Proof.* Consider the ISS Lyapunov function $V = (1/2)x^T L x$. We have that

$$\dot{V} = x(t)^T L\dot{x}(t) = -x(t)^T LLx(t_{k_1}...t_{k_n}) = -\left(x(t_{k_1}...t_{k_n})^T - e(t)^T\right) LLx(t_{k_1}...t_{k_n})$$
$$= -\sum_i z_i^2(t_{k_i}, t_{k_j}) + \sum_i \sum_{j\in N_i} (e_i(t) - e_j(t)) z_i(t_{k_i}, t_{k_j})$$

where $x(t) = [x_1(t) \ldots x_n(t)]^T$ and $e(t) = [e_1(t) \ldots e_n(t)]^T$. By using the inequality $|xy| \le \frac{a}{2}x^2 + \frac{1}{2a}y^2$, for $a$>0, we have:

$$\dot{V} \le -\sum_i z_i^2(t_{k_i}, t_{k_j}) + \sum_i |N_i| \left(\frac{a}{2} z_i^2(t_{k_i}, t_{k_j}) + \frac{1}{2a} e_i^2(t)\right) + \sum_i \sum_{j\in N_i} \left(\frac{a}{2} z_i^2(t_{k_i}, t_{k_j}) + \frac{1}{2a} e_j^2(t)\right). \tag{7}$$

By symmetry of the undirected communication graph and using (6), we have:

$$\dot{V} \le -\sum_i \left(1 - a|N_i|\right) z_i^2(t_{k_i}, t_{k_j}) + \sum_i \frac{1}{a}|N_i| e_i^2(t) \le \sum_i \left(\sigma_i - 1\right)\left(1 - a|N_i|\right) z_i^2(t_{k_i}, t_{k_j}) \tag{8}$$

which implies $\dot{V} \le 0$ for $0 < a < (1/|N_i|)$ and $0 < \sigma_i < 1$.

Because $V \ge 0$, $\dot{V} \le 0$ implies that $V$ has a finite limit and $\dot{V} \to 0$ as $t \to \infty$. We have:

$$0 = \lim_{t\to\infty} \dot{V} \le \sum_i \left(\sigma_i - 1\right)\left(1 - a|N_i|\right) z_i^2(t_{k_i}, t_{k_j}) \le 0. \tag{9}$$

Since $\left(\sigma_i - 1\right)\left(1 - a|N_i|\right) < 0$ and $z_i^2(t_{k_i}, t_{k_j}) \ge 0$ then $\left(\sigma_i - 1\right)\left(1 - a|N_i|\right) z_i^2(t_{k_i}, t_{k_j}) \le 0$ for $i$=1…$n$. Thus, from (9), we have $z_i(t_{k_i}, t_{k_j}) = 0$ as $t \to \infty$ for $i$=1…$n$. In view of (6) and (2), when $z_i(t_{k_i}, t_{k_j}) = 0$ then all errors $e_i(t)$ reset and remain equal to zero, that is, since $z_i(t_{k_i}, t_{k_j}) = 0$ as $t \to \infty$ for $i$=1…$n$, then we have that $\lim_{t\to\infty} e_i(t) = 0$ for $i$=1…$n$. We can also write

$$\dot{V} = x(t)^T L\dot{x}(t) = -x(t)^T LL(x(t) + e(t)) = -z(t)^T z(t) - z(t)^T Le(t) \le 0. \tag{10}$$

Similarly,

$$0 = \lim_{t\to\infty} \dot{V} = \lim_{t\to\infty} \left(-z(t)^T z(t) - z(t)^T Le(t)\right). \tag{11}$$

Because $\lim_{t\to\infty} e_i(t) = 0$ for $i$=1…$n$, it follows from (10) and (11) that

$$\lim_{t\to\infty}\left(-z(t)^T z(t)\right) = -\lim_{t\to\infty}\sum_i z_i^2(t) = 0 \tag{12}$$

that is $\lim_{t\to\infty} z_i(t) = 0$ for $i$=1…$n$. Recall the definition of $z_i(t)$ in (5), we have $\lim_{t\to\infty}\sum_{j\in N_i}(x_i(t)-x_j(t)) = 0$ for $i$=1…$n$ which can be written in vector form as

$$\lim_{t\to\infty} Lx(t) = 0_n. \tag{13}$$

When the interaction graph is connected the Laplacian $L$ has a simple zero eigenvalue with the associated eigenvector $1_n$. Therefore

$$\lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} x_j(t), \quad i, j = 1...n. \tag{14}$$

For undirected graphs it can be shown that the initial average remains constant. Define the average $\overline{x}(t) = \frac{1}{N}\sum_i x_i(t)$, we have the following:

$$\dot{\overline{x}}(t) = \frac{1}{N}\sum_i \dot{x}_i(t) = -\frac{1}{N}\sum_i\sum_{j\in N_i}(x_i(t)-x_j(t)) - \frac{1}{N}\sum_i\sum_{j\in N_i}(e_i(t)-e_j(t)) = 0 \tag{15}$$

and $\overline{x}(t) = \overline{x}(0) = \frac{1}{N}\sum_i x_i(0)$, then the initial average remains constant. ■

The authors of (Dimarogonas, Frazzoli, and Johansson 2012) were able to show that at any given time there exists at least one agent in the network for which its inter-event time is strictly positive. In this note we show that the inter-event times, not for at least one, but for all agents, are always strictly positive.

**Corollary 2.** Consider a group of agents $\dot{x}_i = u_i$, $i$=1…$n$, with control inputs given by (2) and with updates (3). Assume that the communication graph is connected. Then the inter-event times for each agent $i$=1…$n$, are strictly positive.

*Proof.* Consider the evolution of the term $e_i^2(t)$ over the interval $t \in [t_{k_i}, t_{k_i+1})$ when $e_i(t)$ is continuous:

$$\frac{d}{dt}e_i^2(t) = 2e_i(t)\dot{e}_i(t) = 2e_i(t)z_i(t_{k_i}, t_{k_j}) \le 2\left|e_i(t)z_i(t_{k_i}, t_{k_j})\right| \le ae_i^2(t) + \frac{1}{a}z_i^2(t_{k_i}, t_{k_j}) \tag{16}$$

and consider the differential equation:

$$\dot{\phi}_i = a\phi_i + \frac{1}{a}z_i^2(t_{k_i}, t_{k_j}) \tag{17}$$

with initial condition $\phi_i(t_{k_i}) = e_i^2(t_{k_i}) = 0$. Then, we have:

$$e_i^2(t) \le \phi_i(t) = \frac{1}{a}\int_{t_{k_i}}^{t} e^{a(t-\tau)}z_i^2(\tau)d\tau, \quad t \in [t_{k_i}, t_{k_i+1}^i). \tag{18}$$

A lower bound for the inter-event times of agent *i* is obtained by finding the minimum time *t* such as $\phi_i(t) > \chi_i z_i^2(t_{k_i}, t_{k_j})$, where $\chi_i = \frac{\sigma_i a(1-a|N_i|)}{|N_i|} > 0$.

We analyze two cases here, the first case is when $z_i(t_{k_i}, t_{k_j}) \neq 0$ at the last update instant $t_{k_i}$. In this case, from (18), $\phi_i(t)$ takes a finite time *t*>0 to grow from zero to $\chi_i z_i^2(t_{k_i}, t_{k_j})$ since $z_i^2(t_{k_i}, t_{k_j}) > 0$. The second case is when $z_i(t_{k_i}, t_{k_j}) = 0$. In this case $\phi_i(t) = 0$ for $t \in [t_{k_i}, t_{k_j})$, $t_{k_i} < t_{k_j}$ and we have that (6) holds, therefore agent *i* does not generate any event during that time interval. When agent *i* receives an update from its neighbors then $z_i(t_{k_i}, t_{k_j}) \neq 0$ and the first case holds, i.e. the error takes a finite time *t*>0 to grow from zero to $\chi_i z_i^2(t_{k_i}, t_{k_j})$. ■

*Remark 1.* The selection of threshold (3) is intuitive because it really is a function of local information and it is also related to how fast the error will grow at any given time and trigger the next event. In fact, (2), (3), and (18) tell us a clear picture of the communication pattern. Because $z_i(t_k)$ is used in (2), it determines how fast the corresponding agent moves with respect to its previous transmitted value and a proportional threshold is used for the same agent as seen in (3). If

$z_i(t_{k_i}, t_{k_j}) = 0$ for some agent $i$ at some update instant $t_{k_i}$ then $\dot{e}_i(t) = -\dot{x}_i(t) = z_i(t_{k_i}, t_{k_j}) = 0$, this means that the agent will not move, the error remains equal to zero, and the current $x_i(t)$ remains equal to the last update $x_i(t_{k_i})$. It is clear that there is no need to send additional updates if the current information has not changed and no events should be triggered. This is the main reason that the error is compared using 'strictly greater than' in (3) instead of 'equal' as in (Dimarogonas, Frazzoli, and Johansson 2012). The main benefit is that we are able to lower bound the inter-event times, not for at least one agent, but for all of them.

Recent work (Seyboth, Dimarogonas, and Johansson 2013) proposes a different threshold that does not require continuous access to the states of neighbors. The threshold is a function of time and other tuning parameters. The approach in this note preserves the decentralized nature of the solution compared to (Seyboth, Dimarogonas, and Johansson 2013) since one of the tuning parameters depends on global information, i.e. on the second eigenvalue of *L*. Algorithms for estimation of the second eigenvalue of the Laplacian have been presented in (Aragues, *et.al.* 2012), (Franceschelli, *et.al.* 2009), and (Yang, *et.al.* 2010). The algorithm in (Aragues *et.al.* 2012) is especially practical for implementation in the event-based approach in (Seyboth, Dimarogonas, and Johansson 2013) since the estimate of the second eigenvalue always remains smaller than the true second eigenvalue of *L*. This is a condition on the tuning parameter stated in (Seyboth, Dimarogonas, and Johansson 2013) for convergence of the consensus algorithm.

## 3. Decentralized Consensus with Quantization

It was assumed in the last section that the sensor is able to measure the state of the system with infinite precision. In reality, however, the measured variables have to be quantized in order to be represented by a finite number of bits to be used in processor operations and to be transmitted over a digital communication channel. In this section we study the effects of signal quantization on the convergence of the event-triggered control approach previously described in this paper.

We define a uniform quantizer as a function $q:\mathbb{R}\rightarrow\mathbf{V}$ such that:

$$\begin{aligned} & q(\mu)=\nu_i \quad \text{if } \mu\in\left[(\nu_i-\tfrac{\delta}{2}),(\nu_i+\tfrac{\delta}{2})\right) \\ & \left|\mu-q(\mu)\right|\leq\tfrac{\delta}{2} \end{aligned} \tag{19}$$

where $\delta$ represents the quantization step and $\nu_i\in\mathbf{V}=\{...-2\delta,-\delta,0,\delta,2\delta,...\}$. The above quantizer represents an infinite rate, uniform, passive quantizer with bounded quantization error.

The only variables that are available to compute the control inputs and the state errors for each agent are the quantized states of the agents. The control inputs are now given by:

$$u_i(t)=-\sum_{j\in N_i}\left(q(x_i(t_{k_i}))-q(x_j(t_{k_j}))\right),\quad i=1...n. \tag{20}$$

and the quantized state error is defined as follows:

$$e_i^q(t)=q(x_i(t_{k_i}))-q(x_i(t)),\quad i=1...n. \tag{21}$$

**Theorem 3.** Consider a group of agents $\dot{x}_i=u_i$ for $i$=1…$n$, and each agent transmits it quantized output $q(x_i(t_{k_i}))$ to its neighbors at some time instants $t_{k_i}$. The control inputs are given by (20) and the event-based updates are triggered when

$$(e_i^q(t))^2>\frac{\sigma_i a(1-a)}{|N_i|}M_i\,(t_{k_i},t_{k_j}) \tag{22}$$

is satisfied, where $0<a,\sigma_i<1$, $M_i(t_{k_i},t_{k_j})=\sum_{j\in N_i}\left(q(x_i(t_{k_i}))-q(x_j(t_{k_j}))\right)^2$. Assume that the communication graph is connected and undirected. Then all agents asymptotically stabilize to a bounded region around their initial average given by:

$$\lim_{t\rightarrow\infty}\left|x_i(t)-\overline{x}(0)\right|\leq\delta,\quad i=1...n. \tag{23}$$

and the average remains constant, i.e. $\overline{x}(t) = \frac{1}{N}\sum_i x_i(t) = \frac{1}{N}\sum_i x_i(0)$.

*Proof*. Consider the candidate Lyapunov function $V = \sum_i V_i$ where $V_i = \int_0^{x_i} q(\upsilon)d\upsilon$ with $\dot{V}_i = q(x_i(t))\dot{x}_i(t) = q(x_i(t))u_i(t)$, for $i$=1,…,$n$. Note that $V_i \geq 0$ since the series interconnection of a single integrator and a passive memoryless function is lossless (Khalil 2002). We have

$$\begin{aligned}\dot{V} &= \sum_i u_i(t)q(x_i(t)) = \sum_i \sum_{j\in N_i} -\left(q(x_i(t_{k_i})) - q(x_j(t_{k_j}))\right)\left(q(x_i(t_{k_i})) - e_i^q(t)\right) \\ &= \sum_i \sum_{j\in N_i}\left[-q(x_i(t_{k_i}))^2 + q(x_i(t_{k_i}))q(x_j(t_{k_j})) + \left(q(x_i(t_{k_i})) - q(x_j(t_{k_j}))\right)\left(e_i^q(t)\right)\right]\end{aligned} \tag{24}$$

and consider the following relation:

$$\sum_i \sum_{j\in N_i} q(x_i(t_{k_i}))^2 = \frac{1}{2}\sum_i \sum_{j\in N_i}\left[q(x_i(t_{k_i}))^2 + q(x_j(t_{k_j}))^2\right]. \tag{25}$$

Then we have:

$$\begin{aligned}\dot{V} &= \sum_i \sum_{j\in N_i}\left(q(x_i(t_{k_i})) - q(x_j(t_{k_j}))\right)\left(e_i^q(t)\right) - \frac{1}{2}\sum_i \sum_{j\in N_i}\left(q(x_i(t_{k_i})) - q(x_j(t_{k_j}))\right)^2 \\ &\leq \frac{a}{2}\sum_i M_i(t_{k_i}, t_{k_j}) + \frac{1}{2a}\sum_i \sum_{j\in N_i}(e_i^q(t))^2 - \frac{1}{2}\sum_i M_i(t_{k_i}, t_{k_j}) \\ &\leq -\frac{(1-a)}{2}\sum_i M_i(t_{k_i}, t_{k_j}) + \frac{1}{2a}\sum_i |N_i|(e_i^q(t))^2\end{aligned} \tag{26}$$

where the inequality $|xy| \leq \frac{a}{2}x^2 + \frac{1}{2a}y^2$, for $a$>0, has been used to obtain the second line in (26). By using the threshold (22) we can guarantee that

$$(e_i^q(t))^2 \leq \frac{\sigma_i a(1-a)}{|N_i|} M_i\,(t_{k_i}, t_{k_j}) \tag{27}$$

holds. Then we obtain

$$\dot{V} \leq \frac{(1-a)}{2}\sum_i (\sigma_i - 1) M_i\,(t_{k_i}, t_{k_j}) \tag{28}$$

which implies $\dot{V} \leq 0$ for $0 < a < 1$ and $0 < \sigma_i < 1$.

Because $V \geq 0$, $\dot{V} \leq 0$ implies that $V$ has a finite limit and $\dot{V} \to 0$ as $t \to \infty$. We have the following:

$$0 = \lim_{t\to\infty} \dot{V} \leq \tfrac{(1-a)}{2} \sum_i (\sigma_i - 1) M_i \, (t_{k_i}, t_{k_j}) \leq 0. \tag{29}$$

We also have that $(\sigma_i - 1) < 0$ and $M_i \, (t_{k_i}, t_{k_j}) \geq 0$ $i$=1…$n$; consequently $(\sigma_i - 1) M_i \, (t_{k_i}, t_{k_j}) \leq 0$ for $i$=1…$n$. From (29) we can see that $\sum_i (\sigma_i - 1) M_i \, (t_{k_i}, t_{k_j}) = 0$ as $t \to \infty$ which means that $(\sigma_i - 1) M_i \, (t_{k_i}, t_{k_j}) = 0$ as $t \to \infty$ for $i$=1…$n$. Since $(\sigma_i - 1) \neq 0$ we have $M_i \, (t_{k_i}, t_{k_j}) = 0$ as $t \to \infty$ for $i$=1…$n$. By definition $M_i(t_{k_i}, t_{k_j}) = \sum_{j \in N_i} \left( q(x_i(t_{k_i})) - q(x_j(t_{k_j})) \right)^2$, which consists of a summation of quadratic terms. Then we have that

$$\lim_{t\to\infty} q(x_i(t_{k_i})) = \lim_{t\to\infty} q(x_j(t_{k_j})), \quad i, j = 1...n. \tag{30}$$

In view of (20), (27), and (30) all errors $e_i^q(t)$ reset and remain equal to zero, that is, we have that $\lim_{t\to\infty} e_i^q(t) = 0$ for $i$=1…$n$, which is equivalent to

$$\lim_{t\to\infty} q(x_i(t_{k_i})) = \lim_{t\to\infty} q(x_i(t)), \quad i = 1...n \tag{31}$$

therefore, it follows from (30) and (31) that

$$\lim_{t\to\infty} q(x_i(t)) = \lim_{t\to\infty} q(x_j(t)), \quad i, j = 1...n. \tag{32}$$

It can be shown that for undirected graphs, and using quantization of the states in this case, the initial average remains constant. Define the average $\overline{x}(t) = \frac{1}{N} \sum_i x_i(t)$, we have the following:

$$\dot{\overline{x}}(t) = \tfrac{1}{N} \sum_i \dot{x}_i(t) = -\tfrac{1}{N} \sum_i \sum_{j \in N_i} \left( q(x_i(t)) - q(x_j(t)) \right) - \tfrac{1}{N} \sum_i \sum_{j \in N_i} \left( e_i^q(t) - e_j^q(t) \right) = 0 \tag{33}$$

and $\overline{x}(t) = \overline{x}(0) = \frac{1}{N} \sum_i x_i(0)$.

Let $\overline{q} = \lim_{t\to\infty} q(x_i(t))$, from (32) and (33) we have that

$$\left|\overline{q} - \overline{x}(0)\right| \le \frac{\delta}{2}. \tag{34}$$

The last statement can be shown by contradiction. Assume that $\left|\overline{q} - \overline{x}(0)\right| > \frac{\delta}{2}$ then, from (19), we have that either $x_i(t) > \overline{x}(0)$ for $i$=1…$n$, or $x_i(t) < \overline{x}(0)$ for $i$=1…$n$, and in both cases (33) does not hold and we have a contradiction.

From (34) and (19) we obtain (23) and the proof is complete. ■

It is important to note that inter-event times are not lower bounded when using quantization. It is still possible to define solutions for this type of trajectories in the sense of Krasowskii, as it is shown in (Ceragioli, De Persis, and Frasca 2011), by introducing ideal sliding modes for trajectories that contain accumulation points. On the other hand, the computations associated with the event triggered communication policy require continuous sensing, quantizing, and computing and comparing errors and time-varying thresholds. In practice, all these operations can be performed locally by each agent's processor unit frequently but not continuously. This implementation disassociates trajectories from ideal sliding modes and creates a chattering effect.

In order to prevent the undesired chattering effect that may be present when a system transmits updates very frequently in the boundary of a quantization level and its associated Zeno behavior we introduce a minimum update interval $\tau^* > 0$. The minimum update interval is useful not only for avoiding Zeno behavior but also for a practical implementation of a non-continuous sensing and quantizing scheme. In the following we relax the assumption that the errors need to be calculated continuously; instead, we introduce a sampling time $T$, $0 < T \le \tau^*$ which allows for a practical implementation of the event triggered approach. Note that the sampling time $T$ is only used to check the error periodically but communication between agents is still event based since at every sampling time each agent decides if transmission of information is needed based on the size of the current error. In selecting $\tau^*$ we want to ensure that the error $e_i(t) = x_i(t_{k_i}) - x_i(t)$ remains bounded in a desired region $|e_i(t)| < \delta$ for the time interval $t \in [t_{k_i}, t_{k_i} + \tau^*]$. This means that if an update is triggered by the *i*-th agent at time $t_{k_i}$ then we have:

$$\left|e_i^q(t)\right| = \left|q(x_i(t_{k_i})) - q(x_i(t))\right| \le \delta, \quad t \in [t_{k_i}, t_{k_i} + \tau^*] \tag{35}$$

Let us consider in this case a fixed threshold. An event is triggered when:

$$\left|e_i^q(t)\right| \ge p\delta \tag{36}$$

where $p \ge 1$ is an integer since $e_i^q(t)$ varies in increments of $\delta$. In general, asymptotic convergence of the quantized outputs is not achieved in this case, but convergence to a bounded region around the initial average can be shown by evaluating the difference of the states of any one agent and the remaining agents in the network. Choose, without loss of generality, an agent and re-label it as $x_c$ and the remaining agents as $x_1 \ldots x_{n-1}$. Let $\underline{A}$ and $\underline{L}$ represent the adjacency and the Laplacian matrices associated with the communication graph corresponding to the remaining agents.

**Corollary 4.** Consider a group of agents $\dot{x}_i = u_i$ for $i=1\ldots n$, and each agent transmits its quantized output $q(x_i(t_{k_i}))$ to its neighbors at some time instants $t_{k_i}$. The control inputs are given by (20) and the event-based updates are triggered according to (36). Consider a sampled non-continuous event implementation. Assume that the original (before choosing an agent) communication graph is connected and undirected. Then all agents stabilize to a bounded region around their initial average and the following is satisfied:

$$\lim_{\kappa \to \infty} \left\| x_i[\kappa] - x_c[\kappa] \right\|_\infty \le \delta \left(1 + \left(p + \tfrac{1}{2}\right) \|G\|_\infty \right) \left\| (\underline{L} + diag\{A_c\})^{-1} \right\|_\infty \tag{37}$$

when *T* is designed to satisfy

$$T < \min\left(\min_{i=1,\dots,n} 1/|N_i|, \tau^*\right), \tag{38}$$

where $A_c$ is a row vector containing the entries, other than the $a_{c,c}$ entry, in the $c$-th row of the original adjacency matrix $A$ and $G = [A_c^T \ \underline{L}]$.

*Proof*. First, given control inputs (20), triggering condition (36), and for any configuration with finite initial conditions and for fixed and connected topologies (such as the ones considered here) there always exists a finite and positive constant $S$ such that $|\dot{x}_i| \le S$. Consider the behavior of the error $e_i(t) = x_i(t_{k_i}) - x_i(t)$ as follows:

$$\frac{d}{dt}|e_i(t)| \le |\dot{e}_i(t)| = |\dot{x}_i(t)| \le \dot{\phi}_i = S. \tag{39}$$

Solving (39) with initial condition $\phi_i(t_{k_i}) = 0$ we obtain

$$\left|e_i(t_{k_i} + \tau^*)\right| \le \phi_i(t_{k_i} + \tau^*) = S\tau^* = c \tag{40}$$

for 0<$c$<$\delta$. Then $\tau^* = \frac{c}{S} > 0$ is a lower bound on the inter-event times that the $i$-th agent uses to broadcast its measurements. Additionally, using the minimum update interval $\tau^*$ estimated by (40), we guarantee that $\left|q(x_i(t_{k_i} + \tau^*)) - q(x_i(t_{k_i}))\right| \le \delta$. Also note that the initial average using quantization remains constant as it was shown in Theorem 3.

Define $\xi_i[\kappa] = x_i[\kappa] - x_c[\kappa]$, which represents the difference between the chosen agent and any other remaining agent at the $T$-discretized time instants indexed by $\kappa$. We have the following:

$$\begin{aligned} \xi_i[\kappa+1] &= x_i[\kappa] - T\sum_{j\in N_i}\left(q(x_i[\kappa_{k_i}]) - q(x_j[\kappa_{k_j}])\right) - x_c[\kappa+1] \\ &= \xi_i[\kappa] - T\sum_{j\in N_i}\left(\xi_i[\kappa] - \xi_j[\kappa] - \varepsilon_i[\kappa] + e_i^q[\kappa] + \varepsilon_j[\kappa] - e_j^q[\kappa]\right) - x_c[\kappa+1] + x_c[\kappa] \end{aligned} \tag{41}$$

for $i$=1…$n$-1, where $T\kappa_{k_i} = t_{k_i}$ and $\varepsilon_i[\kappa] = x_i[\kappa] - q(x_i[\kappa])$. It is clear that in this case the event times $t_{k_i}$ take place at some of the discrete time instants $\kappa$ labeled as $\kappa_{k_i}$.

Equation (41) can be written in a compact form:

$$\xi[\kappa+1] = Q\xi[\kappa] + TG(\varepsilon[\kappa] - e^q[\kappa]) + X_c[\kappa] \tag{42}$$

where $Q = I_n - T\underline{L} - T \bullet diag\{A_c\}$, $X_c[\kappa] = (-x_c[\kappa+1] + x_c[\kappa])1_n$, $\xi[\kappa] = [\xi_1[\kappa] ... \xi_{n-1}[\kappa]]^T$, $\varepsilon[\kappa] = [\varepsilon_c[\kappa], \varepsilon_1[\kappa] ... \varepsilon_{n-1}[\kappa]]^T$, $e^q[\kappa] = [e^q_c[\kappa], e^q_1[\kappa] ... e^q_{n-1}[\kappa]]^T$. The response of (42) to initial condition $\xi[0]$ is given by:

$$\xi[\kappa] = Q^\kappa \xi[0] - \sum_{l=1}^{\kappa} Q^{\kappa-l} TG(\varepsilon[l-1] - e^q[l-1]) + \sum_{l=1}^{\kappa} Q^{\kappa-l} X_c[l-1]. \tag{43}$$

The norm of $\xi[\kappa]$ satisfies:

$$\|\xi[\kappa]\|_\infty \le \|Q^\kappa\|_\infty \|\xi[0]\|_\infty + \delta(p + \tfrac{1}{2}) T \|G\|_\infty \left\| \sum_{l=0}^{\kappa-1} Q^l \right\|_\infty + T\delta \left\| \sum_{l=0}^{\kappa-1} Q^l \right\|_\infty. \tag{44}$$

Since the original communication graph is connected then agent $x_c$ has directed paths to all followers and $0 < T < \min_{i=1,\dots,n} 1/|N_i|$ then, by lemma 8.3 in (Ren and Cao 2011), $Q$ has all its eigenvalues within the unit circle and $\lim_{\kappa\to\infty} Q^\kappa = 0$. Additionally, from Lemma 1.26 and Lemma 1.28 in (Ren and Cao 2011), we have that $\lim_{\kappa\to\infty} \left\| \sum_{l=0}^{\kappa-1} Q^l \right\|_\infty = \|(I_n - Q)^{-1}\|_\infty$ and

$$\lim_{\kappa\to\infty} \|\xi[\kappa]\|_\infty \le \delta\left(1 + \left(p + \tfrac{1}{2}\right)\|G\|_\infty\right) \|(\underline{L} + diag\{A_c\})^{-1}\|_\infty \tag{45}$$

which is equivalent to (37). ■

*Remark 2*. Any agent in the network can be selected as $x_c$ resulting in different expressions in (37) according to the remaining agents' communication graph. The minimum of these expressions holds as a bound in (37). Since the initial average is constant the agents converge around the initial average.

*Remark 3.* Threshold (36) is constant once we choose a quantization parameter. This threshold choice makes sense because the error (21) varies in increments of $\delta$. Additionally, from (37), the region of convergence can be reduced by choosing a smaller $\delta$, by trading off sampling-inter-event time.

## 4. Examples

*Example 1.* Consider eight agents exchanging quantized measurements of positions using a negligible sampling time for computing the error. The quantization parameter is $\delta$=0.5. Simulation results are shown in Fig. 1. This figure shows that the non-quantized states of the agents converge to a region around the initial average which is 3.875. In this example all the quantized states reach a common value and the bound (23) is satisfied. In addition, since the quantized states reach a common value, the agents do not move and no additional events are generated after approximately 13 seconds as it can be seen in the center and right plots of Fig. 1, where the broadcasting periods for each agent are shown.

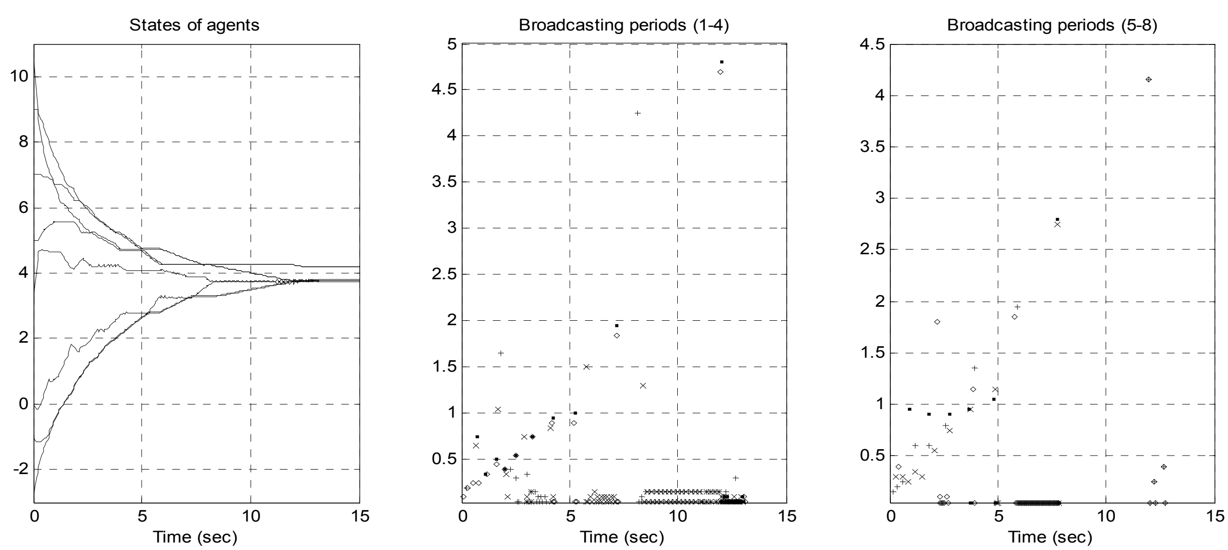


Fig. 1. Quantized consensus. Left: states of eight agents. Center: Broadcasting periods agents 1-4, agent 1(◊), agent 2(■), agent 3(x), agent 4(+). Right: Broadcasting periods agents 5-8, agent 5(◊), agent 6(■), agent 7(x), agent 8(+).

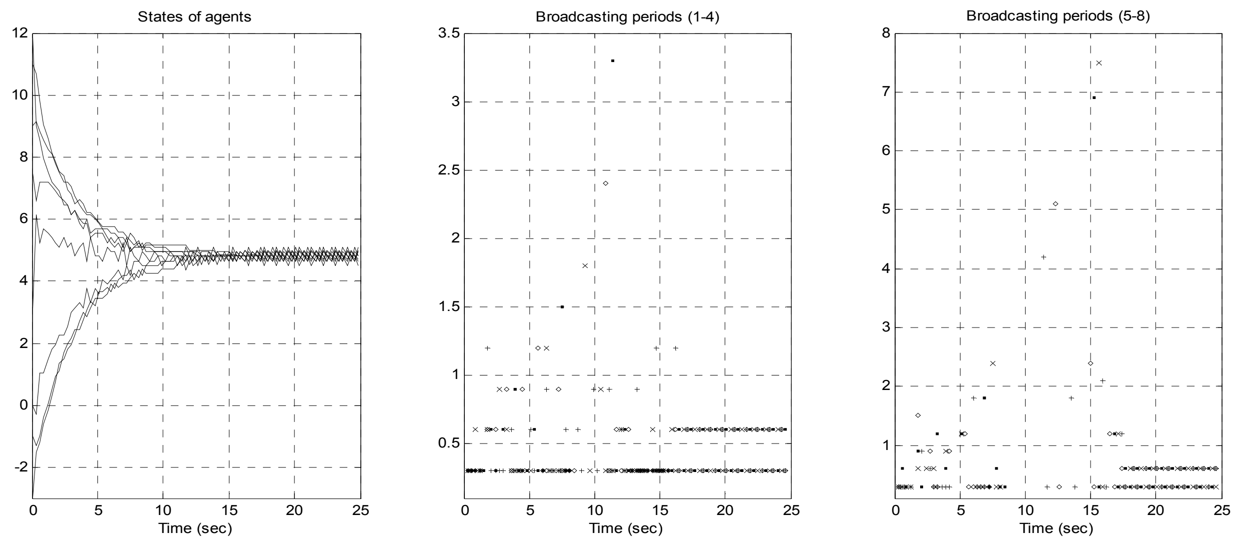


Fig. 2. Quantized consensus with strictly positive inter-event times. Left: states of eight agents. Center: Broadcasting periods agents 1-4, agent 1(◊), agent 2(■), agent 3(x), agent 4(+). Right: Broadcasting periods agents 5-8, agent 5(◊), agent 6(■), agent 7(x), agent 8(+).

*Example 2*. We consider the same system as in Example 1 but the difference is that we introduce a minimum inter-event time equal to 0.3 seconds which also serves as a sampling interval for calculating the error. We select $p$=1 and $\delta$=0.5. Simulation results are shown in Fig. 2.

In this case the average of the non-quantized states also remains constant over time, The quantized values do not reach a common value although the difference of any pair of them remains bounded. The agents keep sending updates when they reach this region but using event times equal or greater than the minimum update interval which is 0.3 seconds as it can be observed in the center and right plots of Fig.2. The agents converge to a bounded region around the initial average equal to 4.8125 and the bound (37) is satisfied. The minimum theoretical bound for this example is equal 17 which is conservative since, from Fig. 2 and after transient response, the difference between any two agents is less than 0.7.

## 5. Conclusions

Decentralized control and broadcasting laws for consensus were presented in this note. The main advantage of this formulation compared to similar work is that we were able to reduce both actuation and transmission updates; continuous monitoring of states of neighbors is no longer needed. Asymptotic convergence to initial average was shown. We offered an important extension to consider transmission of quantized measurements. In order to avoid Zeno behavior in this case we introduced a strictly positive minimum update interval and convergence to a bounded region around the initial average was obtained.

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